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Quantum Schubert polynomials for the G_2 flag manifold

Rachel E. Elliott, Mark E. Lewers and Leonardo C. Mihalcea

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We study some combinatorial objects related to the flag manifold X of Lie type G_2. Using the moment graph of X, we calculate all the curve neighborhoods for Schubert classes. We use this calculation to investigate the ordinary and quantum cohomology rings of X. As an application, we obtain positive Schubert polynomials for the cohomology ring of X and we find quantum Schubert polynomials which represent Schubert classes in the quantum cohomology ring of X.

1. Introduction

One of the major theorems in algebra is the classification of complex semisimple Lie algebras. There are four classical infinite series (of types A_n, B_n, C_n, D_n) and five exceptional finite series (of types E_6, E_7, E_8, F_4, G_2). To each algebra, one can associate a group and to each group a certain geometric object called a flag manifold. In type A_n, the points of this flag manifold are sequences $V_1 \subset V_2 \subset \cdots \subset \mathbb{C}^n$ of vector spaces V_i of dimension i. The algebra of type G_2 is considered the simplest among the exceptional series, and we denote by X the flag manifold for type G_2. The study of flag manifolds has a long and rich history starting in the 1950s, and it lies at the intersection of algebraic geometry, combinatorics, topology and representation theory.

One can associate a ring to the flag manifold X called the cohomology ring $\mathrm{H}^*(X)$. This ring has a distinguished basis given by Schubert classes σ_w, indexed by the elements w in the Weyl group W of type G_2; see Section 4 below. We recall that W is actually isomorphic to the dihedral group with 12 elements, although we will use a different realization of it which is more suitable for our purposes. This ring is generated by Schubert classes $\sigma_{s_1}, \sigma_{s_2}$ for the simple reflections s_1, s_2 in W. Therefore, at least in principle, the full multiplication table in the ring is determined by a formula to multiply one Schubert class by another for either s_1 or s_2. This is called a *Chevalley formula*. There has been a substantial amount of work to find

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Chevalley formulas for this ring, starting with Chevalley [1994] in the 1950s. This formula can be expressed combinatorially in terms of the root system and the Weyl group for type G_2. Alternatively, the cohomology ring has a "Borel" presentation $\mathrm{H}^*(X) = \mathbb{Q}[x_1, x_2]/I$, where I is the ideal generated by $x_1^2 - x_1x_2 + x_2^2$ and x_1^6. A natural question is to find out what is the relation between this "algebraic" presentation and the "geometric" one which involves the Schubert basis. In other words, one needs to find a polynomial in $\mathbb{Q}[x_1, x_2]$ which represents a Schubert class σ_w under the isomorphism $\mathrm{H}^*(X) = \mathbb{Q}[x_1, x_2]$. This is called a *Schubert polynomial*. Such polynomials are not unique, as their class in $\mathbb{Q}[x_1, x_2]/I$ is unchanged if one changes a polynomial by elements in I. In Section 5, we use the Chevalley rule to find Schubert polynomials for σ_w. Some of our polynomials coincide with similar Schubert polynomials found by D. Anderson [2011], via different methods. The polynomials we found are homogeneous and have *positive* coefficients. Given that the positivity of Schubert polynomial coefficients has geometric interpretations in type A_n (see the paper of A. Knutson and E. Miller [2005]), this is a desirable property.

The current paper also focuses on a deformation of the ring above called the *quantum cohomology ring* $\mathrm{QH}^*(X)$. It is a deformation of $\mathrm{H}^*(X)$ with the addition of quantum parameters $q^{\boldsymbol{d}} = q_1^{d_1}q_2^{d_2}$ for degrees $\boldsymbol{d} = (d_1, d_2)$. If $\boldsymbol{d} = (0, 0)$, or equivalently $q_1 = q_2 = 0$, the product reduces to the corresponding calculation in $\mathrm{H}^*(X)$. More detail will be given in Section 4. See [Fulton and Pandharipande 1997] for more information about the background/history of this ring. Similar to the ring $\mathrm{H}^*(X)$, the quantum cohomology ring has a $\mathbb{Z}[q]$-basis consisting of Schubert classes σ_w (where $q = (q_1, q_2)$ are the quantum parameters), and it is generated as a ring by the classes σ_{s_1} and σ_{s_2} for the simple reflections s_1 and s_2.

The *quantum Chevalley formula* is a formula for the quantum multiplication $\sigma_w \star \sigma_{s_i}$ $(i = 1, 2)$. An explicit form of this formula, which uses combinatorics of the root system of Lie type G_2, was obtained by Fulton and Woodward [2004]. In this paper, we use the "curve neighborhoods" method to write down the explicit Chevalley formula. This alternative method, obtained by Buch and Mihalcea [2015], involves an interesting graph associated to the flag manifold, called the *moment graph*. Its definition and properties are found in Section 3. It also has the advantage that it leads to a conjectural Chevalley formula in a further deformation of the quantum cohomology ring, called *quantum K-theory*. This will be addressed in a follow-up paper.

Our main application is to obtain a quantum version of the Schubert polynomials. More precisely, it is known [Fulton and Pandharipande 1997, Proposition 11] that $\mathrm{QH}^*(X) = \mathbb{Q}[x_1, x_2, q_1, q_2]/\tilde{I}$, where $\tilde{I}$ a certain ideal which deforms I. Then, as in the classical case, we would like to find the polynomials in $\mathbb{Q}[x_1, x_2, q_1, q_2]$ which represent each Schubert class σ_w via the isomorphism $\mathrm{QH}^*(X) = \mathbb{Q}[x_1, x_2, q_1, q_2]/\tilde{I}$. These are called *quantum Schubert polynomials*. As before, these polynomials are not unique, but we can impose some natural conditions that they satisfy, such as the fact that they deform the ordinary Schubert polynomials, and that they are homogeneous with respect to a certain grading. To our knowledge, such polynomials have not been explicitly calculated in the literature. As a byproduct, we also use the quantum Chevalley formula to recover the ideal $\tilde{I}$ of quantum relations. This ideal has been, in principle, calculated by Kim [1999] using different techniques, but the explicit polynomials generating this ideal do not seem to appear in the literature. Our results are stated in Theorem 5.2 below.

