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Differentiation properties of the perimeter-to-area ratio for finitely many overlapped unit squares

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In this paper we examine finite unions of unit squares in same plane and consider the ratio of perimeter to area of these unions. In 1998, T. Keleti published the conjecture that this ratio never exceeds 4. Here we study the continuity and differentiability of functions derived from the geometry of the union of those squares. Specifically we show that if there is a counterexample to Keleti's conjecture, there is also one where the associated ratio function is differentiable.

1. Introduction

The purpose of this paper is to introduce several functions associated with the *perimeter-to-area conjecture* (*PAC*) of Tamás Keleti [1998] and to investigate the smoothness properties of those functions.

Keleti's perimeter-to-area conjecture (PAC). *The perimeter-to-area ratio of the union of finitely many unit squares in a plane does not exceed* 4.

The problem of showing such a ratio is bounded first seems to have appeared as Problem 6 in the 1998 edition of the famous Miklós Schweitzer Competition in Hungary [Competition 1998]. Later that same year, Keleti published his perimeter-to-area conjecture that this bound is actually 4. To date, the best known bound is slightly less than 5.6. This bound was achieved by Keleti's student Zoltán Gyenes [2005] in his master's thesis. A special case of the theorem, where all of the squares are axis oriented, is known to be true; Gyenes also presents a proof of this case in the above work, and the authors present two additional proofs in [Humke et al. 2015]. The PAC is particularly intriguing as some of its obvious generalizations are false. Gyenes [2005] showed that the corresponding ratio for unions of congruent convex sets need not be bounded by the ratio for a single copy of the set.

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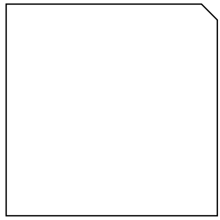

Figure 1. A convex set counterexample to the PAC for general convex sets.

Gyenes's example. There exist congruent convex sets $E_1 \cong E_2 \subset \mathbb{R}^2$ such that the perimeter-to-area ratio for $E_1 \cup E_2$ exceeds the perimeter-to-area ratio for either one of them.

The Gyenes example is disarmingly straightforward. The convex set template is an origin-centered unit square with one judiciously chosen isosceles corner triangle removed. That corner triangle is chosen so that the perimeter-to-area ratio of the resulting figure is less than 4. But the union of this template with a rotated copy is simply the original unit square whose perimeter-to-area ratio is exactly 4. See Figure 1.

In this paper, we build machinery for analyzing the PAC, showing that for almost all finite unions of squares, the perimeter to area ratio is differentiable in the usual Euclidean sense. If a counterexample exists, then there exists a counterexample where the derivative exists. These results provide inroads toward understanding the PAC by potentially relating it to large body of discrete geometric work, including the Kneser–Poulsen theorem and results by Ho-Lun Cheng and Herbert Edelsbrunner [2003] on derivatives when translating circles in the plane.

2. Notation and setting

Let $H = \bigcup_{i=1}^{n} H_i$ be the finite union of unit squares H_i in $\mathbb{R}^2$. Let the perimeter and area functions, $p(\,\cdot\,)$ and $\alpha(\,\cdot\,)$ respectively, take a closed, bounded polygonal figure in the plane as input and return that figure's perimeter and area respectively. If S is a set, we denote the boundary of S by bd S. Throughout we will be interested in the boundary of polygonal regions, and one focus of our attention will be the (maximal) segments comprising that boundary. We refer to these maximal segments as *component segments* of the boundary. The ϵ-ball about a set S will be denoted by $B_\epsilon(S)$ and the convex hull of a set of points $\{p_i : i = 1, 2, \dots, k\} \subset \mathbb{R}^2$ is denoted by $[p_1, p_2, \dots, p_k]$.

Any point $(s_i, t_i, \phi_i)_{i=1}^{n} \in \mathbb{R}^{3n}$ may be mapped to an ordered union of n squares by taking (s_i, t_i) to be the rectangular coordinates of the center of the i-th square H_i and ϕ_i to be the smallest angle between the horizontal and a side of H_i. For notational convenience, we will also denote a single component square H_i by its coordinates, i.e., $H_i = (s_i, t_i, \phi_i)$. This correspondence between $\mathbb{R}^{3n}$ and ordered unions of n squares is surjective and throughout this paper will serve as the domain for corresponding perimeter and area functions. As a general convention, when we refer to a figure $H \subset \mathbb{R}^{3n}$, we shall mean that H is the ordered union of n unit squares determined according to the correspondence described above. Define the function

$$\mathfrak{r} : \mathbb{R}^{3n} \to \mathbb{R}, \quad \mathfrak{r}(H) = \frac{p(H)}{\alpha(H)}.$$

That is, $\mathfrak{r}$ takes an ordered $3n$-tuple of identifiers as input and returns the ratio we've been examining for the figure identified by H. We'll refer to the vector $(\phi_1, \phi_2, \ldots, \phi_n) \in \mathbb{R}^n$ as the *rotational displacement* of H. A figure $H \subset \mathbb{R}^2$ is said to have *distinct rotational displacement* if $\phi_i \neq \phi_j$ when $i \neq j$, is *vertex-free* if no vertex of H_i lies on the boundary of H_j whenever $i \neq j$, and is *triple-free* if no point lies on the boundaries of three distinct H_i. H is said to be in *standard position* provided:

(1) H has distinct rotational displacement,

(2) H is vertex-free, and

(3) H is triple-free.

