involve

a journal of mathematics

Strong depth and quasigeodesics in finitely generated groups

Brian Gapinski, Matthew Horak and Tyler Weber

msp

2016 vol. 9, no. 3

INVOLVE 9:3 (2016)
dx.doi.org/10.2140/involve.2016.9.367

Strong depth and quasigeodesics in finitely generated groups

Brian Gapinski, Matthew Horak and Tyler Weber

(Communicated by Kenneth S. Berenhaut)

A “dead end” in the Cayley graph of a finitely generated group is an element beyond which no geodesic ray issuing from the identity can be extended. We study the so-called “strong dead-end depth” of group elements and its relationship with the set of infinite quasigeodesic rays issuing from the identity. We show that the ratio of strong depth to word length is bounded above by $\frac{1}{2}$ in every finitely generated group and that for any element g in a finitely generated group G, there is an infinite $(3, 0)$-quasigeodesic ray issuing from the identity and passing through g. Applying the Švarc–Milnor lemma to a finitely generated group acting geometrically on a geodesically connected metric space, we obtain the result that for any two points in such a space, there is an infinite quasigeodesic ray starting at one and passing through the other with quasigeodesic constants independent of the points selected.

1. Introduction

Background and summary of results. Let G be a group and X a finite generating set for G. The Cayley graph for G with respect to X is the graph with vertex set G and an edge from g to gx for every $x \in X \cup X^{-1}$. Throughout, we will use $\Gamma(G, X)$ or simply Γ to denote the Cayley graph of G with respect to X. Assigning all edges in $\Gamma(G, X)$ length 1 determines a metric on $\Gamma(G, X)$, and therefore on G, which we denote by $d(\cdot, \cdot) = d_X(\cdot, \cdot)$. The metric d determines a length function $l_X = l$ on G defined by $l(x) = d(e, x)$, where e is the identity of G.

Many results on discrete groups rely upon an understanding of the structure of geodesics in the Cayley graph. In particular, the question arises of the existence (or nonexistence) of elements g beyond which no geodesic ray from the identity to g can be extended to a longer geodesic. In the current literature, such elements are called *dead ends*. Dead ends have been applied to, for example, the construction in [Lyons

MSC2010: 20F65.
Keywords: Cayley graph, dead end, quasigeodesic.
This research was supported by the NSF REU grant DMS-1062403.

et al. 1996] of a random walk on the lamplighter group that is biased towards the identity but that escapes from the identity faster than a simple random walk. Dead ends also played a role in the proof that infinite commensurable hyperbolic groups are bi-Lipschitz equivalent [Bogopol'skiĭ 1997].

A property of arbitrary metric spaces similar to the nonexistence of dead ends in a group is the geodesic extension property, which states that every finite geodesic segment is contained in an infinite geodesic line. The geodesic extension property appears frequently in the study of nonpositively curved spaces, and especially in the study of CAT(0) spaces and CAT(0) groups. For example, it is shown in [Bridson and Haefliger 1999; Hosaka 2012] that if X is a CAT(0) space with the geodesic extension property, then any geometric action on X of a group of the form $G = G_1 \times G_2$ induces a splitting of X as $X = X_1 \times X_2$ with a geometric action of G_1 on X_1 and of G_2 on X_2. With further assumptions on G, the action of $G_1 \times G_2$ on $X_1 \times X_2$ is the product action. We refer the reader to Chapter II.6 of the book [Bridson and Haefliger 1999] for a thorough discussion of the role of infinite geodesics in the study of the geometry of nonpositively curved spaces.

One difficulty of extending the above results involving dead ends or geodesic extension to larger classes of groups is that quasi-isometries take geodesics to quasi-geodesics, not geodesics. Therefore, it is possible for a group to have dead ends with respect to one generating set but not another. Even worse, there exist groups with unbounded dead-end depth with respect to one generating set, but no dead ends with respect to another [Riley and Warshall 2006]. This quasi-isometry noninvariance prevents one from using or studying dead ends by way of a geometric action of the group in question on a space, since such an action provides only the quasi-isometric equivalence of the group with the space.

