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Factor posets of frames and dual frames in finite dimensions

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We consider frames in a finite-dimensional Hilbert space, where frames are exactly the spanning sets of the vector space. A factor poset of a frame is defined to be a collection of subsets of I, the index set of our vectors, ordered by inclusion so that nonempty $J \subseteq I$ is in the factor poset if and only if $\{f_i\}_{i\in J}$ is a tight frame. We first study when a poset $P \subseteq 2^I$ is a factor poset of a frame and then relate the two topics by discussing the connections between the factor posets of frames and their duals. Additionally we discuss duals with regard to ℓ^p-minimization.

1. Introduction

A frame for a finite-dimensional Hilbert space is a possibly redundant spanning set. The concept of frames was introduced by Duffin and Schaeffer [1952]. Daubechies [1992] popularized the use of frames. Many of the modern signal processing algorithms used in mobile phones or digital televisions are developed using the concept of frames. Redundancy in frames plays a pivotal role in the construction of stable signal representations and in mitigating the effect of losses in transmission of signals through communication channels [Goyal et al. 2001; 1998]. A tight frame is a special case of a frame, which has a reconstruction formula similar to that of an orthonormal basis. Because of this simple formulation of reconstruction, tight frames are employed in a variety of applications such as sampling, signal processing, filtering, smoothing, denoising, compression, and image processing.

A factor poset $\mathbb{F}_F$ of a frame $F = \{f_i\}_{i\in I}$ is the collection of subsets $J \subseteq I$ such that $\{f_j\}_{j\in J}$ is a tight frame for a finite-dimensional Hilbert space $\mathcal{H}^n$. We find necessary conditions for a given poset to be a factor poset of a frame. We show that a factor poset is determined entirely by its empty cover (the sets $J \in \mathbb{F}_F$ that have no proper subset in $\mathbb{F}_F$). Moreover, we show that if P is the factor poset of

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a frame $F \subseteq \mathbb{R}^2$ then it is also the factor poset of another frame $G \subseteq \mathbb{R}^2$ whose vectors are multiples of the standard orthonormal basis vectors e_1 and e_2.

We study the relationship among the factor posets of dual frame pairs. Also we study when the dual frame could be tight and when a dual frame can be scaled to be a tight frame. Finally we consider the group structure among all duals of a frame. It is known that a dual is a canonical dual frame if and only if the ℓ^2-sum of the frame coefficients is a minimizer among ℓ^2-sums of frame coefficients of all dual frames. We find new inequalities among ℓ^p-sums of these frame coefficients when $p = 1$ and $p > 2$.

2. Preliminaries

Throughout this paper $\mathcal{H}^n$ denotes either $\mathbb{R}^n$ or $\mathbb{C}^n$. A sequence $F = \{f_i\}_{i=1}^k \subseteq \mathcal{H}^n$ is called a frame for $\mathcal{H}^n$ with frame bounds $A, B > 0$ if for any $f \in \mathcal{H}^n$,

$$A\|f\|^2 \le \sum_{i=1}^{k} |\langle f, f_i\rangle|^2 \le B\|f\|^2. \tag{1}$$

When $A = B = \lambda$, we say that F is a λ-tight frame. For a sequence $F = \{f_i\}_{i=1}^k \subseteq \mathcal{H}^n$, define the analysis operator θ_F from $\mathcal{H}^n$ to $\mathcal{H}^k$ by

$$\theta_F x = \sum_{i=1}^{k} \langle x, f_i\rangle e_i,$$

where $\{e_i\}_{i=1}^k$ is an orthonormal basis for $\mathcal{H}^k$. The adjoint of θ_F, denoted by $\theta_F^* : \mathcal{H}^k \to \mathcal{H}^n$, is defined by $\theta_F^*(e_i) = f_i$ and is called the synthesis operator. The frame operator $\sigma_F : \mathcal{H}^n \to \mathcal{H}^n$ associated to F is defined by $\sigma_F = \theta_F^* \theta_F$, is a positive definite, self-adjoint, invertible operator and all of its eigenvalues belong to the interval $[A, B]$.

Given a frame F, another frame $G = \{g_i\}_{i=1}^k \subseteq \mathcal{H}^n$ is said to be a dual frame of F if the following reconstruction formula holds:

$$f = \sum_{i=1}^{k} \langle f, f_i\rangle g_i \quad \text{for all } f \in \mathcal{H}^n.$$

The canonical dual frame $\widetilde{F}$ associated with $F = \{f_i\}_{i=1}^k$ is given by $\widetilde{F} = \{\sigma_F^{-1} f_i\}_{i=1}^k$.