The set of points in $\mathbb{R}^{3n}$ that do not have distinct rotational displacement lie on finitely many linear curves of the form $\phi_i = \phi_j + k\pi/2$, where $i \neq j$ and $k = 1, 2, 3$. Points which are not vertex-free lie on finitely many curves that are quadratic in the variables $\{s_i, t_i, \sin\phi_i, \cos\phi_i : i = 1, 2, \ldots, n\}$, and points which are not triple-free lie on finitely many quartic curves in the same variables. It follows that the set of points which are in standard position is the complement of a *sparse set* in the sense that they are the complement of a countable union of monotonic curves and so are both residual and of full measure in $\mathbb{R}^{3n}$.

3. Continuity of perimeter and area

Here we give elementary geometric proofs that both the perimeter and area functions as we've defined them are continuous at configurations that are in standard position.

Lemma 1. *The perimeter function p is continuous at every point $H \in \mathbb{R}^{3n}$ which is in standard position.*

Proof. To show that perimeter is continuous, let $[a, b] \subset \operatorname{bd} H$ be a segment of maximal length on $\operatorname{bd} H$. As H has distinct rotational displacement there is a unique component square, say H_{i_o}, such that $[a, b] \subset \operatorname{bd} H_{i_o}$. To simplify notation, we assume $\phi_{i_o} = s_{i_o} = t_{i_o} = 0$, $a = (x_1, -1/2)$ and $b = (x_2, -1/2)$, where $-1/2 \leq x_1 < x_2 \leq 1/2$. We'll examine in some detail the case where neither a nor b are vertices of H_{i_o}; the other cases are similar.

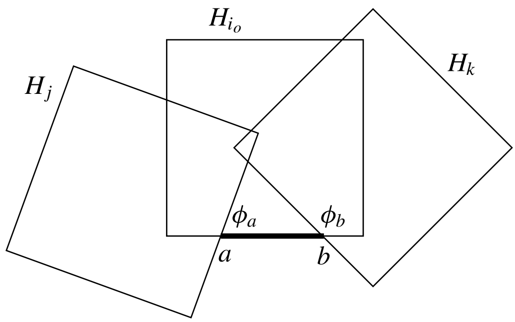

Figure 2. Since H is in standard position, a and b are uniquely determined by two additional component squares H_j and H_k.

Since H is in standard position, a and b are uniquely determined by two additional component squares H_j and H_k in the sense that $a = \operatorname{bd} H_{i_o} \cap \operatorname{bd} H_j$ and $b = \operatorname{bd} H_{i_o} \cap \operatorname{bd} H_k$. Let ϕ_a denote the angle determined by the intersection of the boundaries of H_{i_o} and H_j at a measured counterclockwise from the boundary of H_{i_o} to that of H_j; the angle ϕ_b is defined analogously. See Figure 2.

As $\phi_{i_o} = 0$, either $\phi_j = \phi_a$ or $\phi_j = \phi_a - \pi/2$. Also, $\phi_k = \phi_b$ or $\phi_k = \phi_b - \pi/2$. For definiteness we've supposed that $\phi_j = \phi_a$ and $\phi_k = \phi_b - \pi/2$ so that a is the intersection of the line ℓ given by $y = -1/2$ and ℓ_a given by $y = \tan\phi_a(x - x_1) + 1/2$. Similarly, b is the intersection of ℓ with the line ℓ_b given by $y = \tan\phi_b(x - x_2) + 1/2$. Using the fact that H is in standard position, there is an $\epsilon > 0$ such that $B_\epsilon([a, b])$ intersects no component square of H except H_{i_o}, H_j, and H_k.

Our immediate aim is to show that small perturbations of H result in small local perturbations of $\operatorname{bd} H$. To this end, suppose $\delta > 0$ and $d = (u_i, v_i, w_i)_{i=1}^n$ is a unit vector in $\mathbb{R}^{3n}$. Let $H^* = H + \delta \cdot d$ and denote its component squares by $H_i^* = H_i + \delta \cdot (u_i, v_i, w_i)$. Then, for δ sufficiently small, $B_\epsilon([a, b]) \cap H_i^* \neq \varnothing$ if and only if $i = i_o$, j or k. Let

(1) ℓ^* denote the line ℓ rotated by w_{i_o} about the center of H_{i_o}, $(0, 0)$, then translated by (u_{i_0}, v_{i_0}),

(2) ℓ_a^* denote the line ℓ_a rotated by w_j about the center of H_j, (s_j, t_j), then translated by (u_j, v_j),

(3) ℓ_b^* denote the line ℓ_b rotated by w_k about the center of H_k, (s_k, t_k), then translated by (u_k, v_k).

Finally, let $a^* = \ell^* \cap \ell_a^*$ and $b^* = \ell^* \cap \ell_b^*$. Then $[a^*, b^*]$ is a maximal segment on $\operatorname{bd} H^*$ and is the sole portion of $\operatorname{bd} H^*$ in $B_\epsilon([a, b])$. An elementary estimate shows that $\big| |[a^*, b^*]| - |[a, b]| \big| < 6\delta$.

As there are only finitely many such segments $[a, b] \subset \operatorname{bd} H_i$ comprising the boundary of H, and for δ sufficiently small, there is a one-to-one correspondence between these segments and those comprising the boundary of H^*, it follows that p is continuous at each point of standard position. □

The actual situation is that the perimeter function is continuous at a much larger set of points than those in standard position. The proof given above can be easily adapted to show that p is continuous at points having distinct rotational displacement; however, p is also continuous at most points that do not have distinct rotational displacement. Typical of points at which p is discontinuous is the point $H = (0, 0, 0, 1, .5, 0)$, where the perimeter is 7. If $H_n = (0, 0, 0, 1 + 1/n, .5, 1/n)$, then for every $n \in \mathbb{N}$, $p(H_n) = 8$ and yet $\{H_n\} \to H$.

A bit more can be said about the continuity of p at all points whether in standard position or not.

