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We prove a Harnack-type inequality for nonnegative solutions of second order ordinary differential inequalities. Maximum principles are the main tools used, and to make the paper self-contained, we provide alternative proofs to those available in the literature.

1. Introduction

The aim of this paper is to present a self-contained discussion of the Harnack and Harnack-type inequalities for nonnegative solutions of second order linear ordinary differential inequalities of the form

$$Lu \leq f(x), \quad x \in I := (A, B), \tag{1-1}$$

where, for $u \in C^2(I)$,

$$Lu := u''(x) + p(x)u'(x) + q(x)u. \tag{1-2}$$

Here and in the sequel, the notation $C^2(I)$ stands for the class of twice continuously differentiable real-valued functions on the open interval I. Likewise, we write $C(I)$ for the class of continuous real-valued functions on I. Throughout, we will assume, without further mention, that $p, q, f \in C(I)$. In this case, (1-2) can be rewritten as

$$Lu = \frac{1}{r(x)}(r(x)u')' + q(x)u, \quad \text{where } r(x) := \exp\left(\int^x p(t)\,dt\right). \tag{1-3}$$

Let $\mathcal{H}$ be a class of nonnegative and locally bounded functions in the open interval $I = (A, B)$. We say that Harnack's inequality holds for the class $\mathcal{H}$ if and only if given any closed interval $[a, b] \subseteq I$, there is a positive constant C such that

$$\sup_{x \in [a,b]} u(x) \leq C \inf_{x \in [a,b]} u(x) \quad \text{for all } u \in \mathcal{H}. \tag{1-4}$$

The important point here is that C is independent of $u \in \mathcal{H}$. The class $\mathcal{H}$ is usually a collection of nonnegative (or nonpositive) solutions of some differential equations.

This type of inequality is named after Carl Gustav Axel von Harnack (1851–1888) who first derived the inequality for nonnegative harmonic functions in the plane. The inequality became a very important tool in the study of solutions to second order linear and nonlinear elliptic partial differential equations. We refer the interested reader to the article [Kassmann 2007] for a detailed account on some history and theoretical developments of this fascinating inequality, as well as an extensive bibliography of articles and monographs related to Harnack's inequality. We direct the reader to the paper [Berhanu and Mohammed 2005] for a simple application of Harnack's inequality to the ordinary differential equation $Lu = f$. The same paper also provides an example that shows the explicit dependence of the constant C in (1-4) on the differences $a - A$ and $B - b$.

2. On maximum principles

To develop a version of Harnack's inequality for nonnegative solutions of (1-1), we need several results on maximum principles which can be found in [Protter and Weinberger 1984]. To make the paper self-contained and for the readers' convenience, we provide alternative proofs to these maximum principles under the assumption that p and q are continuous on I.

We first introduce an auxiliary function that will be used in our proof of a basic theorem on maximum principles. We use the notation $J := (\alpha, \beta)$ for $\alpha < \beta$. Consider the following auxiliary function, with $\sigma > 0$ to be chosen:

$$z(x) = \sigma(x - \alpha) - e^{\sigma(x - \alpha)}. \tag{2-1}$$

We observe that

$$z(\alpha) = -1 \quad \text{and} \quad z'(\alpha) = 0.$$

Direct computation shows

$$Lz = -\sigma^2 e^{\sigma(x-\alpha)}\left(1 + \frac{p(x)}{\sigma}(1 - e^{-\sigma(x-\alpha)}) + \frac{q(x)}{\sigma}\left(\frac{1}{\sigma} - (x-\alpha)e^{-\sigma(x-\alpha)}\right)\right).$$

If p and q are bounded on $[\alpha, \beta]$, we see that

$$\lim_{\sigma \to \infty}\left(\frac{p(x)}{\sigma}(1 - e^{-\sigma(x-\alpha)}) + \frac{q(x)}{\sigma}\left(\frac{1}{\sigma} - (x-\alpha)e^{-\sigma(x-\alpha)}\right)\right) = 0,$$

uniformly on $[\alpha, \beta]$. Therefore, in this case we can choose $\sigma > 0$ large enough such that

$$Lz \leq -c\sigma^2 e^{\sigma(x-\alpha)} \quad \text{in } J$$

for some constant $c > 0$.