2. Preliminaries: the root system and the Weyl group of type G_2

2A. ***The G_2 root system.*** Denote by R the root system of type G_2. It consists of 12 roots, which are nonzero vectors in the hyperplane in $\mathbb{R}^3$ given by the equation $\xi_1+\xi_2+\xi_3=0$; our main reference is [Bourbaki 2002]. The roots are displayed in Table 1, in terms of the natural coordinates in $\mathbb{R}^3$. Each root α can be written uniquely as $\alpha = c_1\alpha_1 + c_2\alpha_2$, where α_1, α_2 are *simple roots* and $c_1c_2 \geq 0$. A root is *positive* (*negative*) if both c_1, c_2 are nonnegative (resp. nonpositive). The set of simple roots is denoted by $\Delta = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = -2\epsilon_1 + \epsilon_2 + \epsilon_3$. For later purposes, we need to expand each root in terms of the simple roots. The full results are shown in Table 1. The root vectors in the Δ-basis can be seen in Figure 1.

We also need the *dual* root system consisting of *coroots* $\alpha^\vee$. The coroot $\alpha^\vee$ of a root α is defined as $\alpha^\vee = 2\alpha/(\alpha,\alpha)$, where (α,α) is the standard inner product in $\mathbb{R}^3$. Note that the coroots satisfy the properties $(\alpha^\vee)^\vee = \alpha$ and $(-\alpha)^\vee = -\alpha^\vee$. We denote the full set of coroots by $R^\vee$ and define the set $\Delta^\vee$, which holds the *simple coroots* $\alpha_1^\vee$ and $\alpha_2^\vee$ for $R^\vee$. Table 1 shows the values for each of the coroots.

2B. ***The Weyl group of G_2.*** The *Weyl group of* G_2, denoted W, is the group generated by reflections s_α, where $\alpha \in R$. Let $s_i := s_{\alpha_i}$. Geometrically, s_α is the reflection across the line perpendicular to the root α. For example, the reflection s_1

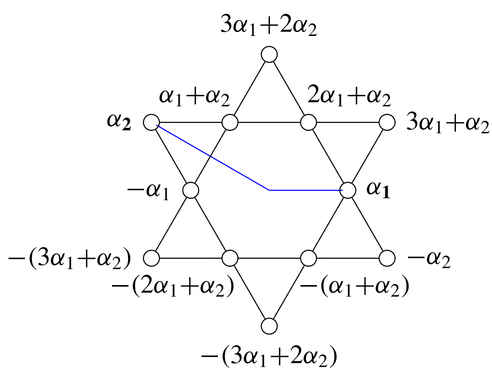

Figure 1. *The root system for G_2. Each node is a root. The blue lines represent the coordinate system using the Δ-basis.*

natural coordinates E-basis $(\epsilon_1, \epsilon_2, \epsilon_3)$	$\pm$	simple roots basis (α_1, α_2)	coroot $\alpha^\vee$ $\alpha^\vee = \lambda\alpha_1 + \mu\alpha_2$
$\epsilon_1 - \epsilon_2$	$+$	α_1	$\alpha_1^\vee = \alpha_1$
$\epsilon_3 - \epsilon_1$	$+$	$\alpha_1 + \alpha_2$	$(\alpha_1+\alpha_2)^\vee = \alpha_1+\alpha_2$
$\epsilon_3 - \epsilon_2$	$+$	$2\alpha_1 + \alpha_2$	$(2\alpha_1+\alpha_2)^\vee = 2\alpha_1+\alpha_2$
$\epsilon_2 + \epsilon_3 - 2\epsilon_1$	$+$	α_2	$\alpha_2^\vee = \frac{1}{3}\alpha_2$
$\epsilon_1 + \epsilon_3 - 2\epsilon_2$	$+$	$3\alpha_1 + \alpha_2$	$(3\alpha_1+\alpha_2)^\vee = \alpha_1 + \frac{1}{3}\alpha_2$
$-\epsilon_1 - \epsilon_2 + 2\epsilon_3$	$+$	$3\alpha_1 + 2\alpha_2$	$(3\alpha_1+2\alpha_2)^\vee = \alpha_1 + \frac{2}{3}\alpha_2$
$-(\epsilon_1 - \epsilon_2)$	$-$	$-\alpha_1$	$(-\alpha_1)^\vee = -\alpha_1$
$-(\epsilon_3 - \epsilon_1)$	$-$	$-(\alpha_1 + \alpha_2)$	$(-\alpha_1-\alpha_2)^\vee = -\alpha_1-\alpha_2$
$-(\epsilon_3 - \epsilon_2)$	$-$	$-(2\alpha_1 + \alpha_2)$	$(-2\alpha_1-\alpha_2)^\vee = -2\alpha_1-\alpha_2$
$-(\epsilon_2 + \epsilon_3 - 2\epsilon_1)$	$-$	$-\alpha_2$	$(-\alpha_2)^\vee = -\frac{1}{3}\alpha_2$
$-(\epsilon_1 + \epsilon_3 - 2\epsilon_2)$	$-$	$-(3\alpha_1 + \alpha_2)$	$(-3\alpha_1-\alpha_2)^\vee = -\alpha_1 - \frac{1}{3}\alpha_2$
$-(-\epsilon_1 - \epsilon_2 + 2\epsilon_3)$	$-$	$-(3\alpha_1 + 2\alpha_2)$	$(-3\alpha_1-2\alpha_2)^\vee = -\alpha_1 - \frac{2}{3}\alpha_2$

Table 1. The root system of type G_2. For each root, we give its sign, the root in terms of Δ-basis, and the corresponding coroot.