Proposition 2. *The function $p : \mathbb{R}^{3n} \to \mathbb{R}$ is lower semicontinuous.*

Proof. To see this, suppose $H = \bigcup_{i=1}^{n} H_i$ is an arbitrary configuration with component squares H_i. A segment $[a, b] \subset \operatorname{bd} H$ is called *proper* if $[a, b]$ is of maximal length under the restriction that no vertex of a component square of H lies on $[a, b]\backslash\{a, b\}$. The fact that H is the finite union of squares means that the boundary of H can be uniquely written as the nonoverlapping union of proper boundary segments. Suppose now that $.1 > \epsilon > 0$ is given and that $S = [a, b]$ is any proper segment on the boundary of H. Let $a^*, b^* \in S$ such that both $|a - a^*| = \epsilon|b - a|$ and $|b - b^*| = \epsilon|b - a|$ and set $S^* = [a^*, b^*]$. Let U be a ball about S^* such that $U \cap \operatorname{bd} H \subset S$ and the radius of the ball is less than $\epsilon/2$. For small ΔH, we wish to use $p(H)$ to estimate $p(H + \Delta H)$. First note that if S lies on a unique component square, then $S^* + \Delta H \subset (S + \Delta H) \cap U \subset \operatorname{bd}(H + \Delta H)$, and if all proper boundary segments have this property, we obtain an easy estimate of $p(H + \Delta H)$. However, should S be common to several component squares of H, then $S + \Delta H$ is the union of several segments and $\operatorname{bd}(H + \Delta H) \cap U$ is a piecewise linear selection from $S + \Delta H$. We handle this situation as follows. Let N_{a^*} denote the line segment in U that contains a^* and is normal to S; N_{b^*} is defined analogously for b^*. Let $a^{**} = (a + a^*)/2$, $b^{**} = (b + b^*)/2$ and $S^{**} = [a^{**}, b^{**}]$. We may take ΔH sufficiently small so that:

(1) $S^{**} + \Delta H \subset U$,

(2) $(S^{**} + \Delta H) \cap N_{a^*} \neq \varnothing \neq (S^{**} + \Delta H) \cap N_{b^*}$, and

(3) if T^{**} is analogous to S^{**}, but derived from another proper boundary segment, then $(T^{**} + \Delta H) \cap U = \varnothing$.

These conditions imply that a proper boundary segment for H yields a portion, but not necessarily a segment, of the boundary of $H + \Delta H$ that extends from N_{a^*} to N_{b^*}. As such, its length is at least $(1 - 2\epsilon)|b - a|$. Moreover, it follows from (3) above that distinct proper boundary segments of H yield disjoint boundary portions of $H + \Delta H$. Hence,

$$p(\mathrm{bd}\, H + \Delta H) \geq p(\mathrm{bd}\, H)(1 - 2\epsilon).$$

Since ϵ is arbitrary, it follows that

$$\liminf_{\Delta H \to 0} p(\mathrm{bd}\, H + \Delta H) \geq p(\mathrm{bd}\, H),$$

or that $p : \mathbb{R}^{3n} \to \mathbb{R}$ is lower semicontinuous. □

The minimizer for p is 4, occurring when all component squares of H coincide. The fact that p is lower semicontinuous, coupled with the continuity of the area function α, implies that the ratio p/α is lower semicontinuous and so has a minimizer. Establishing the minimizer for the ratio p/α and minimizers of similar configurations is interesting, but uses completely different methods from those of the current paper and is the topic of a separate study. It is not known if a maximizer of p/α exists.

We turn now to consider the area function.

Lemma 3. *The area function α is continuous at every point in $\mathbb{R}^{3n}$.*

Proof. As the area of each component square H_i is 1, it follows that the area function $\alpha : \mathbb{R}^3 \to \mathbb{R}$ is Lipschitz in each coordinate with a Lipschitz constant of 1. Hence, α itself is Lipschitz with Lipschitz constant $\sqrt{3n}$. □

The following theorem now follows immediately from Lemmas 1 and 3.

Theorem 4. *The function $\mathfrak{r}$ is continuous at every point $H \in \mathbb{R}^{3n}$ which is in standard position.*

4. A derivative computation for perimeter

Next, we investigate the differentiability of the perimeter and area functions. Our goal is to prove the following theorem.

Theorem 5. *The perimeter function $p : \mathbb{R}^{3n} \to \mathbb{R}^+$ is differentiable at every point $H \in \mathbb{R}^{3n}$ in standard position.*

Proof. We show the partial derivatives of p exist and are continuous at each point of standard position. Fix $0 \leq i_o \leq n$ and consider the three partial derivatives $\partial p/\partial s_{i_o}$, $\partial p/\partial t_{i_o}$ and, initially, $\partial p/\partial \phi_{i_o}$.

Part 1: $\partial p/\partial \phi_{i_o}$. As in Lemma 1, we take a particular component segment $[a, b]$ on the boundary of H and again note that $[a, b]$ must lie on the boundary of a single H_j. There are two cases depending on whether $H_j = H_{i_o}$.