Most of the theorems on maximum principles will be easy consequences of the following basic and useful result.

Theorem 2.1. *Let $p, q \in C(\bar{J})$ and $q \leq 0$ in J. Let $u \in C^2(J) \cap C(\bar{J})$ be a solution of $Lu \leq 0$ in J. Suppose u has a nonpositive minimum at $x_0 \in \{\alpha, \beta\}$. If u is differentiable at x_0, and $u'(x_0) = 0$, then u is a constant in J.*

Proof. We consider the case $x_0 = \alpha$ first. Suppose u has a nonpositive minimum $u(\alpha)$ at α. Furthermore, assume that u is differentiable at α, and $u'(\alpha) = 0$. We consider the auxiliary function z in (2-1) with $\sigma > 0$ such that $Lz \leq 0$ in $\bar{J}$. We note that $z(\alpha) = -1$ and $z'(\alpha) = 0$. We fix $\varepsilon > 0$, and set $w := u + \varepsilon z$. We note that $Lw = Lu + \varepsilon Lz = Lu \leq 0$. On recalling that $u(\alpha) \leq 0$, we have $w(\alpha) = u(\alpha) - \varepsilon < 0$, and $w'(\alpha) = 0$. By continuity of w on $[\alpha, \beta]$, we see that $w(x) < 0$ on $[\alpha, \tau)$ for some $\tau > 0$. Let

$$\eta := \sup\{\rho \in [\alpha, \beta] : w(s) < 0 \ \forall 0 \leq s < \rho\}.$$

Then we note that

$$\begin{aligned}
(r(x)w')' &= (r(x)w')' + r(x)q(x)w - r(x)q(x)w(x) \\
&= r(x)Lw - r(x)q(x)w \\
&\leq -r(x)q(x)w(x) \leq 0, \quad \alpha < x < \eta. \quad (2\text{-}2)
\end{aligned}$$

Thus rw' is decreasing on $[\alpha, \eta]$ so that $r(x)w'(x) \leq r(\alpha)w'(\alpha) = 0$ on $[\alpha, \eta]$. In particular, this implies that w is decreasing on $[\alpha, \eta]$. Hence $w(x) \leq w(\alpha) < 0$ for all $\alpha \leq x \leq \eta$. This and the continuity of w on $[\alpha, \beta]$ would contradict the definition of η if $\eta < \beta$. Therefore we must have $\eta = \beta$, so that w is decreasing on $[\alpha, \beta]$. In particular, we have

$$u(x) + \varepsilon z(x) \leq u(\alpha) + \varepsilon z(\alpha), \quad \alpha \leq x \leq \beta.$$

Letting $\varepsilon \to 0$, we find that $u(x) \leq u(\alpha)$ on $[\alpha, \beta]$. This, together with the fact that $u(x) \geq u(\alpha)$, shows that $u(x) = u(\alpha)$ on $[\alpha, \beta]$.

Now suppose u has a nonpositive minimum at β and $u'(\beta) = 0$. Let $w(x) = u(2\beta - x)$ for $x \in I := [\beta, 2\beta - \alpha]$. Then clearly $w \in C^2(I) \cap C(\bar{I})$, and moreover, w is differentiable at β with $w'(\beta) = -u'(\beta) = 0$. Furthermore, w satisfies the inequality

$$\tilde{L}w = w'' + \tilde{p}(x)w' + \tilde{q}(x)w \leq 0, \quad x \in I,$$

where

$$\tilde{p}(x) = -p(2\beta - x) \quad \text{and} \quad \tilde{q}(x) = q(2\beta - x), \quad x \in I.$$

Finally we also note that w has a nonpositive minimum at β. Therefore, by the above result, we must have $w(y) = w(\beta)$ for all $y \in [\beta, 2\beta - \alpha]$. Thus for any $x \in [\alpha, \beta]$, we have $w(\beta) = w(2\beta - x) = u(x)$, that is, $u(x) = u(\beta)$, as was to be shown. $\square$

As consequences of Theorem 2.1, we have the following immediate and useful theorems on maximum principles.