(corresponding to s_{α_1}) is the reflection across the line perpendicular to the α_1-axis (see Figure 2). As Figure 2 shows, for any root α, we have $s_\alpha = s_{-\alpha}$. Therefore only six unique reflections exist for the G_2 root system.

It is known (see, e.g., [Humphreys 1972]) that W has the presentation

$$W = \langle s_1, s_2 \,:\, {s_1}^2 = {s_2}^2 = 1,\, (s_1 s_2)^6 = 1 \rangle.$$

From this it follows easily that W is isomorphic to the dihedral group with 12 elements. In order to determine the reflections in W, we need the following definitions.

Definition 2.1. Consider $w \in W$. A *reduced expression* for w is one involving products of s_1 and s_2 in as short a way as possible (via the relations in the presentation). If $w \in W$, where w is a reduced expression, the *length* of w, denoted by $\ell(w)$, is the number of simple reflections (s_1 and s_2) that show up in the reduced expression.

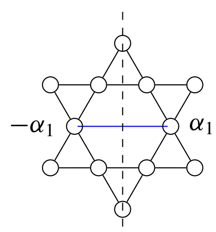

Figure 2. The reflection s_{α_1} (dashed line) which is perpendicular to the α_1-axis (blue line).

root (in Δ-basis)	reflection ($w \in W$)
$\pm\alpha_1$	s_1
$\pm\alpha_2$	s_2
$\pm(3\alpha_1+\alpha_2)$	$s_1s_2s_1$
$\pm(\alpha_1+\alpha_2)$	$s_2s_1s_2$
$\pm(2\alpha_1+\alpha_2)$	$s_1s_2s_1s_2s_1$
$\pm(3\alpha_1+2\alpha_2)$	$s_2s_1s_2s_1s_2$

Table 2. The root reflection corresponding to each root in G_2.

Example 2.2. Consider $w = s_1s_1s_1s_2s_1s_2$. From the presentation of W, we know that ${s_1}^2 = s_1s_1 = 1$ and so this expression is not reduced. However, $(s_1s_1)s_1s_2s_1s_2 = (1)s_1s_2s_1s_2 = s_1s_2s_1s_2$. The latter is a reduced expression and $\ell(w) = 4$.

The 12 reduced expressions of the elements in W are

$$W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2, s_1s_2s_1s_2, s_2s_1s_2s_1, s_1s_2s_1s_2s_1, s_2s_1s_2s_1s_2, s_1s_2s_1s_2s_1s_2\}.$$

We denote by w_0 the longest element $s_1s_2s_1s_2s_1s_2$. Notice that among the twelve elements, only six of them are the *root reflections* from the root system of G_2. Because any reflection has order 2, it is easy to check that the root reflections correspond to the reduced expressions of odd length.

Since the reflections s_1 and s_2 generate W, every reflection s_α in the G_2 root system can be expressed as a reduced expression product of s_1s and s_2s. Consider the action of W on the root system R given by the natural action of reflections on vectors in $\mathbb{R}^3$. Explicitly, this action is given by $s_\alpha \cdot \beta = s_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha$ (see [Humphreys 1972, p. 43]). The following lemma in proved in [loc. cit.].

Lemma 2.3. *Let $w \in W$ and $\alpha \in R$. Then $ws_\alpha w^{-1} = s_{w\cdot\alpha}$.*

Example 2.4. Consider $w = s_1s_2s_1$. We want to find a reflection s_α that corresponds to w. By Lemma 2.3, $s_1s_2s_1 = s_{s_1(\alpha_2)}$, where s_1 is its own inverse and the action is

$$s_1(\alpha_2) = \alpha_2 - (\alpha_2, {\alpha_1}^\vee)\alpha_1 = \alpha_2 - \Big(\alpha_2, \frac{2\alpha_1}{(\alpha_1,\alpha_1)}\Big)\alpha_1.$$

We know $(\alpha_1, \alpha_1) = 2$, (see Table 1) so

$$\alpha_2 - \Big(\alpha_2, \frac{2\alpha_1}{(\alpha_1,\alpha_1)}\Big)\alpha_1 = \alpha_2 - \Big(\alpha_2, \frac{2\alpha_1}{2}\Big)\alpha_1 = \alpha_2 - (\alpha_2, \alpha_1)\alpha_1$$
$$= \alpha_2 - (-3)\alpha_1.$$

Thus $s_1(\alpha_2) = 3\alpha_1 + \alpha_2$. The reflection $s_1s_2s_1$ is the reflection $s_{3\alpha_1+\alpha_2}$.

Table 2 shows the reflection across the line perpendicular to each root. Notice that roots α and $-\alpha$ have the same reflection and all reflections listed have *odd* length.

coroot $\alpha^\vee = d_1{\alpha_1}^\vee + d_2{\alpha_2}^\vee$	degree $\boldsymbol{d}$ (d_1, d_2)
${\alpha_1}^\vee$	$(1, 0)$
${\alpha_2}^\vee$	$(0, 1)$
$(3\alpha_1 + \alpha_2)^\vee$	$(1, 1)$
$(\alpha_1 + \alpha_2)^\vee$	$(1, 3)$
$(2\alpha_1 + \alpha_2)^\vee$	$(2, 3)$
$(3\alpha_1 + 2\alpha_2)^\vee$	$(1, 2)$

Table 3. The degree for each coroot in the moment graph.

3. The moment graph and curve neighborhoods

3A. ***Finding the moment graph.*** Using the properties of the elements in the Weyl group for G_2, it is possible to define the following graph.