Definition 2.1. For any vector

$$f = \begin{bmatrix} f(1) \\ \vdots \\ f(n) \end{bmatrix} \in \mathbb{R}^n,$$

we define the diagram vector of f, denoted $\tilde{f}$, by

$$\tilde{f} = \frac{1}{\sqrt{n-1}} \begin{bmatrix} f^2(1) - f^2(2) \\ \vdots \\ f^2(n-1) - f^2(n) \\ \sqrt{2n} f(1) f(2) \\ \vdots \\ \sqrt{2n} f(n-1) f(n) \end{bmatrix} \in \mathbb{R}^{n(n-1)},$$

where the difference of squares $f^2(i) - f^2(j)$ and the product $f(i) f(j)$ occur exactly once for $i < j$, with $i = 1, \ldots, n-1$.

Definition 2.2. For any vector $f \in \mathbb{C}^n$, we define the diagram vector $\tilde{f}$ of f to be

$$\tilde{f} = \frac{1}{\sqrt{n-1}} \begin{bmatrix} |f(1)|^2 - |f(2)|^2 \\ \vdots \\ |f(n-1)|^2 - |f(n)|^2 \\ \sqrt{n}\, f(1)\overline{f(2)} \\ \sqrt{n}\, \overline{f(1)} f(2) \\ \vdots \\ \sqrt{n}\, f(n-1)\overline{f(n)} \\ \sqrt{n}\, \overline{f(n-1)} f(n) \end{bmatrix} \in \mathbb{C}^{3n(n-1)/2},$$

where the difference of the form $|f(i)|^2 - |f(j)|^2$ occurs exactly once for $i < j$, with $i = 1, 2, \ldots, n-1$, and the product of the form $f(i)\overline{f(j)}$ occurs exactly once for $i \neq j$.

Using these definitions, a characterization of tight frames in $\mathcal{H}^n$ is given in [Copenhaver et al. 2014].

Theorem 2.3 [Copenhaver et al. 2014]. *Let $\{f_i\}_{i \in I}$ be a sequence of vectors in $\mathcal{H}^n$, not all of which are zero. Then $\{f_i\}_{i \in I}$ is a tight frame if and only if $\sum_{i \in I} \tilde{f}_i = 0$. Moreover, for any $f, g \in \mathcal{H}^n$, we have $(n-1)\langle \tilde{f}, \tilde{g} \rangle = n |\langle f, g \rangle|^2 - \|f\|^2 \|g\|^2$.*

3. Factor posets

In [Lemvig et al. 2014], a tight frame $F = \{f_i\}_{i \in I}$ in $\mathcal{H}^n$ is said to be *prime* if no proper subset of F is a tight frame for $\mathcal{H}^n$. One of the main results in [loc. cit.] is that for $k \geq n$, every tight frame of k vectors in $\mathcal{H}^n$ is a finite union of prime tight frames called *prime factors* of F. Thus to study the structure of prime factors, we use a well-known combinatorial object, the poset. A nonempty set P with a partial ordering is called a partially ordered set, or poset. A poset can be represented by a Hasse diagram. We define a poset related to frames as follows:

Definition 3.1. Let $F = \{f_i\}_{i \in I} \subseteq \mathcal{H}^n \setminus \{0\}$ be a finite frame, where $I = \{1, \ldots, k\}$. The *factor poset* of F, denoted $\mathbb{F}_F$, is defined to be a collection of subsets of I

ordered by set inclusion so that nonempty $J \subseteq I$ is in $\mathbb{F}_F$ if and only if $\{f_j\}_{j\in J}$ is a tight frame for $\mathcal{H}^n$. We always assume $\varnothing \in \mathbb{F}_F$.

Example 3.2. Let $F=\{e_1, e_2, e_2\}\subseteq\mathbb{R}^2$ and $I=\{1,2,3\}$. Then $\mathbb{F}_F=\{\varnothing, \{1,2\}, \{1,3\}\}$ and the Hasse diagram is

$$\begin{tikzcd}
\{1,2\} \arrow[dr, dash] & & \{1,3\} \arrow[dl, dash] \\
& \{\} &
\end{tikzcd}$$

Example 3.3. Let $F=\{e_1, e_2, -e_1, -e_2\}\subseteq\mathbb{R}^2$ and $I=\{1, 2, 3, 4\}$. Then the Hasse diagram of $\mathbb{F}_F$ is

$$\begin{tikzcd}
& & \{1,2,3,4\} \arrow[dll, dash] \arrow[dl, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
\{1,2\} \arrow[drr, dash] & \{2,3\} \arrow[dr, dash] & & \{3,4\} \arrow[dl, dash] & \{4,1\} \arrow[dll, dash] \\
& & \{\} & &
\end{tikzcd}$$

The next lemma gives us three equivalent conditions for when the union of elements of $\mathbb{F}_F$ is an element of $\mathbb{F}_F$.