Theorem 2.2. *Let $p, q \in C(\bar{J})$ and $q \leq 0$ in J. Suppose u satisfies the differential inequality $Lu \leq 0$ in an interval J. If u assumes a nonpositive minimum value at an interior point x_0 of J, then $u(x) \equiv u(x_0)$.*

Proof. Suppose u attains its nonpositive minimum at $x_0 \in J = (\alpha, \beta)$. Then $u'(x_0) = 0$. Consider the intervals $[\alpha, x_0]$ and $[x_0, \beta]$. By Theorem 2.1, we see that $u(x) = u(x_0)$ for all $x \in [\alpha, x_0]$ and $u(x_0) = u(x)$ for all $x \in [x_0, \beta]$. That is, $u(x) = u(x_0)$ for all $x \in [\alpha, \beta]$. $\qquad\square$

Theorem 2.3. *Let $p, q \in C(\bar{J})$ and $q \leq 0$ in J. Suppose $u \in C^2(J) \cap C(\bar{J})$ satisfies the differential inequality $Lu \leq 0$ in an interval $J := (\alpha, \beta)$. If u assumes a nonpositive minimum value at $x_0 \in \{\alpha, \beta\}$ and u is differentiable at x_0, then $u'(x_0) > 0$ if $x_0 = \alpha$, and $u'(x_0) < 0$ if $x_0 = \beta$ unless u is a constant on J.*

Proof. Suppose u satisfies $Lu \leq 0$ in J, and u has a nonpositive minimum at $x_0 \in \{\alpha, \beta\}$. By hypothesis, u is differentiable at x_0. Let us take the case $x_0 = \alpha$. Then clearly $u'(\alpha) \geq 0$. If $u'(\alpha) = 0$, then by Theorem 2.1, we conclude u is a constant. Therefore, if u is nonconstant, we must have $u'(\alpha) > 0$. If $x_0 = \beta$, here again we have $u'(\beta) \leq 0$. If u is nonconstant, then again by Theorem 2.1, we must have $u'(\beta) < 0$. $\qquad\square$

Theorem 2.4. *Let $p, q \in C(\bar{J})$ and $q \leq 0$ in J. Suppose $u \in C^2(J) \cap C(\bar{J})$ satisfies the differential inequality $Lu \leq 0$ in an interval $J := (\alpha, \beta)$. Suppose $u(\gamma) \leq 0$ for some $\gamma \in \bar{J}$. In case $\gamma \in \{\alpha, \beta\}$, we assume that u is differentiable at γ:*

(i) *If $u'(\gamma) \leq 0$, then $u(x) \leq 0$ for all $x \in [\gamma, \beta]$.*

(ii) *If $u'(\gamma) \geq 0$, then $u(x) \leq 0$ for all $x \in [\alpha, \gamma]$.*

(iii) *If $u'(\gamma) = 0$, then $u(x) \leq 0$ for all $x \in \bar{J}$.*

Proof. Suppose $u'(\gamma) \leq 0$. We assume that $\gamma < \beta$, for otherwise there is nothing to prove. Suppose that $u(c) > 0$ for some $c \in (\gamma, \beta)$. Since $u(\gamma) \leq 0$, and $u(c) > 0$, we note that u has a nonpositive minimum on $[\gamma, c]$ at some $\gamma \leq d < c$. If $\gamma < d < c$, then $u'(d) = 0$ and we invoke Theorem 2.2 to conclude that u is a constant in $[\gamma, c]$. If $d = \gamma$, then the assumption $u'(\gamma) \leq 0$ and Theorem 2.3 lead us to conclude that u is a constant on $[\gamma, c]$. In any case, we see that $u(c) > 0$ for some $c \in (\gamma, \beta]$ implies that u is a constant on $[\gamma, c]$. But then $u(c) = u(\gamma) \leq 0$, which contradicts the assumption that $u(c) > 0$. This proves statement (i).