Case 1a: $[a, b]$ lies on the boundary of H_{i_o}. We adopt the notation of Lemma 1 in its entirety so that $\phi_{i_o} = s_{i_o} = t_{i_o} = 0$, $a = (x_1, -1/2)$ and $b = (x_2, -1/2)$, where $-1/2 \leq x_1 < x_2 \leq 1/2$. The lines ℓ, ℓ_a and ℓ_b are also as before. However, the unit

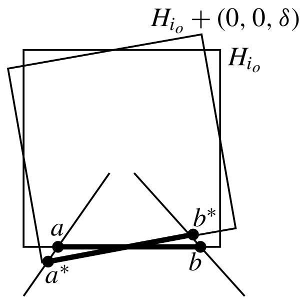

Figure 3. $[a, b]$ and $[a^*, b^*]$ in the case that $[a, b]$ lies on the boundary of H_{i_o}.

vector we consider is more specific in this case; $d \in \mathbb{R}^{3n}$ is the vector with $w_{i_o} = 1$ and all remaining components 0. The set $H^* = H + \delta \cdot d$ is comprised of precisely the same component squares as H with the exception of H_{i_o}, which is replaced by $H_{i_o} + (0, 0, \delta)$, a rotation of H_{i_o} about its center by an angle of δ. The segment on bd H^* corresponding to $[a, b]$ is $[a^*, b^*]$, where $a* = \ell^* \cap \ell_a$ and $b^* = \ell^* \cap \ell_b*$ since $\ell_a = \ell_a^*$ and $\ell_b = \ell_b^*$. See Figure 3.

A computation yields

$$\begin{aligned} a^* &= \left(\frac{x_1 \tan \phi_a}{\tan \phi_a - \tan \delta}, \frac{x_1 \tan \phi_a \tan \delta}{\tan \phi_a - \tan \delta} - \frac{1}{2} \right), \\ b^* &= \left(\frac{x_2 \tan \phi_b}{\tan \phi_b + \tan \delta}, \frac{x_2 \tan \phi_b \tan \delta}{\tan \phi_b + \tan \delta} - \frac{1}{2} \right). \end{aligned} \tag{1}$$

Consequently, the x-coordinate of $b^* - a^*$ is

$$x(b^* - a^*) = \frac{(x_2 - x_1) \tan \phi_a \tan \phi_b - \tan \delta (x_2 \tan \phi_b + x_1 \tan \phi_a)}{(\tan \phi_a - \tan \delta)(\tan \phi_b + \tan \delta)}.$$

However, $x(b^* - a^*)/|b^* - a^*| = \cos \delta$ and so

$$|b^* - a^*| = \frac{(x_2 - x_1) \tan \phi_a \tan \phi_b - \tan \delta (x_2 \tan \phi_b + x_1 \tan \phi_a)}{(\tan \phi_a - \tan \delta)(\tan \phi_b + \tan \delta) \cos \delta}. \tag{2}$$

We're now in a position to complete the computation of the contribution of $[a, b]$ to $\partial p / \partial \phi_{i_o}$ at H_{i_o}:

$$\lim_{\delta \to 0} \frac{|b^* - a^*| - |b - a|}{\delta} = (x_2 - x_1)(\cot \phi_b - \cot \phi_a). \tag{3}$$

Case 1b: $[a, b]$ lies on the boundary of H_j with $i_o \neq j$. If $[a, b] \cap H_{i_o} = \varnothing$, then the partial derivative of that portion of p with respect to ϕ_{i_o} is 0. Hence, we may assume that $[a, b] \cap H_{i_o} \neq \varnothing$. As $[a, b] \subset$ bd H is maximal, it follows that either $[a, b] \cap H_{i_o} = \{a\}$ or $[a, b] \cap H_{i_o} = \{b\}$. We suppose the former and for purposes

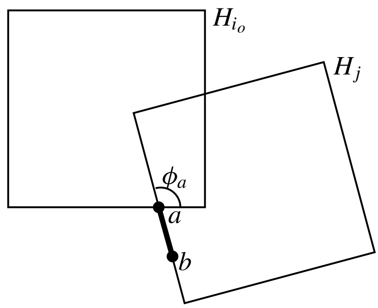

Figure 4. $[a, b] \not\subset \operatorname{bd} H_{i_o}$.

of computation, we again adopt some of the notation of Lemma 1. Specifically we suppose that $H_{i_o} = (0, 0, 0)$, $a = (x_1, -1/2)$, ϕ_a, ℓ_a, ℓ and ℓ^* are as before. Then the segment $[a, b]$ lies on the line ℓ_a. See Figure 4 where a lies to the right of the center of H_{i_o} (or $0 \le x_1 \le 1/2$).

In this case, the change in perimeter due to $[a, b]$ is $\big||[a^*, b^*]| - |[a, b]|\big| = |a^* - a|$. Then according to the law of sines,

$$|a^* - a| = \frac{|c - a| \sin\delta}{\sin(\phi_\delta)} = \frac{|c - a| \sin\delta}{\sin\phi_a \cos\delta - \cos\phi_a \sin\delta},$$

where $c = \ell \cap \ell^*$. See Figure 5.

Hence, in this case, the contribution of $[a, b]$ to $\partial p/\partial\phi_{i_o}$ at H_{i_o} is

$$\lim_{\delta\to 0} \frac{|a^* - a|}{\delta} = \lim_{\delta\to 0} \frac{|c - a|(\sin\delta)/\delta}{\sin\phi_a \cos\delta - \cos\phi_a \sin\delta} = \frac{x_1}{\sin\phi_a}.$$

Since H is in standard position, this is well-defined and continuous. The case in which a lies to the left of the center of H_{i_o} is the same, but is negative since $-1/2 \le x_1 < 0$ in that case.

To summarize, we have shown that the rate of change of each component segment of $\operatorname{bd} H$ with respect to ϕ_{i_o} is continuous. As $\operatorname{bd} H$ is comprised of finitely many

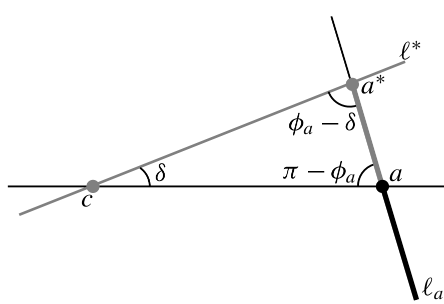

Figure 5. $\big||[a^*, b^*]| - |[a, b]|\big| = |a^* - a|$ in the case that $[a, b] \not\subset \operatorname{bd} H_{i_o}$.