Definition 3.1. The *moment graph* is an oriented graph that consists of a pair (V, E), where V is the set of vertices and E is the set of edges. To each Weyl group element $v \in W$ there corresponds a vertex $v \in V$ in this graph. For $x, y \in V$, an edge exists from x to y, denoted by

$$x \xrightarrow{\alpha^\vee} y,$$

if there exists a reflection s_α such that $y = xs_\alpha$ and $\ell(y) > \ell(x)$.

Definition 3.2. A *degree* $\boldsymbol{d}$ is a nonnegative combination $d_1\alpha_1^\vee + d_2\alpha_2^\vee$ of simple coroots. We will denote it as $\boldsymbol{d} = (d_1, d_2)$.

Since any coroot $\alpha^\vee$ is a linear combination in terms of $\alpha_1^\vee$ and $\alpha_2^\vee$, it determines a degree. These degrees are given in Table 3.

Example 3.3. An edge exists from s_1 to s_2s_1. This is so because

$$\ell(s_2s_1) > \ell(s_1) \qquad \text{and} \qquad s_2s_1 = s_1s_\alpha, \quad \text{where } s_\alpha = s_1s_2s_1.$$

Example 2.4 shows $s_1s_2s_1 = s_{3\alpha_1+\alpha_2}$. The edge corresponding to these two edges has degree $(3\alpha_1 + \alpha_2)^\vee$; i.e.,

$$s_1 \xrightarrow{(3\alpha_1+\alpha_2)^\vee} s_2s_1.$$

Notice that $(3\alpha_1 + \alpha_2)^\vee = 1\alpha_1^\vee + 1\alpha_2^\vee$, so $\boldsymbol{d} = (1, 1)$. The edge from s_1 to s_2s_1 can be represented by the degree $(1, 1)$.

We depict the moment graph as oriented upward, as in Figure 3. To help read the moment graph, a color code has been set up to represent the different edges. We review some of the relevant properties of the moment graph:

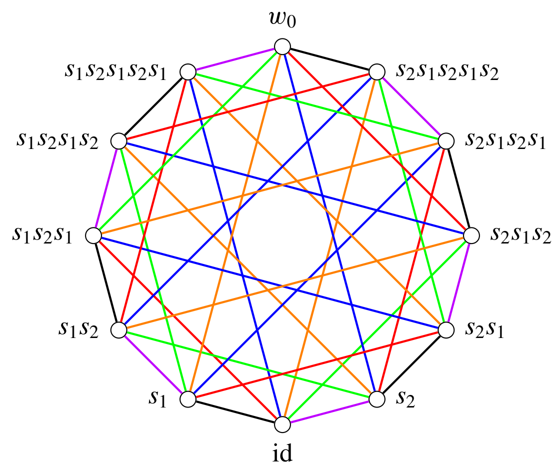

Figure 3. The moment graph for G_2. The color code for the degrees is black = (1, 0), violet = (0, 1), red = (1, 1), green = (1, 3), blue = (2, 3), orange = (1, 2).

- The vertices correspond to the 12 Weyl group elements.
- The edges represent the root reflections associated to the G_2 root system. There are six different types of edges (different degree values) because there are exactly six reflections in the G_2 root system. Note that edges exist between Weyl group elements if the difference between lengths is odd.
- The bottom vertex is the element with the smallest length (id, where $\ell(\text{id}) = 0$). The vertices in the next "row" have length 1 (s_1 and s_2). The length of these elements increases by one as you travel up the graph. The top vertex is the element with the largest length, w_0, where $\ell(w_0) = 6$.
- For any vertex, there are six edges connected to it, corresponding to the six different coroots in $R^\vee$.
- For any $w_1, w_2 \in W$, where $\ell(w_1) = \ell(w_2)$, both w_1 and w_2 will have edges connecting to the same six vertices.

3B. ***Curve neighborhoods.*** In Section 3A, we defined the degree $\boldsymbol{d}$ to help simplify the moment graph for use in future calculations. The importance of the moment graph can be realized with the following concept defined by A. Buch and L. Mihalcea [2015]:

Definition 3.4. Fix a degree $\boldsymbol{d} = (d_1, d_2)$ and an element u of the Weyl group W. The *curve neighborhood*, $\Gamma_d(u)$, is a subset of W which consists of the maximal elements in the moment graph which can be reached from u with a path of total degree at most $\boldsymbol{d}$.

Example 3.5. Consider $w = \mathrm{id}$ and $\boldsymbol{d} = (1, 1)$. We want to determine the "highest" path (starting at the identity) where the total degree traveled is at most $(1, 1)$. By inspecting the moment graph, we see that there are three initial paths starting from id:

• *path* $\boldsymbol{d} = (1, 0)$, which goes from id to s_1. Upon reaching s_1, one is not allowed to travel more than $\boldsymbol{d}' = (0, 1)$ upwards. Further inspection of the moment graph shows that a path exists with degree $(0, 1)$ from s_1 to s_1s_2. We now have traveled a total degree of (1,1). Thus we are done and s_1s_2 is the largest element on this path.

• *path* $\boldsymbol{d} = (0, 1)$ which goes from id to s_2. Upon reaching s_2, one is not allowed to travel more than $\boldsymbol{d}' = (1, 0)$ upwards. Further inspection gives a path with degree $(1, 0)$ from s_2 to s_2s_1. We now have traveled a total degree of $(1, 1)$. Thus we are done and s_2s_1 is the largest element on this path.

• *path* $\boldsymbol{d} = (1, 1)$ which goes from id to $s_1s_2s_1$. Since we traveled a total degree of $(1, 1)$, we are done, and $s_1s_2s_1$ is the largest element on this path.

We now take the maximal element that can be reached from id with degree $(1, 1)$. The largest of the three elements above is $s_1s_2s_1$; thus $\Gamma_{(1,1)}(\mathrm{id}) = \{s_1s_2s_1\}$.