To prove (ii), let us assume that $u'(\gamma) \geq 0$, and that $\gamma > \alpha$. Assume that $u(c) > 0$ for some $c \in [\alpha, \gamma)$. Since $u(\gamma) \leq 0$, as in the previous case we note that u takes a nonpositive minimum on $[c, \gamma]$ at some $c < d \leq \gamma$. If $c < d < \gamma$, then $u'(d) = 0$, and by Theorem 2.2, we see that u is a constant on $[c, \gamma]$. If, on the other hand, $d = \gamma$,

then since $u'(\gamma) \geq 0$, we conclude that u is a constant on $[c, \gamma]$ by Theorem 2.3. In either case, we conclude that u is a constant on $[c, \gamma]$. But this implies that $u(c) = u(\gamma) \leq 0$, which again contradicts the assumption that $u(c) > 0$. Therefore statement (ii) holds as well.

Finally statement (iii) follows from statements (i) and (ii).   □

3. The Harnack and Harnack-type inequalities

We start with following existence and uniqueness theorem for solutions of $Lu = f$ that satisfy initial conditions. This theorem is usually taught in a first course on ordinary differential equations in undergraduate curriculum (see [Boyce and DiPrima 1965] for instance), and will be needed in our proof of Harnack's inequality.

Theorem E (existence and uniqueness). *Suppose $p, q, f \in C(I)$. Let $x_0 \in I$ and let c_0 and c_1 be arbitrary real constants. Then there exists a unique solution $u \in C^2(I)$ of equation $Lu = f$ such that $u(x_0) = c_0$ and $u'(x_0) = c_1$.*

We now begin our considerations of Harnack's inequality with respect to the class of nonnegative solutions of the differential inequality

$$Lu \leq 0 \quad \text{in } I := (A, B). \tag{3-1}$$

To proceed further, we fix some notations, some of which are fairly standard. For any function $h : (A, B) \to \mathbb{R}$, we write

$$h^+(x) := \max\{h(x), 0\} \quad \text{and} \quad h^-(x) := \max\{-h(x), 0\}, \quad x \in (A, B).$$

Note that we have

$$h = h^+ - h^-.$$

In the sequel, we will also use the following notation repeatedly.

$$L_0 u := u'' + p(x)u' - q^-(x)u, \quad x \in I.$$

Remark 3.1. We first make note of the following:

(1) If u is a nonnegative solution of (3-1), then u is a solution of $L_0 u \leq 0$ in I.

(2) If u is a nonnegative solution of (3-1) with $u \not\equiv 0$ on I, then $u > 0$ in I, for if $u(x_0) = 0$ for some $x_0 \in I$, then $u'(x_0) = 0$. Since $L_0 u \leq 0$ in I, we invoke Theorem 2.2 and conclude that $u(x) \equiv 0$ in I.

We start with the following theorem on Harnack's inequality for nonnegative solutions of (3-1).

Theorem 3.2. *Given $[a, b] \subseteq I$, there is a positive constant C that depends on the coefficients p, q and the constants A, B, a and b only, such that*

$$\max_{a \leq x \leq b} u(x) \leq C \min_{a \leq x \leq b} u(x) \tag{3-2}$$

for any nonnegative solution u of (3-1).

We break down the proof into two lemmas, each of which may be of independent interest. The proof closely follows the method in [Berhanu and Mohammed 2005].

Lemma 3.3. *Given $[a, b] \subseteq I$, there are constants C_a and C_b that depend on the coefficients p, q and the constants A, B, a and b only, such that*

$$\frac{u'(a)}{u(a)} \le C_a \quad and \quad \frac{u'(b)}{u(b)} \ge C_b \tag{3-3}$$

for all positive solutions u of (3-1).

Proof. Let w_1 and w_2 be solutions of

$$L_0 w := w'' + p(x)w' - q^-(x)w = 0, \quad A < x < B,$$

such that $w_1(a) = 1$, $w_1'(a) = 0$, and $w_2(a) = 0$, $w_2'(a) = 1$.