In this paper, we address this problem in two ways. We first analyze the behavior of the *strong depth*, $\sigma(g)$, of an element g, introduced by Lehnert [2009]. Informally, this is the minimum distance back towards the identity that any path in $\Gamma(G, X)$ from g to an element of greater length must travel. Warshall [2011] introduced a similar notion, the *retreat depth*, which is the minimum distance, d, towards the identity that one must travel to enter an unbounded component of the complement of the ball of radius $l(g) - d$. Strong depth and retreat depth are similar and seem to behave roughly the same. Therefore, although we have chosen to phrase all of our theorems in terms of strong depth, they can be restated in terms of retreat depth.

Even though strong depth depends on the generating set, just as for ordinary dead ends, its ratio to length turns out to be well-behaved for all generating sets. In Section 2 we prove the following:

Theorem 2.2. *Let G be an infinite group and X a finite generating set for G. Then for all $g \in G \setminus \{e\}$, we have $\sigma(g)/l(g) \le \frac{1}{2}$.*

The second way in which we address the strong dependence of dead ends on the quasi-isometry class or the generating set is to relax the question of extending a geodesic path between two elements and instead ask whether there exist universal constants L and A for which one can find an infinite (L, A)-quasigeodesic ray passing through any arbitrary pair of points. If so, then we say that the space in question has *uniform quasigeodesic ray extension*. In Section 3, we show that every infinite, finitely generated group has uniform quasigeodesic ray extension:

Theorem 3.3. *Let G be an infinite group and X a finite generating set for G. Then for all $g \in G$ there exists an infinite $(3,0)$-quasigeodesic ray in $\Gamma(G, X)$ starting at the identity of G and passing through g.*

Applying the Švarc–Milnor lemma to a “nice” metric space X admitting a geometric action of a finitely generated group G, we obtain the following corollary:

Corollary 3.8. *Let (X, d_X) be a metric space in which any two points can be joined by a geodesic segment and G a finitely generated group acting by isometries on X. If there exists a ball $B(x_0, R)$ in X whose G-translates cover X with the property that for every $r > 0$ the set $\{g : B(x_0, r) \cap g \cdot B(x_0, r) \neq \varnothing\}$ is finite, then X has uniform quasigeodesic ray extension.*

Definitions. In this section, we review the definitions of the various types of dead ends and dead-end depths that we deal with and summarize some of their basic properties. In what follows, all graphs are assumed to be endowed with the metric d induced by declaring each edge to have length 1.

Definition 1.1. Let Γ be a graph. A *path* in Γ is a function $\rho : I \to \Gamma$, where I is the intersection of a (possibly infinite) interval of the real line with $\mathbb{Z}$ such that for each $i, j \in I$ with $|i - j| = 1$, we have that $\rho(i)$ and $\rho(j)$ span an edge in Γ.

For convenience and to aid the memory and imagination, we often express a path as

$$\rho = \dots, a_k, a_{k+1}, a_{k+2}, a_{k+3}, \dots,$$

where $a_i = \rho(i)$. We use similar notation for finite paths and paths infinite on only one end.

Definition 1.2. If $\rho = a_0, a_1, a_2, \dots, a_m$ and $\tau = x_0, x_1, x_2, \dots, x_n$ are paths with $a_m = x_0$, then the *concatenation* of ρ and τ is $\rho\tau = a_1, a_2, \dots, a_m, x_1, \dots, x_n$.

Definition 1.3. Let $\rho = a_1, a_2, \dots, a_n$ be a path in a graph Γ. The *path length* between two vertices a_i and a_j in ρ is defined as $p_\rho(a_i, a_j) = |i - j|$.

Definition 1.4. Let $\gamma = a_1, a_2, \dots, a_n$ be a (possibly infinite) path in the graph Γ. We say that γ is a *geodesic* in Γ if for all i, j, we have $p_\gamma(a_i, a_j) = d(a_i, a_j)$.

We now specialize to the case where G is a finitely generated group, X is a fixed generating set for G and $\Gamma(G, X)$ is the Cayley graph of G with respect to X. All of these definitions are dependent on the generating set X, but if there is only one generating set in question, we often omit it from the notation.

Definition 1.5. For $g \in G$ the *word length* of g (with respect to X) is $l(g) = l_X(g) = d(e, g)$ with distance measured in $\Gamma(G, X)$.

Definition 1.6. Let G be a group and X a finite generating set for G. Let $g \in G$ and $n \in \mathbb{N}$. The *sphere of radius n centered at g* (with respect to generating set X) is $S_g(n) := \{h \in G : d_X(g, h) = n\}$.