It is clear that for any $w \in W$, there exists some degree (a, b) where $\Gamma_{(a,b)}(w) = w_0$. Then for any larger degree (a', b'), where $a' \geq a$ and $b' \geq b$, we have $\Gamma_{(a',b')}(w) = w_0$. Table 6 in the Appendix shows the curve neighborhoods for every element of the Weyl group. For all the examples given, the curve neighborhood for some degree $\boldsymbol{d}$ at $u \in W$ is always unique, a fact which was initially proved in [Buch and Mihalcea 2015] for all Lie types.

4. Quantum cohomology ring for flag manifold X

Recall that X denotes the flag manifold of type G_2. The cohomology ring, denoted by $\mathrm{H}^*(X)$, consists of elements that can each be written uniquely as finite sums $\sum_{w\in W} a_w\sigma_w$, where $a_w \in \mathbb{Z}$ and σ_w is a (geometrically defined) *Schubert class*. Addition in this ring is given by

$$\sum_{w\in W} a_w\sigma_w + \sum_{w\in W} b_w\sigma_w = \sum_{w\in W} (a_w + b_w)\sigma_w.$$

The quantum cohomology ring $\mathrm{QH}^*(X)$ is a deformation of $\mathrm{H}^*(X)$ by adding quantum parameters, $q^{\boldsymbol{d}} = q_1^{d_1} q_2^{d_2}$ for degrees $\boldsymbol{d} = (d_1, d_2)$. If $\boldsymbol{d} = (0, 0)$ for any calculation in $\mathrm{QH}^*(X)$, we reduce down to the corresponding calculation in $\mathrm{H}^*(X)$. Similarly to $\mathrm{H}^*(X)$, the elements of $\mathrm{QH}^*(X)$ can each be written uniquely as finite sums $\sum_{w\in W} a_w(\boldsymbol{d})q^{\boldsymbol{d}}\sigma_w$, where $a_w(\boldsymbol{d}) \in \mathbb{Z}$. The addition in this ring is also straightforward:

$$\sum_{w\in W} a_w(\boldsymbol{d})q^{\boldsymbol{d}}\sigma_w + \sum_{w\in W} b_w(\boldsymbol{d})q^{\boldsymbol{d}}\sigma_w = \sum_{w\in W} (a_w(\boldsymbol{d}) + b_w(\boldsymbol{d}))q^{\boldsymbol{d}}\sigma_w.$$

The multiplication in this ring is given by certain integers $c_{u,v}^{w,\boldsymbol{d}}$ called the *Gromov–Witten invariants*:

$$\sigma_u \star \sigma_v = \sum_{w,\boldsymbol{d}} c_{u,v}^{w,\boldsymbol{d}} q^{\boldsymbol{d}} \sigma_w,$$

where the sum is over $w \in W$ and degrees $\boldsymbol{d}$ which have nonnegative components. The (quantum) cohomology ring has two generators, namely σ_{s_1} and σ_{s_2}, corresponding to the simple reflections $s_1, s_2 \in W$. As a result, every element is a sum of monomials in the σ_{s_i}, and the quantum multiplication $\sigma_u \star \sigma_{s_i}$ by generators σ_{s_i} determines the entire ring multiplication. The formula for $\sigma_w \star \sigma_{s_i}$, the *(quantum) Chevalley rule*, is illustrated in Section 4A. We list below a few properties that will help to understand this ring and we refer, e.g., to [Fulton and Pandharipande 1997] for full details.

(1) The multiplication of quantum parameters is given by $q_i^{d_s} q_i^{d_s'} = q_i^{d_s+d_s'}$.

(2) The quantum multiplication $\star$ is associative, commutative and has unit $1 = \sigma_{\mathrm{id}}$.

(3) The quantum multiplication is graded by imposing $\deg(\sigma_w) = \ell(w)$ and for $\boldsymbol{d} = (d_1, d_2)$, we have $\deg q^{\boldsymbol{d}} = 2(d_1 + d_2)$. This implies that $\deg(\sigma_u \star \sigma_v) = \deg(\sigma_u) + \deg(\sigma_v)$ and that $c_{u,v}^{w,\boldsymbol{d}} = 0$ unless $\ell(u) + \ell(v) = \ell(w) + \deg q^{\boldsymbol{d}}$.

(4) If we impose the substitution $q_1 = q_2 = 0$ in $\sigma_u \star \sigma_v$ then we obtain the multiplication $\sigma_u \cdot \sigma_v$ in the ordinary cohomology ring $\mathrm{H}^*(X)$.

4A. ***Quantum Chevalley rule via curve neighborhoods.*** Recall that each coroot $\alpha^\vee$ can be written as a linear combination $\alpha^\vee = d_1\alpha_1^\vee + d_2\alpha_2^\vee$, where $\alpha_1^\vee, \alpha_2^\vee$ are the simple coroots and $d_1, d_2 \in \mathbb{Z}$. It follows that each $\alpha^\vee$ can be identified with the unique degree $\boldsymbol{d} = (d_1, d_2)$. Let $\boldsymbol{d}[i]$ denote the i-th component of the degree $\boldsymbol{d}$ in the decomposition $\boldsymbol{d} = \boldsymbol{d}[1]\alpha_1^\vee + \boldsymbol{d}[2]\alpha_2^\vee$. In other words, $\boldsymbol{d}[i] = d_i$. Note that $\alpha^\vee[i]$ means the same thing as $\boldsymbol{d}[i]$.

The classical *Chevalley rule* [1994] (see also [Fulton and Woodward 2004]) is a formula for the products $\sigma_u \cdot \sigma_{s_i} \in \mathrm{H}^*(X)$:

$$\sigma_u \cdot \sigma_{s_i} = \sum_{\alpha} (\alpha^\vee[i]) \sigma_{us_\alpha}, \tag{1}$$

where the sum is over positive roots α such that $\ell(us_\alpha) = \ell(u) + 1$.