We define

$$v(x) := u(x) - u(a)w_1(x) - u'(a)w_2(x), \quad A < x < B.$$

Then recalling that $L_0 u \le 0$, and $L_0 w_1 = 0 = L_0 w_2$ in (A, B), we see that $L_0 v \le 0$ in (A, B). Moreover, we have $v(a) = 0$ and $v'(a) = 0$. By Theorem 2.4, we conclude that $v \le 0$ on (A, B). Thus

$$0 \le u(x) \le u(a)w_1(x) + u'(a)w_2(x), \quad A < x < B,$$

whence

$$\frac{u'(a)}{u(a)} w_2(x) + w_1(x) \ge 0, \quad A < x < B. \tag{3-4}$$

Since $w_2'(a) = 1$, we note that there is a small interval centered at a on which w_2 is increasing. So we fix a^* with $\alpha < a^* < a$ such that $w_2(a^*) < 0$. Therefore, on taking $x = a^*$ in (3-4), we conclude that

$$\frac{u'(a)}{u(a)} \le -\frac{w_1(a^*)}{w_2(a^*)} = C_a. \tag{3-5}$$

Next we establish the second estimate in (3-3). This is very similar to the previous case, and hence we will be brief. Let z_1 and z_2 be solutions of

$$L_0 z_1 = 0, \quad z_1(b) = 1, \ z_1'(b) = 0 \quad and \quad L_0 z_2 = 0, \quad z_2(b) = 0, \ z_2'(b) = 1.$$

Let us consider the function

$$v(x) := u(x) - u(b)z_1(x) - u'(b)z_2(x), \quad A < x < B.$$

Then $Lv \le 0$ in (A, B) and $v(b) = 0$, $v'(b) = 0$. Arguing as before, we can show that $v \le 0$ on (A, B), from which we conclude

$$\frac{u'(b)}{u(b)} z_2(x) + z_1(x) \ge 0, \quad A < x < B.$$

Since z_2 is increasing in some interval centered at b, we can find $b < b^* < B$ such that $z_2(b^*) > 0$. Thus we find that

$$\frac{u'(b)}{u(b)} \geq -\frac{z_1(b^*)}{z_2(b^*)} = C_b. \tag{3-6}$$

This completes the proof of the lemma.   □

Lemma 3.4. *Given $[a, b] \subseteq I$, there is a positive constant C, depending on the coefficients p, q and the constants A, B, a and b only, such that*

$$|u'(x)| \leq Cu(x), \quad a \leq x \leq b \tag{3-7}$$

for all nonnegative solutions u of (3-1) *in I.*

Proof. Let u be a nonnegative solution of (3-1) in I with $u \not\equiv 0$ so that $u > 0$ in I. Direct computation shows that

$$\begin{aligned}
\left(\frac{u'}{u}\right)' &= \frac{u''}{u} - \left(\frac{u'}{u}\right)^2 \\
&\leq \frac{1}{u}(-p(x)u' - q(x)u) \\
&= -p(x)\left(\frac{u'}{u}\right) - q(x).
\end{aligned}$$

Therefore,

$$\left(\frac{u'}{u}\right)' + p(x)\left(\frac{u'}{u}\right) \leq q^-(x).$$

This leads to

$$\left(\exp\left(\int_a^x p(t)\, dt\right)\frac{u'}{u}\right)' \leq q^- \exp\left(\int_a^x p(t)\, dt\right).$$

This gives

$$\left(r(x)\frac{u'}{u} - \int_a^x r(t)\, q^-(t)\, dt\right)' \leq 0, \quad x \in (a, b),$$

where we have set

$$r(x) := \exp\left(\int_a^x p(t)\, dt\right).$$

Thus for any $a \leq x \leq b$, we have

$$r(b)\frac{u'(b)}{u(b)} - \int_a^b r(t)\, q^-(t)\, dt \leq r(x)\frac{u'}{u} - \int_a^x r(t)\, q^-(t)\, dt \leq r(a)\frac{u'(a)}{u(a)}.$$