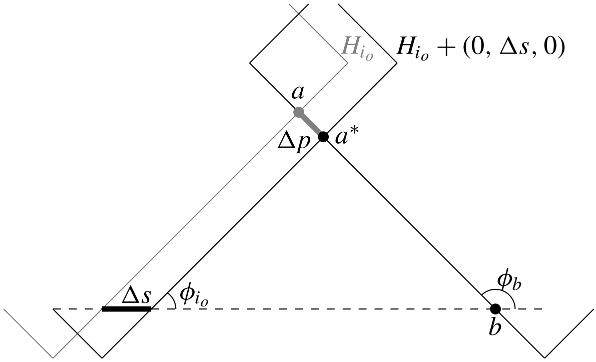

Figure 6. Translating H_{i_o} by $(0, \Delta s, 0)$.

such segments, it follows that $\partial p/\partial \phi_{i_o}$ exists and is continuous at each point in standard position.

Part 2: $\partial p/\partial s_i$. For notational convenience, we denote Δs_{i_o} simply by Δs. As in the previous case, we take a particular segment $[a, b]$ on the boundary of H and consider three cases:

(a) $[a, b]$ does not lie on $\operatorname{bd} H_{i_o}$,

(b) $[a, b]$ lies on $\operatorname{bd} H_{i_o}$ and contains a vertex of H_{i_o}, and

(c) $[a, b]$ lies on $\operatorname{bd} H_{i_o}$ and neither a nor b are vertices of H_{i_o}.

Case 2a: Suppose that $[a, b] \not\subset \operatorname{bd} H_{i_o}$. If $[a, b] \cap H_{i_o} = \varnothing$, a sufficiently small translation of H_{i_o} will leave $[a, b]$ unchanged. Suppose then that $[a, b] \cap H_{i_o} \neq \varnothing$. As H is in standard position, it follows that either $[a, b] \cap H_{i_o} = \{a\}$ or $[a, b] \cap H_{i_o} = \{b\}$. Suppose $\operatorname{bd} H_i \cap [a, b] = \{a\}$. If Δs is sufficiently small, the point b remains on $\operatorname{bd} H$ when translating H_{i_o} by Δs. Because a lies on the intersection of two component squares and H is in standard position, a is not the vertex of any square. Hence, $\operatorname{bd}(H_{i_o} + (0, \Delta s, 0)) \cap [a, b] \neq \varnothing$. Let $\operatorname{bd}(H_{i_o} + (0, \Delta s, 0)) \cap [a, b] = a^*$. The segments $[a^*, b]$ and $[a, b]$ differ by a length of Δp and it is this distance we wish to compute. See Figure 6.

We consider two cases depending on if $\phi_{i_o} < \phi_b$ or $\phi_{i_o} > \phi_b$. Supposing that $\phi_{i_o} < \phi_b$, the relevant triangle is illustrated in Figure 7.

Using the law of sines, we compute $\Delta p/\Delta s = \sin \phi_{i_o}/\sin(\phi_b - \phi_{i_o})$. If $\phi_{i_o} > \phi_b$, a similar computation yields $\Delta p/\Delta s = -\sin \phi_{i_o}/\sin(\phi_b - \phi_{i_o})$. As H is in standard position, $\phi_{i_o} \neq \phi_b$, so these are the only cases.

Case 2b: Suppose that $[a, b] \subset \operatorname{bd} H_{i_o}$ and that a is a vertex of H_{i_o}. If b is also a vertex of H_{i_o}, then as H is in standard position, neither a nor b lie on the boundary of another component square. Consequently, $[a + \Delta s, b + \Delta s] \subset \operatorname{bd}(H + (0, \Delta s, 0))$, given a sufficiently small translation Δs.

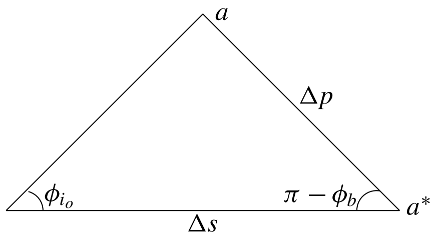

Figure 7. The relevant triangle in the case that $\phi_{i_o} < \phi_b$.

Suppose then that b is not a vertex of H_{i_o}. Then, given a sufficiently small translation Δs, the segment $[a + \Delta s, b + \Delta s]$ will intersect bd H at point b^* and $[b, b^*] \subset \text{bd } H$. See Figure 8.

At this point the geometry and subsequent computation of $\Delta p/\Delta s$ are analogous to that found in Figure 7. In the case illustrated in Figure 8, $\Delta p/\Delta s = \cos\phi_{i_o} - \sin\phi_{i_o} \tan\phi_b$.

Case 2c: Finally, suppose that $[a, b]$ lies on bd H_{i_o} and neither a nor b are vertices of H_{i_o}. Then the endpoints a and b of this boundary segment lie on uniquely determined segments on bd H that lie on the boundaries of component squares other than H_{i_o}. This is similar to the analysis done in Case 1a above. As H is in standard position, if Δs is sufficiently small, then as H_{i_o} is translated to $H_{i_o} + (0, \Delta s, 0)$, the boundary segment $[a, b]$ is translated to a new boundary segment $[a^*, b^*]$. Moreover, a^* lies on the same boundary segment of H as does a, and b^* lies on the same boundary segment of H as does b. See Figure 9.

In order to facilitate the required computation, we establish the notation found in Figure 10.