Definition 1.7. Let G be a group and X a finite generating set for G. Let $g \in G$ and $n \in \mathbb{N}$. The *ball of radius n centered at g* (with respect to generating set X) is $B_g(n) := \{h \in G : d_X(g, h) \leq n\}$.

Definition 1.8. Let G be a group and X a finite generating set for G. The *dead-end depth* (with respect to X) of an element $g \in G$ with $l_X(g) = n$ is the least integer k such that there exists a path of length k in $\Gamma(G, X)$ from g to $S_e(n+1)$. We denote the dead-end depth of g as $\delta_X(g)$ or simply $\delta(g)$ if only one generating set is under investigation. An element $g \in G$ with $\delta(g) > 1$ is called a *dead end*.

Definition 1.9. Let G be a group and X a finite generating set for G. We say that G has *bounded dead-end depth with respect to* X if there exists $N \in \mathbb{N}$ such that, for all $g \in G$, we have $\delta(g) \leq N$. If no such N exists, we say that G has *unbounded dead-end depth with respect to* X.

As previously mentioned, the dead-end elements, dead-end depth, and retreat depth of a group are strongly dependent on the generating set. Riley and Warshall [2006] constructed a group that has bounded dead-end depth with respect to one finite generating set but unbounded dead-end depth with respect to another finite generating set. Lehnert [2009] introduced the following notion of strong depth and showed that Houghton's group H_2 has unbounded strong depth.

Definition 1.10. Let Γ be the Cayley graph of a group G with respect to the finite generating set X and let $g \in G$ with $d(e, g) = n$. The *strong depth* of g (with respect to X) is the minimum k such that there exists a path in $\Gamma(G, X)$ from g to an element of $S_e(n+1)$ that does not enter $B_e(n-k)$. We denote the strong depth of g with respect to X as $\sigma_X(g)$ or simply $\sigma(g)$ if the context is clear.

2. Strong depth

There are two "easy" inequalities satisfied by dead-end depth and strong depth. The first inequality follows from the definitions and states that for every element g of a finitely generated group G, we have that $\sigma(g) \leq \frac{1}{2}\delta(g)$. The second inequality

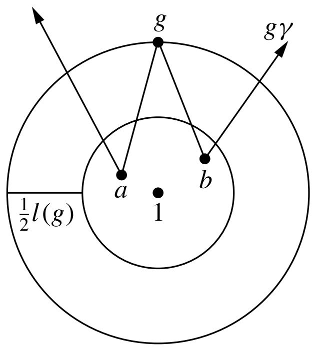

Figure 1. Schematic of a geodesic passing through an element g with $\sigma(g) > \frac{1}{2}l(g)$.

states that for every element g of a finitely generated group G, we have that $\delta(g) \leq 2l(g) + 1$. This is observed by taking a geodesic path from g to the identity and concatenating a geodesic path to an element of greater length than g. To our knowledge, these are the only two inequalities involving dead-end depth known to hold in all finitely generated groups and for all generating sets. In this section, we establish another property of strong depth that holds for every finite generating set of any infinite finitely generated group. Our argument uses the fact that every infinite finitely generated group contains an infinite geodesic line passing through the identity. A sketch of the proof of this fact can be found in [de la Harpe 2000]. We record this as:

Lemma 2.1. *Let G be an infinite group and X a finite generating set for G. Then the Cayley graph $\Gamma(G, X)$ contains a bi-infinite geodesic line passing through the identity.*

Theorem 2.2. *Let G be an infinite group and X a finite generating set for G. Then for all $g \in G \setminus \{e\}$, we have $\sigma(g)/l(g) \leq \frac{1}{2}$.*

Proof. Suppose towards a contradiction that there exists a nonidentity $g \in G$ with $\sigma(g)/l(g) > \frac{1}{2}$. By Lemma 2.1, select an infinite geodesic line

$$\gamma = \ldots, w_2, w_1, e, v_1, v_2, \ldots$$

in G. Since G acts on Γ by isometries, $g \cdot \gamma$ is an infinite geodesic line that passes through g. Let a be the element in $\{g \cdot w_k : k \in \mathbb{N}\}$ of least length. If two or more such elements exist, select a to be the closest such element to g along $g \cdot \gamma$. Similarly, we let b be the element of $\{g \cdot v_k : k \in \mathbb{N}\}$ of least length, again taking the closest such element to g along $g \cdot \gamma$ if more than one least length element exists. A schematic of this is shown in Figure 1.