In conclusion, we have

$$r(b)\frac{u'(b)}{u(b)} - Q(b) \leq r(x)\frac{u'}{u} \leq \frac{u'(a)}{u(a)} + Q(b), \quad x \in (a, b), \tag{3-8}$$

where $Q(b)$ denotes the constant

$$Q(b) := \int_a^b r(t) q^-(t) \, dt.$$

Using Lemma 3.3 in (3-8), we obtain

$$r(x) \left| \frac{u'(x)}{u(x)} \right| \leq C_0, \quad x \in [a, b],$$

for some positive constant C_0, independent of u. Since

$$\frac{1}{r(x)} = \exp\left(-\int_a^x p(t) \, dt \right) \leq \exp(\|p\|_\infty (b - a)), \quad x \in [a, b],$$

we conclude that

$$\left| \frac{u'(x)}{u(x)} \right| \leq C, \quad x \in [a, b], \tag{3-9}$$

for a constant $C > 0$ that is independent of u. $\qquad\square$

Proof of Theorem 3.2. For any $x, y \in [a, b]$, we see that

$$\log\left(\frac{u(x)}{u(y)} \right) = \int_y^x \frac{d}{dt} \log u(t) \, dt$$

$$= \int_y^x \frac{u'(t)}{u(t)} \, dt.$$

Therefore,

$$\frac{u(x)}{u(y)} = \exp\left(\int_y^x \frac{u'(t)}{u(t)} \, dt \right).$$

It follows from this and Lemma 3.4 that

$$\exp(-C|x - y|) \leq \frac{u(x)}{u(y)} \leq \exp(C|x - y|), \quad x, y \in [a, b].$$

Therefore, we finally see that

$$\exp(-C(b - a)) \leq \frac{u(x)}{u(y)} \leq \exp(C(b - a)), \quad x, y \in [a, b],$$

which leads to the inequality stated in (3-2). $\qquad\square$

Remark 3.5. The differential inequality (3-1) with the inequality reversed doesn't satisfy Harnack's inequality as can be seen from the following simple example. Fix $x_0 \in [a, b]$. Then $u_k(x) = e^{k(x - x_0)}$ satisfies the inequality $u'' \geq 0$ in $\mathbb{R}$, and note that

$$e^{k(b - x_0)} \leq \sup_{[a,b]} u_k(x) \leq C \inf_{[a,b]} u_k(x) \leq C u_k(x_0) = C.$$

But there is no single positive constant C, independent of u_k and hence k, such that

$$e^{k(b - x_0)} \leq C.$$

Next we study a Harnack-type inequality for nonnegative solutions of nonhomogeneous equations.

We will start by deriving a Harnack-type inequality for nonnegative solutions of the following equation, assuming that $f \geq 0$ on I:

$$L_0 u = f \quad \text{in } I := (A, B). \tag{3-10}$$

However, it should be noted that Harnack's inequality (3-2) does not hold for nonnegative solutions of (3-10) for general f. This is to be expected as nonnegative solutions of (3-10) are not necessarily positive in (A, B). In fact, the following simple example shows that the inequality (3-2) cannot hold even for positive solutions of (3-10).

Example 3.6. Consider the equation $u'' = 1$ in the interval $(A, B) := (-2, 2)$. For any positive integer k,

$$u_k = \frac{1}{2}\left(x - \frac{1}{k}\right)^2 + \frac{1}{k}$$

is a solution of $u_k'' = 1$, and $u_k > 0$ in $(-2, 2)$ for all k. Suppose there is a constant $C > 0$ such that

$$\max_{[-1,1]} u \leq C \min_{[-1,1]} u \quad \forall u > 0, u'' = 1. \tag{3-11}$$

Then note that

$$u_k(1) = \frac{1}{2}\left(1 - \frac{1}{k}\right)^2 + \frac{1}{k} \quad \text{and} \quad u_k\left(\frac{1}{k}\right) = \frac{1}{k}.$$

If (3-11) were to hold, then

$$\frac{1}{2}\left(1 - \frac{1}{k}\right)^2 + \frac{1}{k} \leq C\left(\frac{1}{k}\right) \quad \forall k = 1, 2, \ldots.$$

Letting $k \to \infty$, we arrive at a contradiction.