The *quantum Chevalley formula* for $\sigma_u \star \sigma_{s_i} = \sum_{w,\boldsymbol{d}} c_{u,s_i}^{w,\boldsymbol{d}} q^{\boldsymbol{d}} \sigma_w$ was first proved by Fulton and Woodward [2004]. See Theorem 4.3 below. We follow here an approach based on curve neighborhoods, recently proved by Buch and Mihalcea [2015]. If $\boldsymbol{d} = (0, 0)$ then the coefficients $c_{u,s_i}^{w,\boldsymbol{d}}$ are those from identity (1) above. If $\boldsymbol{d} \neq (0, 0)$ then the quantum coefficient $c_{u,s_i}^{w,\boldsymbol{d}}$ can be calculated as follows. First, let $w[\boldsymbol{d}] \in W$ be the curve neighborhood $\Gamma_{\boldsymbol{d}}(w)$. Then

$$c_{u,s_i}^{w,\boldsymbol{d}} = \boldsymbol{d}[i] \cdot \delta_{u,w[\boldsymbol{d}]}, \tag{2}$$

where δ_{v_1,v_2} is the Kronecker symbol and w satisfies $\ell(w) + \deg q^{\boldsymbol{d}} = \ell(u) + 1$.

Remark 4.1. Although it is not clear from the definition, it turns out that if $\boldsymbol{d}[i] \neq 0$ then $u = w[\boldsymbol{d}]$ only if $\boldsymbol{d} = \alpha^\vee$ for some α such that $\ell(s_\alpha) = \deg q^{\alpha^\vee} - 1$. This recovers the original quantum Chevalley rule from [Fulton and Woodward 2004].

Example 4.2. Consider $\sigma_{s_1} \star \sigma_{s_1}$.

• Assume $\boldsymbol{d} = (0, 0)$. We need to determine roots α such that $\ell(s_1 s_\alpha) = \ell(s_1) + 1 = 2$. The only possible Weyl group elements to represent s_α are s_2 and $s_1 s_2 s_1$. If $s_\alpha = s_2$ then $\alpha = \alpha_2$. This implies $\alpha^\vee = (0, 1)$, so $\alpha^\vee[1] = 0$. If $s_\alpha = s_1 s_2 s_1$ then $\alpha = 3\alpha_1 + \alpha_2$. This implies $\alpha^\vee = (1, 1)$, so $\alpha^\vee[1] = 1$. Thus

$$\sum_\alpha (\alpha^\vee[i]) \sigma_{us_\alpha} = 0 \cdot \sigma_{s_1 s_2} + 1 \cdot \sigma_{s_1 s_1 s_2 s_1} = \sigma_{s_2 s_1}.$$

• Assume $\boldsymbol{d} \neq (0, 0)$. We need to determine $w \in W$ such that $w[\boldsymbol{d}] = \Gamma_{\boldsymbol{d}}(w) = s_1$. According to the curve neighborhood results table in the Appendix, the only possible $w \in W$ are id and s_1. For both elements, the possible nondegrees are $(N, 0)$, where $N \in \mathbb{N}$. Note that we also need to choose w and $\boldsymbol{d}$ such that $\ell(w) + \deg q^{\boldsymbol{d}} = \ell(s_1) + 1 = 2$. Since $\deg q^{\boldsymbol{d}}$ is never odd, $\ell(w)$ must be even. This eliminates s_1. As for id, $\ell(\mathrm{id}) = 0$ so then $\deg q^{\boldsymbol{d}} = 2$, where $\boldsymbol{d} = (N, 0)$. This implies $N = 1$. Therefore $c_{s_1, s_1}^{\mathrm{id}, (1,0)} = \boldsymbol{d}[1] \cdot \delta_{s_1, s_1} = 1 \cdot 1 = 1$ and this represents the only nonzero quantum term. Thus for $\boldsymbol{d} \neq (0, 0)$,

$$\sum_{w \in W, \boldsymbol{d}} c_{u, s_i}^{w, \boldsymbol{d}} q^{\boldsymbol{d}} \sigma_w = 1 \cdot q^{(1,0)} \cdot \sigma_{\mathrm{id}} = 1 \cdot q_1 \cdot 1 = q_1.$$

Combining the classical (i.e., from $\mathrm{H}^*(X)$) and pure quantum terms gives us $\sigma_{s_1} \star \sigma_{s_1} = \sigma_{s_2 s_1} + q_1$.

Table 4 shows the results of our quantum Chevalley computations.

Theorem 4.3 (the quantum Chevalley rule [Fulton and Woodward 2004; Buch and Mihalcea 2015]). *The following holds in* $\mathrm{QH}^*(X)$:

$$\sigma_u \star \sigma_{s_i} = \sum_\alpha (\alpha^\vee[i]) \sigma_{us_\alpha} + \sum_\beta (\beta^\vee[i]) q^{\beta^\vee} \sigma_{us_\beta}. \tag{3}$$

The first sum is over positive roots α *such that* $\ell(us_\alpha) = \ell(u) + 1$ *and the second sum is over positive roots* β *such that* $\ell(us_\beta) = \ell(u) + 1 - \deg(q^{\beta^\vee})$.