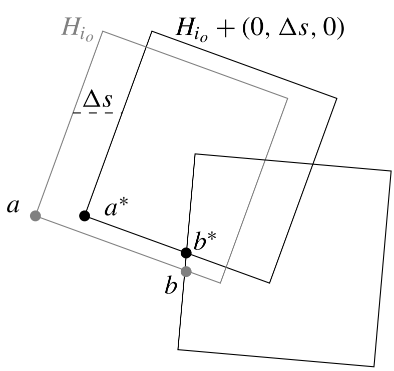

Figure 8. The case in which b is not a vertex of H_{i_o}.

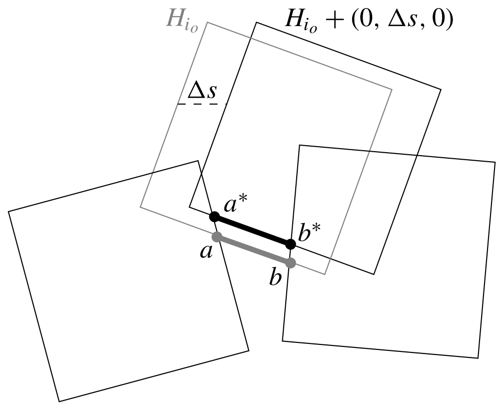

Figure 9. The case in which $[a, b]$ lies on bd H_{i_o} and neither a nor b are vertices of H_{i_o}.

Applying the law of sines to triangles $[a, a^*, c]$ and $[a, a^*, d]$, we find

$$\frac{|a - a^*|}{\sin\phi} = \frac{\Delta s}{\sin(\pi - \phi - \phi_{i_o})},$$
$$\frac{|a - a^*|}{\sin(\phi + \phi_b)} = \frac{\Delta p}{\sin(\phi_{i_o} - \phi_b)}.$$

Hence,

$$\frac{\Delta p}{\Delta s} = \frac{\sin\phi \sin(\phi_{i_o} - \phi_b)}{\sin(\phi + \phi_{i_o}) \sin(\phi + \phi_b)}.$$

In the case illustrated in Figure 10, $\phi = \pi/2 - \phi_{i_o}$. As H is in standard position, this quantity is well-defined and indeed constant.

To obtain $\partial p/\partial s_i$, sum $\Delta p/\Delta s$ for every component segment $[a, b] \subset \operatorname{bd} H$. Since there are finitely many partitioning segments and $\Delta p/\Delta s$ is continuous for each of them, $\partial p/\partial s_i$ exists and is continuous at every point in standard position.

Part 3: $\partial p/\partial y_i$. Rotating the whole figure by 90°, this case becomes exactly the same as the previous one.

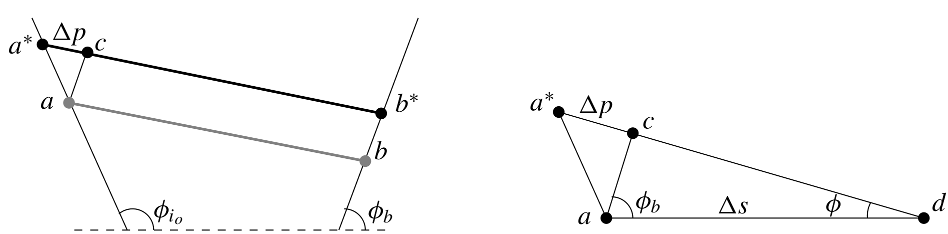

Figure 10. The same case with added detail and notation.

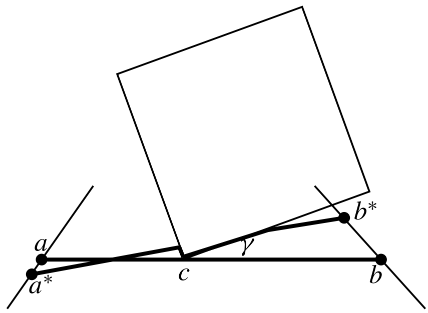

Figure 11. When the segment $[a, b]$ contains the vertex of a new square, the derivative of perimeter changes discontinuously.

Finally, because all of the partial derivatives exist and are continuous at each point in standard position, the perimeter function is differentiable at every point $H \in \mathbb{R}^{3n}$ of standard position, as desired. □

The situation for points that are not in standard position is mixed. For example, at some of these points the perimeter is differentiable; if the configuration H is not vertex-free, but the vertices that lie on edges of remote squares are all interior to H, then the fact that H is not vertex-free has no bearing on the differentiability of perimeter at H. However, if a segment on the boundary of H contains a vertex, then p is not differentiable at H. Figure 11 is a portion of Figure 3 with an additional square added in such a way that a new vertex, c resides on the segment $[a, b]$. The angle this new square makes with the segment $[a, b]$ is important and labeled γ. Also important is the distance $|c-a|$ this vertex is from the endpoint a. In the next paragraph, we adopt the notation established earlier in Lemma 1 and Part 1 in the proof of Theorem 5.

In such a case, several of the partial derivatives do not exist. In particular, both one-sided partial derivatives, $\partial p/\partial \phi_{i_o}^+$ and $\partial p/\partial \phi_{i_o}^-$ exist, but differ. The latter, $\partial p/\partial \phi_{i_o}^-$ is as computed in Part 1 of the proof of Theorem 5. However, a computation similar to this shows that $\partial p/\partial \phi_{i_o}^+$ differs from $\partial p/\partial \phi_{i_o}^-$ by $|c-a|\tan(\gamma)$. See Figure 11. To summarize,

$$\frac{\partial p}{\partial s_{i_o}^-} = (x_2 - x_1)(\cot \phi_b - \cot \phi_a),$$

$$\frac{\partial p}{\partial s_{i_o}^+} = (x_2 - x_1)(\cot \phi_b - \cot \phi_a) + |c-a| \tan \alpha.$$

5. A derivative computation for area

To show $\mathfrak{r}$ is differentiable at every point in standard position, it remains to show that the area function is differentiable at every point in standard position and to show how that derivative can be computed.