Since $\sigma(g) > \frac{1}{2}l(g)$, we have

$$l(a) < \tfrac{1}{2}l(g), \tag{1}$$

$$l(b) < \tfrac{1}{2}l(g). \tag{2}$$

Inequalities (1) and (2), together with the facts that $d(a, g) \geq l(g) - l(a)$ and $d(b, g) \geq l(g) - l(b)$, give

$$d(a, g) > \tfrac{1}{2}l(g), \tag{3}$$

$$d(b, g) > \tfrac{1}{2}l(g). \tag{4}$$

Now consider the distance along $g \cdot \gamma$ between a and b. Since $g \cdot \gamma$ is a geodesic, inequalities (3) and (4) give

$$p_{g\cdot\gamma}(a, g) = d(a, g) > \tfrac{1}{2}l(g), \tag{5}$$

$$p_{g\cdot\gamma}(b, g) = d(b, g) > \tfrac{1}{2}l(g). \tag{6}$$

Since the total path length between a and b is simply the sum of $p_{g\cdot\gamma}(a, g)$ and $p_{g\cdot\gamma}(b, g)$, equations (5) and (6) give

$$\begin{aligned} d(a, b) = p_{g\cdot\gamma}(a, b) &= p_{g\cdot\gamma}(a, g) + p_{g\cdot\gamma}(b, g) \\ &> \tfrac{1}{2}l(g) + \tfrac{1}{2}l(g) = l(g). \end{aligned} \tag{7}$$

By the triangle inequality, the distance between a and b is less than or equal to the sum of their lengths. So, by (1) and (2),

$$\begin{aligned} d(a, b) &\leq l(a) + l(b) \\ &< \tfrac{1}{2}l(g) + \tfrac{1}{2}l(g) = l(g). \end{aligned} \tag{8}$$

Thus (7) and (8) provide a contradiction, which proves that $\sigma(g)/l(g) \leq \frac{1}{2}$ for all $g \in G \setminus \{e\}$. □

In practice, groups containing elements with large ratios of strong depth to length seem difficult to find. Indeed, all elements of sufficiently large length of the families of dead ends studied in the papers referenced on page 368 have ratios of strong depth to length that are less than $\frac{1}{6}$. Moreover, we were able to modify the families or generating sets in question to get families of elements with ratios of strong depth to length only as large as $\frac{1}{4}$. This leads one to consider the "limiting" ratio of strong depth to length

$$\Omega(G) = \limsup_{l(g)\to\infty} \left\{ \frac{\sigma(g)}{l(g)} : g \in G \right\}$$

and ask if there are groups for which $\Omega(G) = \frac{1}{2}$.

Imagining what such a group would look like, one envisions a group G with a sequence of elements (g_n) of increasing length for which it is more and more difficult to reach elements of larger length without retreating closer and closer to halfway back to the identity. If the group had many such elements, one might expect it to be difficult to construct a family of paths, one through each group element, which escape from identity uniformly quickly. In the next section, we examine a property, which we call *uniform quasigeodesic ray extension*, that guarantees the existence of such a family. Unfortunately, however, we cannot establish a connection between $\Omega(G) = \frac{1}{2}$ and a group not having uniform quasigeodesic ray extension. Indeed, the best connection we are able to establish is that uniform quasigeodesic ray extension implies that $\Omega(G) < 1$, which is weaker than the conclusion of Theorem 2.2 for finitely generated groups.

3. Uniform quasigeodesic ray extension

One difficulty in the study of dead ends and in the use of geodesic completeness is that neither existence of dead ends nor the geodesic completeness property is invariant under quasi-isometry. Thus, one cannot apply one of the main strategies of geometric group theory: analyzing a group by understanding its action on a space (or vice versa). In this section we suggest a way of dealing with this difficulty by relaxing the condition of finding geodesic paths to that of finding quasigeodesic paths, all of which have the same multiplicative and additive constants. We begin by reviewing the terminology involved with quasi-isometries and quasigeodesics.

Definition 3.1. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is an (L, A)-*quasi-isometric embedding* if there exist constants $L \geq 1$ and $A \geq 0$ such that, for all $x_1, x_2 \in X$,

$$\frac{1}{L} d_X(x_1, x_2) - A \leq d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2) + A.$$

We refer to L as the *multiplicative constant* and A as the *additive constant* for f. The function f is an (L, A, C)-*quasi-isometry* if there exists an additional constant $C \geq 0$ such that, for all $y \in Y$, there exists $x \in X$ with $d_Y(y, f(x)) \leq C$.