5. Quantum Schubert polynomials

We know that $\mathrm{QH}^*(X)$ is generated as a $\mathbb{Q}[q] = \mathbb{Q}[q_1, q_2]$-algebra by the classes σ_{s_1} and σ_{s_2}. (This means that every element in $\mathrm{QH}^*(X)$ can be written as a sum of monomials in the σ_{s_i} with coefficients in $\mathbb{Q}[q]$.) Then there exists a *surjective* homomorphism of $\mathbb{Q}[q]$-algebras $\Psi : \mathbb{Q}[x_1, x_2; q_1, q_2] \to \mathrm{QH}^*(X)$ sending

$$\Psi(q_i) = q_i, \quad \Psi(x_1) = \sigma_{s_1}, \quad \Psi(x_1 + x_2) = \sigma_{s_2}.$$

w	$\sigma_w \star \sigma_{s_1}$	$\sigma_w \star \sigma_{s_2}$
s_1	$\sigma_{s_2s_1} + q_1$	$\sigma_{s_1s_2} + \sigma_{s_2s_1}$
s_2	$\sigma_{s_1s_2} + \sigma_{s_2s_1}$	$3\sigma_{s_1s_2} + q_2$
s_1s_2	$\sigma_{s_1s_2s_1} + \sigma_{s_2s_1s_2}$	$2\sigma_{s_2s_1s_2} + q_2\sigma_{s_1}$
s_2s_1	$2\sigma_{s_1s_2s_1} + q_1\sigma_{s_2}$	$3\sigma_{s_1s_2s_1} + \sigma_{s_2s_1s_2}$
$s_1s_2s_1$	$\sigma_{s_2s_1s_2s_1} + q_1\sigma_{s_1s_2} + q_1q_2$	$\sigma_{s_1s_2s_1s_2} + 2\sigma_{s_2s_1s_2s_1} + q_1q_2$
$s_2s_1s_2$	$\sigma_{s_2s_1s_2s_1} + 2\sigma_{s_1s_2s_1s_2}$	$3\sigma_{s_1s_2s_1s_2} + q_2\sigma_{s_2s_1}$
$s_1s_2s_1s_2$	$\sigma_{s_1s_2s_1s_2s_1} + \sigma_{s_2s_1s_2s_1s_2}$	$\sigma_{s_2s_1s_2s_1s_2} + q_2\sigma_{s_1s_2s_1}$
$s_2s_1s_2s_1$	$\sigma_{s_1s_2s_1s_2s_1} + q_1\sigma_{s_2s_1s_2} + q_1q_2\sigma_{s_2}$	$\sigma_{s_2s_1s_2s_1s_2} + 3\sigma_{s_1s_2s_1s_2s_1} + q_1q_2\sigma_{s_2}$
$s_1s_2s_1s_2s_1$	$q_1\sigma_{s_1s_2s_1s_2} + q_1q_2\sigma_{s_1s_2}$	$\sigma_{w_0} + q_1q_2\sigma_{s_1s_2}$
$s_2s_1s_2s_1s_2$	$\sigma_{w_0} + q_1{q_2}^2$	$q_2\sigma_{s_2s_1s_2s_1} + 2q_1{q_2}^2$
$w_0 = (s_1s_2)^3$	$q_1\sigma_{s_2s_1s_2s_1s_2} + q_1q_2\sigma_{s_2s_1s_2} + q_1{q_2}^2\sigma_{s_1}$	$q_2\sigma_{s_1s_2s_1s_2s_1} + q_1q_2\sigma_{s_2s_1s_2} + 2q_1{q_2}^2\sigma_{s_1}$

Table 4. The quantum Chevalley table.

Note that for any P, $P' \in \mathbb{Q}[x_1, x_2, q_1, q_2]$, we have $\Psi(P \cdot P') = \Psi(P) \star \Psi(P')$. We call Ψ the *quantization map*. Let $\tilde{I}$ be the kernel of this homomorphism. By the first isomorphism theorem, we have an isomorphism

$$\overline{\Psi} : \mathbb{Q}[x_1, x_2, q_1, q_2]/\tilde{I} \to \mathrm{QH}^*(X),$$

and this gives the presentation of the quantum cohomology ring. A *quantum Schubert polynomial* for the Schubert class σ_w is any polynomial $P_w \in \mathbb{Q}[x_1, x_2, q_1, q_2]$ such that the image of P_w under Ψ gives the class σ_w. Equivalently $\overline{\Psi}(P_w + \tilde{I}) = \sigma_w$.

To find a quantum Schubert polynomial P_w, we proceed by induction on $\ell(w)$, using the quantum Chevalley formula from Table 4, and starting from the "initial conditions" $P_{s_1} = x_1$ and $P_{s_2} = x_1 + x_2$. To obtain the corresponding classical Schubert polynomials for cohomology, set $q_1 = q_2 = 0$.

Example 5.1. In order to calculate $P_{s_2s_1}$, we use the identity $\sigma_{s_1} \star \sigma_{s_1} = \sigma_{s_2s_1} + q_1$ (taken from Table 4). Using that Ψ is an algebra homomorphism, we know that

$$\Psi(x_1^2) = \Psi(x_1) \star \Psi(x_1) = \sigma_{s_1} \star \sigma_{s_1} \quad \text{and} \quad \Psi(q_1) = q_1.$$

Since $\Psi(x_1^2 - q_1) = \Psi(x_1^2) - \Psi(q_1)$, it follows that $\Psi(x_1^2 - q_1) = \sigma_{s_2s_1}$. This shows that $x_1^2 - q_1$ is a quantum Schubert polynomial for $\sigma_{s_2s_1}$. The corresponding ordinary Schubert polynomial is x_1^2, obtained by making $q_1 = 0$.