Proposition 3.4. *Let $F=\{f_i\}_{i\in I}\subseteq\mathcal{H}^n\setminus\{0\}$ be a tight frame. Suppose $\mathbb{F}_F$ is the factor poset and let $C, D\in\mathbb{F}_F$. Then the following are equivalent*:

(i) $C\cup D\in\mathbb{F}_F$.

(ii) $C\cap D\in\mathbb{F}_F$.

(iii) $C\triangle D\in\mathbb{F}_F$.

(iv) $C\setminus D\in\mathbb{F}_F$.

Proof. By the inclusion-exclusion principle, it is easy to verify that for diagram vectors of F, the following hold:

(a) $\sum_{\ell\in C\cup D}\tilde{f}_\ell=\sum_{\ell\in C}\tilde{f}_\ell+\sum_{\ell\in D}\tilde{f}_\ell-\sum_{\ell\in C\cap D}\tilde{f}_\ell$.

(b) $\sum_{\ell\in C\cup D}\tilde{f}_\ell=\sum_{\ell\in C\setminus D}\tilde{f}_\ell+\sum_{\ell\in D\setminus C}\tilde{f}_\ell+\sum_{\ell\in C\cap D}\tilde{f}_\ell$.

(c) $\sum_{\ell\in C}\tilde{f}_\ell=\sum_{\ell\in C\setminus D}\tilde{f}_\ell+\sum_{\ell\in C\cap D}\tilde{f}_\ell$.

Since $C, D\in\mathbb{F}_F$, using Theorem 2.3 we have $\sum_{\ell\in C}\tilde{f}_\ell=\sum_{\ell\in D}\tilde{f}_\ell=0$. Hence, from (a) we see that (i) $\Longleftrightarrow$ (ii). By the definition of symmetric difference $C\triangle D$, the implication (i) $\Longleftrightarrow$ (ii) and (b) above, it follows that (i) $\Longrightarrow$ (iii). Conversely, if (iii) holds, then from (b) we have $\sum_{\ell\in C\cup D}\tilde{f}_\ell=\sum_{\ell\in C\cap D}\tilde{f}_\ell$. But from (a), when $C, D\in\mathbb{F}_F$ we have $\sum_{\ell\in C\cup D}\tilde{f}_\ell=-\sum_{\ell\in C\cap D}\tilde{f}_\ell$. Hence (iii) $\Longrightarrow$ (ii). Thus (i)–(iii) are equivalent. Using (c) above, we conclude that (iv) $\Longleftrightarrow$ (ii). $\square$

The above proposition gives some necessary conditions for a given poset to be a factor poset of a frame.

Proposition 3.5. *Let $F = \{f_i\}_{i\in I} \subseteq \mathcal{H}^n \setminus \{0\}$ be a finite frame with corresponding factor poset $\mathbb{F}_F$. Then for any $m \in \mathbb{N}$ with $m \geq |I| = k$, there exists a frame sequence $G = \{g_j\}_{j\in J} \subseteq \mathcal{H}^n \setminus \{0\}$, where $J = \{1, \ldots, m\}$, such that $\mathbb{F}_F = \mathbb{F}_G$.*

Proof. We show that there exists some $g_{k+1} \in \mathcal{H}^n \setminus \{0\}$ so that $G = F \cup \{g_{k+1}\}$ satisfies $\mathbb{F}_F = \mathbb{F}_G$. Consider

$$T = \left\{ -\sum_{\ell\in L} \tilde{f}_\ell : \varnothing \subsetneq L \subseteq I \right\}.$$

This is a finite collection of vectors in $\mathbb{R}^{n(n-1)}$ or $\mathbb{C}^{3n(n-1)/2}$. Now select $g_{k+1} \in \mathcal{H}^n \setminus \{0\}$ so that $\tilde{g}_{k+1} \notin T$. This completes the proof. $\square$

Definition 3.6. For a frame $F = \{f_i\}_{i\in I} \subseteq \mathcal{H}^n$ and its factor poset $\mathbb{F}_F$, we define the empty cover of $\mathbb{F}_F$, denoted $\mathrm{EC}(\mathbb{F}_F)$, to be the set of $J \in \mathbb{F}_F$ which cover $\varnothing \in \mathbb{F}_F$; that is,

$$\mathrm{EC}(\mathbb{F}_F) = \left\{ J \in \mathbb{F}_F : J \neq \varnothing \text{ and } \nexists J' \in \mathbb{F}_F \text{ with } \varnothing \subsetneq J' \subsetneq J \right\}.$$

Example 3.7. Let $F = \{e_1, e_2, -e_1, -e_2\} \subseteq \mathbb{R}^2$. As seen from Example 3.3,

$$\mathrm{EC}(\mathbb{F}_F) = \left\{ \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\} \right\}.$$

We now show that a factor poset is entirely determined by its empty cover.