Definition 3.2. Let (X, d_X) be a metric space. A *quasigeodesic* is a (λ, ϵ)-quasi-isometric embedding $g : I \to X$, where $I = (a, b) \cap \mathbb{Z}$ for some $a, b \in \mathbb{R} \cup \{\infty, -\infty\}$.

We remark that if γ is a geodesic in a graph Γ in the sense of Definition 1.4 then γ is a $(1, 1)$-quasigeodesic in the sense of Definition 3.2.

Theorem 3.3. *Let G be an infinite group and X a finite generating set for G. Then for all $g \in G$ there exists an infinite $(3, 0)$-quasigeodesic ray in $\Gamma(G, X)$ starting at the identity of G and passing through g.*

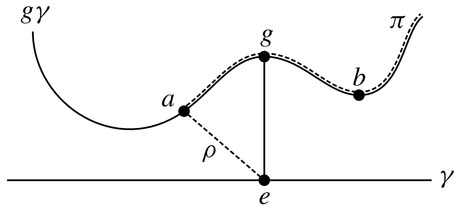

Figure 2. Schematic of the path in Theorem 3.3.

Proof. Let G be an infinite group and X a finite generating set for G and let $\Gamma = \Gamma(G, X)$ be the Cayley graph of G with respect to X. Let γ be an infinite geodesic line in Γ that passes through the identity as given by Lemma 2.1. Let $g \in G$ be an arbitrary element. Since G acts on the Cayley graph by isometries, $g \cdot \gamma$ is an infinite geodesic line passing through g. Let a be an element on $g \cdot \gamma$ such that, for all b on $g \cdot \gamma$, we have $l(a) \le l(b)$.

Let ρ be any geodesic from the identity to a and let π be the geodesic ray along $g \cdot \gamma$ that starts at a and passes through g. Figure 2 shows a schematic illustration of these geodesic segments. Define $\gamma' : \mathbb{N} \cup \{0\} \to G$ by

$$\gamma'(t) = \begin{cases} \rho(t) & \text{if } 0 \le t \le l(a), \\ \pi(t - l(a)) & \text{if } t > l(a). \end{cases}$$

This infinite segment is indicated by the dotted lines in Figure 2. Observe that γ' is infinite by construction. We break the problem into cases to prove that γ' is a $(3, 0)$-quasigeodesic by showing that the following inequality holds for all x and y in the domain of γ':

$$\tfrac{1}{3}|x - y| \le d(\gamma'(x), \gamma'(y)) \le 3|x - y|. \tag{9}$$

Case 1: $x \le l(a)$ and $y \le l(a)$. In this case, $\gamma'(x)$ and $\gamma'(y)$ are defined by ρ. Since ρ is a geodesic, we have

$$\tfrac{1}{3}|x - y| \le |x - y| = d(\rho(x), \rho(y)) = |x - y| \le 3|x - y|.$$

Because $\rho(x) = \gamma'(x)$ and $\rho(y) = \gamma'(y)$, we have $d(\rho(x), \rho(y)) = d(\gamma'(x), \gamma'(y))$. Therefore, inequality (9) is satisfied.

Case 2: $x > l(a)$ and $y > l(a)$. This case is similar to the previous one.

Case 3: $x \le l(a)$ and $y > l(a)$. Note that the right inequality of (9) is trivially true by the definitions of distance and path length because γ' is a path. For the left inequality of (9), first note that $l(\gamma'(x)) \le l(a) \le l(\gamma'(y))$. Therefore any geodesic path between $\gamma'(x)$ and $\gamma'(y)$ must intersect $S_e(l(a))$. Since a lies on $S_e(l(a))$, this implies that $d(\gamma'(x), a) \le d(\gamma'(x), \gamma'(y))$. Because the portion of ρ between

$\gamma'(x)$ and a is a geodesic, we have

$$|x-(l(a))| = d(\gamma'(x), a) \le d(\gamma'(x), \gamma'(y)). \tag{10}$$

By inequality (10) and the triangle inequality,

$$\begin{aligned} |l(a)-y| &= d(a, \gamma'(y)) \\ &\le d(a, \gamma'(x)) + d(\gamma'(x), \gamma'(y)) \le 2d(\gamma'(x), \gamma'(y)). \end{aligned} \tag{11}$$

By inequalities (10) and (11),

$$\begin{aligned} \tfrac{1}{3}|x-y| &= \tfrac{1}{3}\big(|x-(l(a))| + |l(a)-y|\big) \\ &\le \tfrac{1}{3}\big(d(\gamma'(x), \gamma'(y)) + 2d(\gamma'(x), \gamma'(y))\big) = d(\gamma'(x), \gamma'(y)). \end{aligned}$$

Therefore, inequality (9) is satisfied in this case as well.