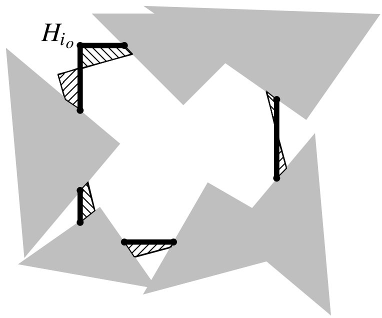

Figure 12. The change in area after rotating H_{i_o}.

Theorem 6. *The area function $\alpha : \mathbb{R}^{3n} \to \mathbb{R}^+$ is differentiable at every point $H \in \mathbb{R}^{3n}$ in standard position.*

Proof. Again we show the partial derivatives of α exist and are continuous at each point of standard position. Fix $0 \le i_o \le n$. If $H_{i_o} \subset \operatorname{int}(H)$, then $\partial a/\partial s_{i_o}(H) = \partial a/\partial t_{i_o}(H) = \partial a/\partial \phi_{i_o}(H) = 0$, so we may assume that a portion of $\operatorname{bd} H_{i_o}$ is contained in $\operatorname{bd} H$; since H is in standard position, $\operatorname{bd} H_{i_o} \cap \operatorname{bd} H$ is the union of closed nonoverlapping intervals.

Part 1: $\partial\alpha/\partial\phi_{i_o}$. There may be several segments common to $\operatorname{bd} H_{i_o}$ and $\operatorname{bd} H$, and for a fixed $\delta = \Delta\phi_{i_o}$, each such segment contributes to a corresponding $\Delta\alpha$. See Figure 12 where those segments common to both $\operatorname{bd} H_{i_o}$ and $\operatorname{bd} H$ are darkened and the components of $\Delta\alpha$ are hatched, northeast for gain and northwest for loss. The gray regions are portions of the other component squares of H that intersect H_{i_o}.

The area change, $\Delta\alpha$ is simply the sum total of the signed area changes at each of the line segment components of $\operatorname{bd} H_{i_o} \cap \operatorname{bd} H$. There are several cases to consider depending on the relative location of a boundary segment of $\operatorname{bd} H_{i_o} \cap \operatorname{bd} H$, but in any case, for purposes of this computation, we may assume $H_{i_o} = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$ and that the boundary segment $[a, b]$ lies on the line $y = -\frac{1}{2}$.

Case 1a: $a = \left(x_1, -\frac{1}{2}\right)$ and $b = \left(x_2, -\frac{1}{2}\right)$ with $-\frac{1}{2} < x_1 < x_2 \le 0$. Then the additional area determined by $[a, b]$ is a net gain (or +) and is the area of the quadrilateral $[a, p, q, b]$. See Figure 13 where the notation is the same as in the Lemma 1 except for a new value of $d = (0, \dots, 0, \Delta\phi_{i_o}, 0, \dots, 0)$. Using the coordinates of a^* and b^* computed in (1), we find the area of the quadrilateral $[a, a^*, b^*, b]$ to be

$$\Delta\alpha([a, b]) = \frac{(x_1^2 - x_2^2)\tan\phi_a \tan\phi_b \tan\delta + \tan^2\delta\,(x_2^2 \tan\phi_b + x_1^2 \tan\phi_a)}{2(\tan\phi_a - \tan\delta)(\tan\phi_b + \tan\delta)}. \qquad (4)$$

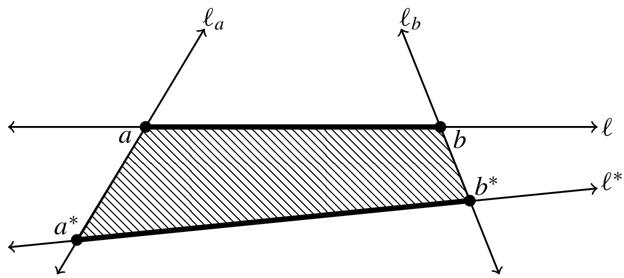

Figure 13. $\Delta\alpha$ at $[a, b]$ in Case 1a.

To find the contribution to $\partial\alpha/\phi_{i_o}$ at $[a, b]$, we divide the quantity found in (4) by δ and take the limit as $\delta \to 0$ to obtain

$$\left.\frac{\partial\alpha}{\phi_{i_o}}\right|_{[a,b]} = \lim_{\delta\to 0} \frac{\Delta\alpha([a, b])}{\delta} = \frac{x_1^2 - x_2^2}{2}.$$

Case 1b: $a = \left(x_1, -\frac{1}{2}\right)$ and $b = \left(x_2, -\frac{1}{2}\right)$ with $0 \le x_1 < x_2 \le \frac{1}{2}$. This case is symmetric to Case 1a, but since $0 \le x_1 < x_2$, the contribution to $\partial\alpha/\phi_{i_o}$ is negative:

$$\left.\frac{\partial\alpha}{\phi_{i_o}}\right|_{[a,b]} = \frac{x_1^2 - x_2^2}{2}.$$

Case 1c: $a = \left(x_1, -\frac{1}{2}\right)$ and $b = \left(x_2, -\frac{1}{2}\right)$ with $-\frac{1}{2} \le x_1 \le 0 \le x_2 \le \frac{1}{2}$. Once again

$$\left.\frac{\partial\alpha}{\phi_{i_o}}\right|_{[a,b]} = \frac{x_1^2 - x_2^2}{2}.$$

To see this, introduce $x_3 = 0$ and add the corresponding amounts computed using the formula from Cases 1a and 1b.