Proposition 3.8. *Let $F = \{f_i\}_{i\in I} \subseteq \mathcal{H}^n$ be a finite frame. If $\mathbb{F}_F$ is the factor poset of F, then for any nonempty J in $\mathbb{F}_F \setminus \mathrm{EC}(\mathbb{F}_F)$, there exists $J_1, J_2 \in \mathbb{F}_F$ with $J_1 \subsetneq J$ and $J_2 \subsetneq J$ so that $J_1 \cap J_2 = \varnothing$ and $J_1 \sqcup J_2 = J$.*

Proof. Let $J \in \mathbb{F}_F \setminus \mathrm{EC}(\mathbb{F}_F)$ be a nonempty set. There must exist some nonempty $J_1 \in \mathbb{F}_F$ so that $J_1 \subsetneq J$, otherwise $J \in \mathrm{EC}(\mathbb{F}_F)$. Using Proposition 3.4, it is easy to see that $J_2 := J \setminus J_1 \in \mathbb{F}_F$. By the assumption on J and J_1, we see that J_2 is nonempty. Hence $J = J_1 \sqcup J_2$. $\square$

Corollary 3.9. *Let $F = \{f_i\}_{i\in I}, G = \{g_i\}_{i\in I}$ be finite frames in $\mathcal{H}^n$ with factor posets $\mathbb{F}_F$ and $\mathbb{F}_G$, respectively. Then $\mathrm{EC}(\mathbb{F}_F) = \mathrm{EC}(\mathbb{F}_G)$ if and only if $\mathbb{F}_F = \mathbb{F}_G$.*

Proof. It is obvious that if $\mathbb{F}_F = \mathbb{F}_G$ then the empty covers are equal, so we restrict our attention to the other direction. It suffices to show that the factor poset of the frame is entirely determined by its empty cover. Let $T = \mathrm{EC}(\mathbb{F}_F) \cup \{\varnothing\}$ for a frame $F = \{f_i\}_{i\in I} \subseteq \mathcal{H}^n$ with $\mathbb{F}_F$ as its factor poset. For every $J_1, J_2 \in T$, if $J_1 \cap J_2 \in T$ then append $J_1 \cup J_2$ to T. Repeat this process until no more unions can be added. This process must terminate after finitely many iterations since I is finite. Clearly the new collection of sets, which we again denote by T, is contained in $\mathbb{F}_F$. From Proposition 3.8, the reverse inclusion holds. Therefore the factor poset $\mathbb{F}_F$ is determined by $\mathrm{EC}(\mathbb{F}_F) \cup \{\varnothing\}$. The desired result follows. $\square$

As a consequence of Corollary 3.9, we get an alternate proof of the following result from [Lemvig et al. 2014].

Corollary 3.10. *Every tight frame $F = \{f_i\}_{i \in I} \subseteq \mathcal{H}^n \setminus \{0\}$ can be written as a union of prime tight frames.*

The proof of Corollary 3.10 follows from observing that if $J \in \mathrm{EC}(\mathbb{F}_F)$ then $\{f_j\}_{j \in J}$ is a prime tight frame. An important case of factor posets occurs when we consider a tight frame $F = \{f_i\}_{i \in I} \subseteq \mathcal{H}^n \setminus \{0\}$. Note that when F is tight, we have that $\varnothing, I \in \mathbb{F}_F$.

Definition 3.11. Suppose $F = \{f_i\}_{i \in I} \subseteq \mathcal{H}^n \setminus \{0\}$ is a frame. Let $\chi(F)$ denote the sequence indexed by I, where $\chi(F)(i)$ is the number of times i occurs in $\mathrm{EC}(\mathbb{F}_F)$ for each $i \in I$. We call $\chi(F)$ the characteristic of F. If $\chi(F)(i) = m$ for all $i \in I$ then F is said to have uniform characteristic.

Example 3.12. Let $F = \{e_1, e_1, e_2\} \subseteq \mathbb{R}^2$. Then $\mathrm{EC}(\mathbb{F}_F) = \{\{1, 3\}, \{2, 3\}\}$ and $\chi(F) = (1, 1, 2)$. Hence $\chi(F)$ need not be uniform.