Therefore, for all x and y in the domain of γ', inequality (9) holds and γ' is an infinite $(3, 0)$-quasigeodesic that starts at the identity and passes through g. □

We have just shown that for any element g in an infinite group G with finite generating set X, there is a $(3, 0)$-quasigeodesic ray in $\Gamma(G, X)$ starting at the identity and passing through g. We now generalize this property by relaxing the additive and multiplicative constants of the quasigeodesic rays and prove that a large class of metric spaces have this property.

Definition 3.4. A metric space (X, d) has *uniform quasigeodesic ray extension* if there exist real numbers $L \ge 1$ and $A \ge 0$ such that for any $x_1, x_2 \in X$, there is an infinite (L, A)-quasigeodesic ray that starts at x_1 and passes through x_2. We refer to L as the multiplicative constant and A as the additive constant.

We note that if G is an infinite finitely generated group, then G has uniform quasigeodesic ray extension with respect to any finite generating set. This is because, for any $g_1, g_2 \in G$, Theorem 3.3 provides an infinite $(3, 0)$-quasigeodesic ray γ starting at e and passing through $g_1^{-1}g_2$. Translating γ by g_1 provides an infinite quasigeodesic ray starting at g_1 and passing through g_2. We record this as:

Corollary 3.5. *Let G be an infinite group and X a finite generating set for G. Then $\Gamma(G, X)$ has the uniform quasigeodesic ray extension property.*

We now prove that *uniform quasigeodesic ray extension* is invariant under quasi-isometry.

Theorem 3.6. *Let (X, d_X) be a metric space having uniform quasigeodesic ray extension with constants L and A. If a metric space (Y, d_Y) is quasi-isometric to (X, d_X), then there exist $L' \ge 1$ and $A' \ge 0$ such that (Y, d_Y) has uniform quasigeodesic ray extension with constants L' and A'.*

The proof of this theorem requires the following lemma, whose proof is a standard exercise in quasi-isometries and quasigeodesics.

Lemma 3.7. *Let $\rho : \mathbb{N} \to X$ be an (L, A)-quasigeodesic ray in a metric space X and $f : X \to Y$ be an (ϵ, λ)-quasi-isometry from X to a metric space Y. Then $\rho' = f \circ \rho : \mathbb{N} \to Y$ is an (L', A')-quasigeodesic ray for constants L' and A' depending only on L, A, ϵ, and λ.*

We now prove Theorem 3.6.

Proof. Let (X, d_X) be a metric space with (L, A)-uniform quasigeodesic ray extension, (Y, d_Y) a metric space and $f : X \to Y$ an (L', A', C)-quasi-isometry. Consider $y_1, y_2 \in Y$. Since f is a quasi-isometry, there exist $x_1, x_2 \in X$ such that $d_Y(y_1, f(x_1)) \le C$ and $d_Y(y_2, f(x_2)) \le C$. Since X has uniform quasigeodesic ray extension, there is an infinite (L, A)-quasigeodesic γ that starts at x_1 and passes through x_2. By Lemma 3.7, $\gamma' = f \circ \gamma$ is an infinite (λ, ϵ)-quasigeodesic that starts at $f(x_1)$ and passes through $f(x_2)$ with λ and ϵ depending only on L, A, L', and A'. Select $l \in \mathbb{N}$ with $\gamma'(l) = f(x_2)$. We define $\gamma'' : \mathbb{N} \to Y$ by

$$\gamma''(t) = \begin{cases} y_1 & \text{if } t = 1, \\ \gamma'(t-1) & \text{if } 2 \le t \le l+1, \\ y_2 & \text{if } t = l+2, \\ \gamma'(l) & \text{if } t = l+3, \\ \gamma'(t-3) & \text{if } t \ge l+4. \end{cases}$$