Part 2: $\partial\alpha/\partial s_{i_o}$. The analysis of this case is much the same as Part 1. Again there may be several segments common to bd H_{i_o} and bd H, and for a fixed $\delta = \Delta s_{i_o}$, each such segment contributes to a corresponding $\Delta\alpha$. See Figure 14 where those segments common to both bd H_{i_o} and bd H are darkened and the components of $\Delta\alpha$ are hatched, northeast for gain and northwest for loss. The gray regions are portions of the other component squares of H that intersect H_{i_o}.

There are two basic cases to consider here. The first concerns a component segment $[a, b] \subset \text{bd}\, H_{i_o} \cap \text{bd}\, H$ that lies on either the bottom or top of H_{i_o} and the second is when that segment lies on one of the other two sides. In both cases we let $\Delta s_{i_o} = \Delta s$ be sufficiently small.

Case 2a: $[a, b]$ lies on the bottom of H_{i_o}. For definiteness, we again suppose that neither a nor b is a vertex of H_{i_o}, as the cases when they are vertices can be handled

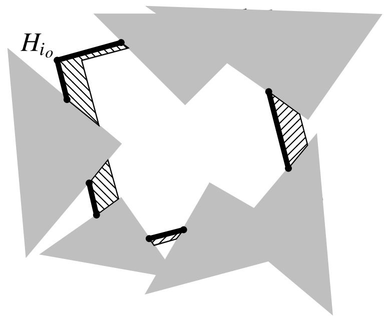

Figure 14. The change in area after translating H_{i_o}.

in much the same manner. As before, there are uniquely defined H_j and H_k such that $a \in \mathrm{bd}\, H_j \cap \mathrm{bd}\, H_{i_o}$ and $b \in \mathrm{bd}\, H_k \cap \mathrm{bd}\, H_{i_o}$. Also let ℓ denote the line containing $[a, b]$, so that the segment $[a, b]$ is that portion of ℓ between H_j and H_k. Similarly, let ℓ^* be the line $\ell + (0, \Delta s, 0)$, and let $[a^*, b^*]$ denote that segment on ℓ^* extending between H_j and H_k. See Figure 15 where the trapezoidal region whose area is $\Delta\alpha$ at $[a, b]$ is shaded and the rotation displacement ϕ_{i_o} as well as Δs are labeled.

Then

$$\Delta\alpha|_{[a,b]} = \frac{|b-a| + |b^* - a^*|}{2} \sin\phi_{i_o} \cdot \Delta s.$$

From this it follows that

$$\left.\frac{\partial\alpha}{\partial s_{i_o}}\right|_{[a,b]} = \lim_{\Delta s \to 0} \frac{|b-a| + |b^* - a^*|}{2} \sin\phi_{i_o} = |b-a| \sin\phi_{i_o}.$$

The case in which $[a, b]$ lies on the top of H_{i_o} is identical with the exception that the sign is negative.

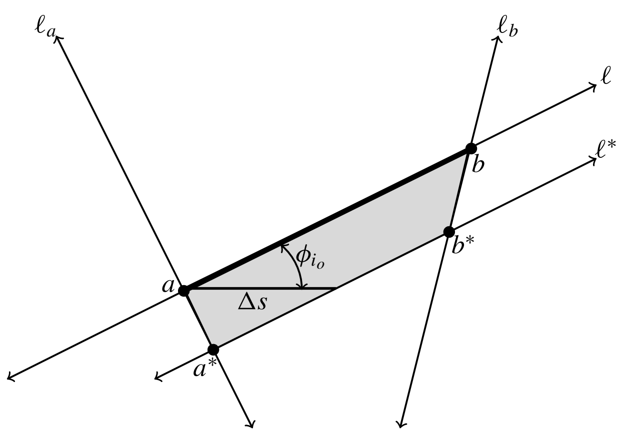

Figure 15. $\Delta\alpha$ at $[a, b]$ in Case 2a.

Case 2b: $[a, b]$ lies on the right (or left) side of H_{i_o}. The right side case is much the same as described in Case 2a above, but with the angle ϕ_{i_o} replaced by $\phi_{i_o} + \pi/2$. The resulting derivative formula becomes

$$\left.\frac{\partial \alpha}{\partial s_{i_o}}\right|_{[a,b]} = \lim_{\Delta s \to 0} \frac{|b-a| + |b^* - a^*|}{2} \cos \phi_{i_o} = |b-a| \cos \phi_{i_o}.$$

As above, the left side case is identical with the exception that the sign is negative.

Part 3: $\partial\alpha/\partial t_{i_o}$. As with the analysis of perimeter, the translation cases are completely analogous.

As each of the partial derivatives is defined and continuous at each point H in standard position, the proof of Theorem 6 is complete. □

Because the perimeter and area functions are differentiable at every point in standard position (and $a \neq 0$), their ratio $\mathfrak{r} : \mathbb{R}^{3n} \to \mathbb{R}^+$ is differentiable and our main result now follows immediately.

Theorem 7. *The function $\mathfrak{r} : \mathbb{R}^{3n} \to \mathbb{R}^+$ is differentiable at every point $H \in \mathbb{R}^{3n}$ in standard position.*

The idea of studying the variation of $p(H)/a(H)$ allows us to consider a vast body of discrete geometric literature to help study the problem. However, nearly all of this literature concerns itself with studying disks in the plane, rather than squares or arbitrary shapes. An example is the famous Kneser–Poulsen theorem concerning disks in the plane. See [Bezdek and Connelly 2002] and [Bollobás 1968] for details.

Kneser–Poulsen theorem. *If a set of disks in the plane are rearranged so that the distance between the centers of any pair of discs decreases, then the area and the perimeter of the union of the discs also decreases.*

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