Proposition 3.13. *If $F = \{f_i\}_{i \in I} \subseteq \mathcal{H}^n \setminus \{0\}$ has positive uniform characteristic, then F is a tight frame.*

Proof. Suppose that F has uniform characteristic $m > 0$. Let $T_1, \dots, T_h$ be the elements of $\mathrm{EC}(\mathbb{F}_F)$. Then $\sum_{i \in T_q} \tilde{f}_i = 0$ for $q = 1, \dots, h$. So $\sum_{q=1}^{h} \sum_{i \in T_q} \tilde{f}_i = 0$. Since $j \in I$ occurs in $\mathrm{EC}(\mathbb{F}_F)$ m times, it follows that $\tilde{f}_j$ occurs m times in the sum $\sum_q \sum_{i \in T_q} \tilde{f}_i = 0$. Hence

$$\sum_{j \in I} \tilde{f}_j = \frac{1}{m} \Bigl(\sum_q \sum_{i \in T_q} \tilde{f}_i \Bigr) = 0.$$

Hence F is a tight frame. □

Remark 3.14. The condition in Proposition 3.13 is sufficient but not necessary. Consider the frame $F = \{e_1, e_1, e_2, e_2, e_1 + e_2, e_1 - e_2\}$, which is a tight frame, but $\chi(F) = (2, 2, 2, 2, 1, 1)$.

The following theorem states that given a factor poset P of a frame in $\mathbb{R}^2$, we can find another frame with vectors parallel to e_1 and e_2 (the standard orthonormal basis vectors) whose factor poset is also P.

Theorem 3.15. *Let $F = \{f_i\}_{i \in I} \subseteq \mathbb{R}^2 \setminus \{0\}$ be a finite frame with $I = \{1, \dots, k\}$. Then there exists a frame $G = \{g_i\}_{i \in I}$ whose vectors are scaled multiples of the standard orthonormal basis vectors e_1 and e_2 such that $\mathbb{F}_F = \mathbb{F}_G$.*

Proof. Let $\{J_\ell : 1 \le \ell \le 2^k\}$ be an enumeration of 2^I and let

$$2^I \setminus \mathbb{F}_F = \Bigl\{ J_{\ell_r} : \sum_{i \in J_{\ell_r}} \tilde{f}_i \neq 0 \Bigr\}.$$

Consider a projection P of rank 1 on $\mathbb{R}^2$ such that range$(P) \neq \left(\text{span}\left\{\sum_{i\in J_\ell} \tilde{f}_i\right\}\right)^\perp$ for any J_ℓ. Let $\widetilde{V} = \{P(\tilde{f}_i) : 1 \le i \le k\}$ and V be the set of vectors of cardinality k in $\mathbb{R}^2$ whose diagram vectors are equal to the set $\widetilde{V}$.

We now claim that $\sum_{i\in J_\ell} \tilde{f}_i = 0$ if and only if $\sum_{i\in J_\ell} P(\tilde{f}_i) = 0$. The forward implication is clear. Now assume $\sum_{i\in J_\ell} P(\tilde{f}_i) = 0$. Then $\sum_{i\in J_\ell} \tilde{f}_i \in \ker(P)$. By the choice of P, we have that $\ker(P) \cap \left(\text{span}\left\{\sum_{i\in J_\ell} \tilde{f}_i\right\}\right) = \{0\}$ for all $J_\ell \in 2^I$. Therefore, $\sum_{i\in J_\ell} \tilde{f}_i \in \ker(P)$ if and only if $\sum_{i\in J_\ell} \tilde{f}_i = 0$. This proves the claim.

Now assume that $\mathbb{F}_F$ contains something other than the empty set. Then F has a tight subframe. Hence there exists some $J' \in 2^I$ such that $\sum_{i\in J'} \tilde{f}_i = 0$. This implies $\sum_{i\in J'} P(\tilde{f}_i) = 0$. Because $\tilde{f}_i \neq 0$ for each $i \in J'$, we know $P(\tilde{f}_i) \neq 0$. By assumption, range$(P) = \text{span}\{v\}$, where v is a unit vector. Then there exist nonzero scalars $\{\alpha_i\}_{i\in J'}$ such that $\alpha_i v = \tilde{f}_i$ for each $i \in J'$. Since $0 = \sum_{j\in J'} P(\tilde{f}_j) = \sum_{j\in J} \alpha_j v$ and $\alpha_j \neq 0$, we have $s, t \in J'$ such that $\text{sgn}(\alpha_s) = -\text{sgn}(\alpha_t)$. Since $\tilde{f}_s = \alpha_s v$ and $\tilde{f}_t = \alpha_t v$, we must have corresponding vectors in V that are nonzero and orthogonal. Since any two nonzero orthogonal vectors span $\mathbb{R}^2$, the vectors in V must span $\mathbb{R}^2$ and hence form a frame.

Suppose $J_\ell \in \mathbb{F}_F$. Then $\sum_{i\in J_\ell} \tilde{f}_i = 0$. From the claim, $\sum_{i\in J_\ell} \tilde{f}_i = 0$ if and only if $\sum_{i\in J_\ell} P(\tilde{f}_i) = 0$. Hence $J_\ell \in \mathbb{F}_V$. The reverse direction is similar, and thus $\mathbb{F}_F = \mathbb{F}_V$.