We now state the following theorem on a Harnack-type inequality for solutions of (3-10). For the remainder of our discussion, we will use the following notations.

$$\alpha := \tfrac{1}{2}(a + A) \quad \text{and} \quad \beta := \tfrac{1}{2}(b + B).$$

We will also find it convenient to use the notation $\|g\|_\infty$ to denote the following number for any g bounded on an interval I:

$$\|g\|_{I,\infty} := \sup_{x \in I} |g(x)|,$$

or simply $\|g\|_\infty$ if I is clear from the context.

Theorem 3.7. *Suppose $f \geq 0$ in I. Given $[a, b] \subseteq I$, there is a positive constant C that depends on the coefficients p, q and the constants A, B, a and b such that*

$$\max_{a \leq x \leq b} u(x) \leq C \left(\min_{a \leq x \leq b} u(x) + \int_{\alpha}^{\beta} f(x) \, dx \right) \qquad (3\text{-}12)$$

for all nonnegative solutions u of (3-10).

Proof. We prove the theorem in three steps. Suppose $u \geq 0$ in (A, B) is a solution of (3-10).

Step 1. Let $u(x_0) = \min\{u(x) : x \in [\alpha, \beta]\}$. By Theorem E, we pick $z_* \in C^2(I) \cap C(\bar{I})$ such that

$$L_0 z_* = f, \quad z_*(x_0) = 0 = z_*'(x_0). \qquad (3\text{-}13)$$

By Theorem 2.4(iii), we note that $z_* \geq 0$ in (A, B). We claim that $u \geq z_*$ in $[\alpha, \beta]$. To see this, we start by observing that

$$L_0(u - z_*) = 0 \quad \text{in } (A, B) \qquad \text{and} \qquad (u - z_*)(x_0) \geq 0.$$

Suppose first that $\alpha < x_0 < \beta$. Then $u'(x_0) = 0$, and therefore $(u - z_*)'(x_0) = 0$. Consequently, by Theorem 2.4(iii), we conclude that $u - z_* \geq 0$ in $[\alpha, \beta]$, as desired. Suppose $x_0 = \alpha$. Then $u'(x_0) = u'(\alpha) \geq 0$, so that $(u - z_*)'(x_0) \geq 0$. By Theorem 2.4(i), we conclude that $u - z_* \geq 0$ in $[x_0, \beta] = [\alpha, \beta]$. Finally, suppose that $x_0 = \beta$. Then $u'(x_0) = u'(\beta) \leq 0$, so that $(u - z_*)'(x_0) \leq 0$. Again, by Theorem 2.4(ii), we conclude that $u - z_* \geq 0$ in $[\alpha, x_0] = [\alpha, \beta]$. Thus, in all cases, we have shown that $u \geq z_*$ in $[\alpha, \beta]$ as claimed.

Step 2. Let $u(\zeta) := \min\{u(x) : a \leq x \leq b\}$. Since $u - z_*$ is a nonnegative solution of $L_0 w = 0$ in (α, β), we invoke Theorem 3.2 to obtain a positive constant C that depends on p, q^- and the constants A, B, a and b only such that the following chain of inequalities hold:

$$\begin{aligned}
\max_{[a,b]} u &= \max_{[a,b]}(z_* + u - z_*) \\
&\leq \max_{[a,b]} z_* + \max_{[a,b]}(u - z_*) \\
&\leq \max_{[a,b]} z_* + C \min_{[a,b]}(u - z_*) \quad \text{(by Theorem 3.2)} \\
&\leq C(u - z_*)(\zeta) + \max_{[a,b]} z_* \\
&\leq C u(\zeta) + \max_{[a,b]} z_* \quad \text{(recall that } u(\zeta) = \min_{[a,b]} u) \\
&= C \min_{[a,b]} u + \max_{[a,b]} z_*. \qquad (3\text{-}14)
\end{aligned}$$