Setting the multiplicative constant $\lambda' = \lambda$ and the additive constant $\epsilon' = \epsilon + 2C + 3/\lambda$, one can readily verify that γ'' is a (λ', ϵ')-quasigeodesic. Since the constants of γ'' depend only on λ, ϵ, and C, and not the particular y_1 and y_2 selected, (Y, d_Y) has (λ', ϵ') uniform quasigeodesic ray extension. □

The Švarc–Milnor lemma is usually phrased in terms of a group G, not known beforehand to be finitely generated, acting properly discontinuously by isometries and with compact quotient on a proper geodesic metric space X, as, for example, in [de la Harpe 2000, Theorem IV.B.23]. In this case, one concludes that G is finitely generated and, when endowed with the word metric with respect to a finite generating set, quasi-isometric with X. However, if one already knows G to be finitely generated, one can drop the requirement that X be proper, replace the condition of a cocompact action with the existence of a ball $B_{x_0}(R)$ of finite radius whose G-translates cover X, and rephrase a properly discontinuous action as one such that, for every $r > 0$, the set $\{g : B_{x_0}(r) \cap g \cdot B_{x_0}(r) \neq \varnothing\}$ is finite. If the action of G satisfies the above conditions, then G is quasi-isometric with X. In this case, by Corollary 3.5 and Theorem 3.6, one may also conclude that X has uniform quasigeodesic ray extension. We formalize this in our final corollary.

Corollary 3.8. *Let (X, d_X) be a metric space in which any two points can be joined by a geodesic segment and G a finitely generated group acting by isometries on X. If there exists a ball $B_{x_0}(R)$ in X whose G-translates cover X with the property that for every $r > 0$ the set $\{g : B_{x_0}(r) \cap g \cdot B_{x_0}(r) \neq \varnothing\}$ is finite then X has uniform quasigeodesic ray extension.*

Acknowledgment

The authors express thanks to Dr. Elizabeth Weaver, the production editor, for her generous help in improving the flow and exposition of the writing.

References

[Bogopol'skiĭ 1997] O. V. Bogopol'skiĭ, "Infinite commensurable hyperbolic groups are bi-Lipschitz equivalent", *Algebra i Logika* **36**:3 (1997), 259–272, 357. In Russian; translated in *Algebra and Logic* **36**:3 (1997), 155–163. MR 1485595 Zbl 0988.53001

[Bridson and Haefliger 1999] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften **319**, Springer, Berlin, 1999. MR 2000k:53038 Zbl 0988.53001

[de la Harpe 2000] P. de la Harpe, *Topics in geometric group theory*, University of Chicago Press, 2000. MR 1786869 Zbl 0965.20025

[Hosaka 2012] T. Hosaka, "On splitting theorems for CAT(0) spaces and compact geodesic spaces of non-positive curvature", *Math. Z.* **272**:3–4 (2012), 1037–1050. MR 2995154 Zbl 1293.20040

[Lehnert 2009] J. Lehnert, "Some remarks on depth of dead ends in groups", *Internat. J. Algebra Comput.* **19**:4 (2009), 585–594. MR 2536193 Zbl 1191.20027

[Lyons et al. 1996] R. Lyons, R. Pemantle, and Y. Peres, "Random walks on the lamplighter group", *Ann. Probab.* **24**:4 (1996), 1993–2006. MR 1415237 Zbl 0879.60004

[Riley and Warshall 2006] T. R. Riley and A. D. Warshall, "The unbounded dead-end depth property is not a group invariant", *Internat. J. Algebra Comput.* **16**:5 (2006), 969–983. MR 2274725 Zbl 1111.20034

[Warshall 2011] A. D. Warshall, "A group with deep pockets for all finite generating sets", *Israel J. Math.* **185** (2011), 317–342. MR 2837139 Zbl 1269.20035

Received: 2014-06-13 Revised: 2015-07-19 Accepted: 2015-07-22

bgapinski@wesleyan.edu — *Department of Mathematics and Computer Science, Wesleyan University, Middletown, CT 06459, United States*

horakmatt@gmail.com — *Department of Mathematics, Statistics and Computer Science, University of Wisconsin-Stout, Menomonie, WI 54751, United States*

webert2575@my.uwstout.edu — *Department of Mathematics, Statistics and Computer Science, University of Wisconsin-Stout, Menomonie, WI 54751, United States*

involve

msp.org/involve

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2016 is US $160/year for the electronic version, and $215/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2016 vol. 9 no. 3