Since rank$(P) = 1$, there exists a unitary operator U such that $Uv = e_1$. Hence

$$UP(\tilde{f}_i) = \begin{bmatrix} \lambda_i \\ 0 \end{bmatrix}$$

for some $\lambda_i \in \mathbb{R}$. Define g as

$$g_i := \begin{cases} [\sqrt{\lambda_i}\ \ 0]^T & \lambda_i \ge 0, \\ [0\ \ \sqrt{-\lambda_i}\,]^T & \lambda_i < 0. \end{cases}$$

Let $G = \{g_i\}_{i\in I}$. It easily follows that $UP(\tilde{f}_i) = \tilde{g}_i$. Moreover $\sum_{i\in J_\ell} P(\tilde{f}_i) = 0$ if and only if $\sum_{i\in J_\ell} UP(\tilde{f}_i) = 0$. Therefore $\mathbb{F}_G = \mathbb{F}_V = \mathbb{F}_F$. $\square$

Remark 3.16. Based on the above Theorem 3.15, we propose the following inverse factor poset problem: given a poset $P \subseteq 2^I$, does there exist a frame $F \subseteq \mathbb{R}^n$ such that $\mathbb{F}_F = P$?

4. Dual frames

For a given frame F, we define the set $\mathcal{W}_F$ as

$$\mathcal{W}_F := \left\{ W = \begin{bmatrix} \leftarrow & w_1 & \rightarrow \\ & \vdots & \\ \leftarrow & w_n & \rightarrow \end{bmatrix} : \bar{w}_i \in \ker(\theta_F^*) \right\}.$$

Then, by the result in [Li 1995; Christensen et al. 2012], we have the following observation.

Observation 4.1. Let F be a frame for $\mathcal{H}^n$. Then any dual frame to a frame F can be expressed as columns of the matrix

$$\sigma_F^{-1}\theta_F^* + W \tag{2}$$

for some $W \in \mathcal{W}_F$.

For a given frame F, let $\mathcal{G} = \{\sigma_F^{-1}\theta_F^* + W : W \in \mathcal{W}_\mathcal{F}\}$ be the set of all matrices whose columns form duals of F. Define the operation $\oplus : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ by

$$(\sigma_F^{-1}\theta_F^* + W_1) \oplus (\sigma_F^{-1}\theta_F^* + W_2) := \sigma_F^{-1}\theta_F^* + W_1 + W_2.$$

Proposition 4.2. *Let F be a frame and let $\mathcal{G} = \{\sigma_F^{-1}\theta_F^* + W : W \in \mathcal{W}_\mathcal{F}\}$. Then $(\mathcal{G}, \oplus)$ defines an abelian group.*

Proof. For any $W_1, W_2 \in \mathcal{W}$, since $\ker(\theta_F^*)$ is a vector space, $W_1 + W_2 \in \mathcal{W}$, which implies that $\mathcal{G}$ is closed under $\oplus$. Associativity and commutativity follow from associativity and commutativity of matrix addition, and the identity is given by $\sigma_F^{-1}\theta_F^*$. Each element $\sigma_F^{-1}\theta_F^* + W \in \mathcal{G}$ has an inverse $\sigma_F^{-1}\theta_F^* - W$. □

Proposition 4.3. *Let F be a tight frame. Suppose that $G \in \{\sigma_F^{-1}\theta_F^* + W : W \in \mathcal{W}_\mathcal{F}\}$ is a matrix whose columns form a tight frame. Then the subgroup generated by G is contained in the set of matrices whose columns form tight duals of F.*

Proof. Let $\sigma_F = \lambda I_n$. Then $G = \frac{1}{\lambda}\theta_F^* + W$ for some $W \in \mathcal{W}_F$, where $WW^* = \alpha I_n$ for some $\alpha \in \mathbb{R}$. If $H = \frac{1}{\lambda}\theta_F^* + mW$ for some $m \in \mathbb{N}$, then $H^*H = \left(\frac{1}{\lambda} + m^2\alpha\right)I_n$, which completes the proof. □

The following example shows that in general, it is not true that the set of matrices in $\mathcal{G}$ whose columns form a tight frame is a subgroup of $(\mathcal{G}, \oplus)$.

Example 4.4. Let F be the frame where the synthesis operator θ_F^* is given by

$$\theta_F^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then the set of all matrices whose columns form a dual of F is

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

We consider the two matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then $A, B \in \mathcal{G}$, and the columns of A and B form a tight dual of F. However, the columns of

$$A \oplus B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

do not form a tight dual of F.