Step 3. We now estimate z_* on $[a, b]$. Recall the notation $\|g\|_\infty := \max_{x \in [\alpha, \beta]} |g(x)|$ for any function $g \in C([\alpha, \beta])$. We recall that

$$f = L_0 z_* = \frac{1}{r(x)} (r(x) z_*')' - q^-(x) z_*, \quad x \in I,$$

where

$$r(x) = \exp\left(\int_a^x p(s)\, ds\right).$$

For $x \in (x_0, b)$, we have

$$z_*(x) = \int_{x_0}^x \frac{1}{r(t)} \int_{x_0}^t r(s)\big(q^-(s) z_*(s) + f(s)\big)\, ds\, dt.$$

Therefore, for $x \in (x_0, b)$,

$$z_*(x) \leq \exp\big((b - a)\|p\|_\infty\big) \int_{x_0}^x \int_{x_0}^t \big(q^-(s) z_*(s) + f(s)\big)\, ds\, dt$$

$$\leq (b - a) \exp\big((b - a)\|p\|_\infty\big) \int_{x_0}^x \big(q^-(t) z_*(t) + f(t)\big)\, dt$$

$$\leq P_0 \int_\alpha^\beta f(t)\, dt + P_0 \|q^-\|_\infty \int_{x_0}^x z_*(t)\, dt,$$

where $P_0 := (b - a) \exp\big((b - a)\|p\|_\infty\big)$.

Denoting the right-hand side of the last inequality by $\vartheta(x)$ for $x_0 < x < b$, and on noting that $z_*(x) \leq \vartheta(x)$ on (x_0, b), we find

$$\vartheta'(x) = P_0 \|q^-\|_\infty z_*(x)$$

$$\leq P_0 \|q^-\|_\infty \vartheta(x) \quad \text{(since } z_*(x) \leq \vartheta(x)\text{)}, \quad x \in (x_0, b),$$

so that

$$\frac{\vartheta'(x)}{\vartheta(x)} \leq P_0 \|q^-\|_\infty, \quad x \in (x_0, b).$$

Integrating on (x_0, x), we find that

$$z_*(x) \leq \vartheta(x) \leq P_0 \exp\big(P_0 \|q^-\|_\infty (b - a)\big) \int_\alpha^\beta f(t)\, dt. \tag{3-15}$$

The same inequality holds if $a < x < x_0$.

Using (3-15) in (3-14) leads to the desired inequality (3-12). $\qquad \square$

Finally we are ready to state and prove the following Harnack-type inequality for nonnegative solutions of the differential inequality (1-1) with the nonhomogeneous term f in $C(I)$, without any sign restrictions.

Theorem 3.8. *Given $[a, b] \subseteq I$, there is a positive constant C, that depends on the coefficients p, q and the constants A, B, a and b only such that the Harnack-type inequality* (3-12), *with f replaced by f^+, holds for all nonnegative solutions of* (1-1).

Proof. Let u be a nonnegative solution of (1-1) in (A, B). Let $u(x_0) = \min_{[a,b]} u$, and consider the solution z of

$$L_0 z = f^+ \quad \text{in } (A, B) \quad \text{and} \quad z(x_0) = u(x_0), \ z'(x_0) = u'(x_0).$$

Then $L_0(u - z) = L_0 u - L_0 z \leq f - f^+ \leq 0$, and $(u - z)(x_0) = 0$ and $(u - z)'(x_0) = 0$. By Theorem 2.4(iii), we conclude that $u - z \leq 0$ in (A, B), so that $0 \leq u \leq z$ in (A, B). Thus

$$\max_{x \in [a,b]} u(x) \leq \max_{x \in [a,b]} z(x)$$

$$\leq C \left(\min_{x \in [a,b]} z(x) + \int_\alpha^\beta f^+(x) \, dx \right) \quad \text{(by Theorem 3.7)}$$

$$\leq C \left(z(x_0) + \int_\alpha^\beta f^+(x) \, dx \right)$$

$$= C \left(u(x_0) + \int_\alpha^\beta f^+(x) \, dx \right)$$

$$= C \left(\min_{x \in [a,b]} u(x) + \int_\alpha^\beta f^+(x) \, dx \right).$$

This is the desired result. $\square$

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