We study the relationship between the factor posets for a tight frame and its canonical dual. Recall that an isomorphism on posets $(P_1, \leq_1), (P_2, \leq_2)$ is a bijection $\phi : P_1 \to P_2$ so that $\phi(a) \leq_2 \phi(b)$ if and only if $a \leq_1 b$ for all $a, b \in P_1$. We define a stronger notion of order isomorphism. We let S_m denote the symmetric group on m elements.

Definition 4.5. We say that two factor posets $\mathbb{F}_F$ and $\mathbb{F}_G$ corresponding to frames $F = \{f_i\}_{i\in I}$ and $G = \{g_j\}_{j\in J}$ are *strongly isomorphic* if there exists some $m \in \mathbb{N}$ and some $\eta \in S_m$ such that $\eta(\mathbb{F}_F) = \eta(\mathbb{F}_G)$, where $\eta(\mathbb{F}_F) = \{\eta(J') : J' \in \mathbb{F}_F\}$ and $\eta(J') = \{\eta(j) : j \in J'\}$.

If F_J is a tight subframe of a λ-tight frame $F = \{f_i\}_{i\in I}$ for some $J \subseteq I$, then $\sum_{i\in J} \tilde{f}_i = 0$ so that we have

$$\sum_{i\in J} \sigma_F^{-1} \tilde{f}_i = \sum_{i\in J} \frac{1}{\lambda^2} \tilde{f}_i = 0.$$

This implies that $\{\sigma_F^{-1} f_i\}_{i\in J}$ is a tight frame. Thus we have the following result.

Proposition 4.6. *A tight frame F and its associated canonical dual frame $\{\sigma_F^{-1} f_i\}_{i=1}^k$ have the same factor posets.*

This result does not hold true for nontight frames and their canonical duals. For example, the factor poset of the following frame F and its canonical dual are not strongly isomorphic:

$$F = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{3989\sqrt{15912321}}{100} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{3989}{10} \end{bmatrix} \right\}.$$

Proposition 4.7. *There exist a frame F such that no dual G of F has a factor poset structure that is strongly isomorphic to $\mathbb{F}_F$.*

Proof. Let $F = \{e_1, e_1, e_2\}$ be a frame for $\mathbb{R}^2$, where $\{e_1, e_2\}$ is the standard orthonormal basis of $\mathbb{R}^2$. Let G be an arbitrary dual of F; then by Observation 4.1,

$$\theta_G^* = \begin{bmatrix} \frac{1}{2} + a & \frac{1}{2} - a & 0 \\ b & -b & 1 \end{bmatrix}$$

for some $a, b \in \mathbb{R}$. We consider the poset $\mathbb{F}_F = \{\varnothing, \{1, 3\}, \{2, 3\}\}$. We know that two vectors in $\mathbb{R}^2$ form a tight frame if and only the vectors are orthogonal and are of equal norm. If F and G have strongly isomorphic poset structures, then two of the vectors of G must be orthogonal to the remaining vector in G, and all three vectors have equal norms. This is impossible. $\square$

From the dual expression given in (2), we obtain a characterization of tight duals of a tight frame. The following result is remarked in [Krahmer et al. 2013]; the proof given here is different.

Theorem 4.8. *Let F be a λ-tight frame with k frame elements for $\mathcal{H}^n$. If $k < 2n$, then the canonical dual is the only tight dual of F. If $k \geq 2n$ then F has an alternate dual that is tight.*

Proof. Let G be a dual frame of F. Since $\theta_G^* = \frac{1}{\lambda}\theta_F^* + W$ for some $W \in \mathcal{W}$, we have that $\theta_G^*\theta_G = \frac{1}{\lambda} I_n + WW^*$. This implies that G is tight if and only if $WW^* = \alpha I_n$ for some $\alpha \in \mathbb{R}$. If $k < 2n$, since $\dim(\ker(\theta_F^*)) < n$, we have $\alpha = 0$. This implies that if $k < 2n$, then the canonical dual is the only tight dual of F. Let $k \geq 2n$ and let $\{\bar{w}_j\}_{j=1}^{k-n}$ be an orthonormal basis for $\ker(\theta^*)$. Then for any $\alpha \in \mathcal{H}\setminus\{0\}$, we consider

$$W = \alpha \begin{bmatrix} \leftarrow & w_1 & \rightarrow \\ & \vdots & \\ \leftarrow & w_n & \rightarrow \end{bmatrix}.$$

Since $W \in \mathcal{W}$, we have $\left(\frac{1}{\lambda}\theta_F^* + W\right)\left(\frac{1}{\lambda}\theta_F^* + W\right)^* = \left(\frac{1}{\lambda} + |\alpha|^2\right) I_n$, which implies that the columns of $\frac{1}{\lambda}\theta_F^* + W$ form a tight dual frame of F. □

Remark 4.9. We close this section with a simple and well-known construction for dual frames. Suppose $F = \{f_i\}_{i\in I}$ is a frame for $\mathcal{H}^n$ and $H = \{f_i\}_{j\in J}$ is a subframe of F. If $K = \{g_i\}_{i\in J}$ is a dual of H, then $G = \{g_i\}_{i\in I}$, where

$$g_i = \begin{cases} g_i & \text{if } i \in J, \\ 0 & \text{if } i \in I\setminus J \end{cases}$$

is a dual of F. Because any frame has a basis subframe, we have that if $F = \{f_i\}_{i=1}^k$ is a frame for $\mathcal{H}^n$, then there exists a dual of F consisting of n basis vectors for $\mathcal{H}^n$ and $(k - n)$ zero vectors. Likewise, if a frame in $\mathcal{H}^n$ has a tight subframe, then it has a tight dual.

5. ℓ^p-norm of the frame coefficients

It is well known that the ℓ^2-norm of the frame coefficients with respect to the canonical dual is smaller that the ℓ^2-norm of the frame coefficients with respect to any other dual. Moreover, this ℓ^2-minimization characterizes the canonical dual of a frame.

Proposition 5.1 [Han et al. 2007]. *Let $\{f_i\}_{i=1}^k$ be a frame for $\mathcal{H}^n$ and let $\{g_i\}_{i=1}^k$ be a dual frame of $\{f_i\}_{i=1}^k$. Then $\{g_i\}_{i=1}^k$ is the canonical dual if and only if*

$$\sum_{i=1}^k |\langle f, g_i\rangle|^2 \leq \sum_{i=1}^k |\langle f, h_i\rangle|^2 \quad \textit{for all } f \in \mathcal{H}^n$$

for all frames $\{h_i\}_{i=1}^k$ which are duals of $\{f_i\}_{i=1}^k$.

Using Newton's generalized binomial theorem and Hölder's inequality, for any two sequences $x = \{x_i\}_{i=1}^k$ and $y = \{y_i\}_{i=1}^k$ and $p \in (1, \infty)$, we have

$$\|x\|_p \leq \|x\|_1 \leq k^{1-1/p}\|x\|_p, \tag{3}$$

where $\|x\|_p = \left(\sum_{i=1}^k |x_i|^p\right)^{1/p}$. The right-side inequality in (3) with $p = 2$ and Proposition 5.1 gives us the following result:

Proposition 5.2. *Let $\{f_i\}_{i=1}^k$ be a frame for $\mathcal{H}^n$ and let $\{h_i\}_{i=1}^k$ be a dual frame of $\{f_i\}_{i=1}^k$. If $\{g_i\}_{i=1}^k$ is the canonical dual, then*

$$\sum_{i=1}^k |\langle f, g_i\rangle| \le \sqrt{k} \sum_{i=1}^k |\langle f, h_i\rangle| \quad \textit{for all } f \in \mathcal{H}^n.$$

From the inequalities (3) and Proposition 5.2, for any $p \in (1, \infty)$, we obtain

$$\sum_{i=1}^k |\langle f, g_i\rangle|^p \le k^{3p/2-1} \sum_{i=1}^k |\langle f, h_i\rangle|^p \quad \text{for all } f \in \mathcal{H}^n.$$

If $p > 2$, we have a better estimation.

Theorem 5.3. *Let $\{f_i\}_{i=1}^k$ be a frame for $\mathcal{H}^n$ and let $\{h_i\}_{i=1}^k$ be a dual frame of $\{f_i\}_{i=1}^k$. If $\{g_i\}_{i=1}^k$ is the canonical dual, then for any $p \in (2, \infty)$, we have*

$$\sum_{i=1}^k |\langle f, g_i\rangle|^p \le k^{p/2-1} \sum_{i=1}^k |\langle f, h_i\rangle|^p \quad \textit{for all } f \in \mathcal{H}^n.$$

Proof. First observe that the right-side inequality with $p/2$ implies that

$$\sum_{i=1}^k |\langle f, h_i\rangle|^p = \sum_{i=1}^k (|\langle f, h_i\rangle|^2)^{p/2} \ge k^{1-p/2}\left(\sum_{i=1}^k |\langle f, h_i\rangle|^2\right)^{p/2}.$$

By Proposition 5.1 and the left-side inequality with $p/2$, we have that

$$\sum_{i=1}^k |\langle f, h_i\rangle|^p \ge k^{1-p/2}\left(\sum_{i=1}^k |\langle f, g_i\rangle|^2\right)^{p/2} \ge k^{1-p/2} \sum_{i=1}^k |\langle f, g_i\rangle|^p. \qquad \square$$

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