Computations of ordinary Schubert polynomials were done for the ordinary cohomology ring $\mathrm{H}^*(X)$ of the G_2 flag manifold in a paper by Anderson [2011]. A classical result of Borel [1953] shows that $\mathrm{H}^*(X) = \mathbb{Q}[x_1, x_2]/I$, where $I = \langle x_1^2 - x_1x_2 + x_2^2, x_1^6 \rangle$. (This can also be deduced from the classical Chevalley formula.) Anderson used this presentation and a different method to obtain different Schubert

σ_{s_α}	our calculation	Anderson's calculation [2011]
w_0	$\frac{1}{2}(x_1^6+x_1^5x_2)$	$\frac{1}{2}x_1^5x_2$
$s_1s_2s_1s_2s_1$	$\frac{1}{2}x_1^5$	$\frac{1}{2}x_1^5$
$s_2s_1s_2s_1s_2$	$\frac{1}{6}(x_1+x_2)^3x_1x_2$	$\frac{1}{2}(x_1^3+x_2x_1^2+x_2^2x_1+x_2^3)x_1x_2$
$s_2s_1s_2s_1$	$\frac{1}{2}x_1^4$	$\frac{1}{2}(4x_1^2-3x_1x_2+3x_2^2)x_1^2$
$s_1s_2s_1s_2$	$\frac{1}{6}(x_1+x_2)^2x_1x_2$	$\frac{1}{2}(x_1^4+x_1^3x_2+x_1^2x_2^2+x_1x_2^3+x_2^4)$
$s_1s_2s_1$	$\frac{1}{2}x_1^3$	$\frac{1}{2}(4x_1^2-3x_1x_1+3x_2^2)x_1$
$s_2s_1s_2$	$\frac{1}{2}(x_1+x_2)x_1x_2$	$2x_1^3+\frac{1}{2}x_1^2x_2+\frac{1}{2}x_1x_2^2+2x_2^3$
s_2s_1	x_1^2	$3x_1^2-2x_1x_2+2x_2^2$
s_1s_2	x_1x_2	$2x_1^2-x_1x_2+2x_2^2$
s_2	x_1+x_2	x_1+x_2
s_1	x_1	x_1
id	1	1

Table 5. Classical Schubert polynomials.

polynomials, but our answers and his must be equal modulo the ideal I. The classical Schubert polynomials we found are shown alongside Anderson's in Table 5. In order to check if our results are equal, we verified that the difference between our resulting classical polynomials was a multiple of one of the elements of the ideal.

We used our quantum Schubert polynomial results, found in Theorem 5.2 below, to compute the ideal $\tilde{I}$ of the quantum cohomology ring $\mathrm{QH}^*(X)$. This ideal is a deformation of the ideal I of $\mathrm{H}^*(X)$. As an example, we will derive the degree-2 relation in $\tilde{I}$. From the quantum Chevalley table on page 447, we know the identities

- $\sigma_{s_1}\star\sigma_{s_1}=\sigma_{s_2s_1}+q_1$,
- $\sigma_{s_1}\star\sigma_{s_2}=\sigma_{s_1s_2}+\sigma_{s_2s_1}$, and
- $\sigma_{s_2}\star\sigma_{s_2}=3\sigma_{s_1s_2}+q_2$.

These three equalities can be combined to obtain

$$3(\sigma_{s_1}\star\sigma_{s_1})+(\sigma_{s_2}\star\sigma_{s_2})=3(\sigma_{s_1}\star\sigma_{s_2})+3q_1+q_2.$$

Now apply the transformation under $\overline{\Psi}$ to get

$$3(x_1\cdot x_1)+((x_1+x_2)\cdot(x_2+x_2))\equiv\big(3(x_1(x_1+x_2))+3q_1+q_2\big)+\tilde{I},$$

which is

$$3x_1^2+x_1^2+2x_1x_2+x_2^2\equiv(3x_1^2+3x_1x_2+3q_1+q_2)+\tilde{I}.$$

Their difference belongs to $\tilde{I}=\ker\Psi$, so (after simplification) we get

$$x_1^2-x_1x_2+x_2^2-(3q_1+q_2)\in\tilde{I}.$$

This is the degree-2 relation in $\tilde{I}$. Notice that this is clearly a deformation of the ideal term $x_1^2 - x_1x_2 + x^2$ in I. To get the degree-6 relation, one does a similar manipulation but using the higher-degree terms in the quantum Chevalley table on page 447. The following is the main result of this paper.

Theorem 5.2. *The quantum cohomology ring of the flag manifold of type G_2 is*

$$\mathrm{QH}^*(X) = \mathbb{Q}[q_1, q_2, x_1, x_2]/\langle R_2, R_6\rangle,$$

where $R_2 := x_1^2 - x_1x_2 + x_2^2 - (3q_1 + q_2)$ *and*

$$R_6 := x_1^6 + q_1\bigl(-2x_1^4 - \tfrac{13}{3}x_2^3x_2 - \tfrac{5}{3}x_1^2x_2^2 - \tfrac{1}{3}x_1x_2^3\bigr) + q_1^2\bigl(-\tfrac{10}{3}x_1^2 - \tfrac{5}{3}x_1x_2 - \tfrac{1}{3}x_2^2\bigr) + q_1q_2\bigl(-2x_1^2 - \tfrac{11}{3}x_1x_2\bigr) - \tfrac{8}{3}q_1^2q_2.$$

Under this presentation, the corresponding quantum Schubert polynomials are

$$w_0 = (s_1s_2)^3:\ \tfrac{1}{2}(x_1^6+x_1^5x_2)+\tfrac{1}{2}(-2x_1^4-6x_1^3x_2-5x_1^2x_2^2-x_1x_2^3)q_1, +\tfrac{1}{2}(-3x_1^2-7x_1x_2-2x_2^2)q_1q_2+\tfrac{1}{2}(-3x_1^2-4x_1x_2-x_2^2)q_1^2-q_1^2q_2,$$

$$s_2s_1s_2s_1s_2:\ \tfrac{1}{6}\bigl((x_1+x_2)^3x_1x_2\bigr)+\tfrac{1}{6}\bigl((x_1+x_2)^3q_1, +(-6x_1^3-4{x_1}^2x_2-x_1x_2^2)q_2+(8x_1+5x_2)q_1q_2\bigr),$$

$$s_1s_2s_1s_2s_1:\ \tfrac{1}{2}x_1^5+\tfrac{1}{2}\bigl((-2x_1^3-4x_1^2x_2-x_1x_2^2)q_1+(-3x_1-2x_2)q_1q_2+(-3x_1-x_2)q_1^2\bigr),$$

$$s_1s_2s_1s_2:\ \tfrac{1}{6}\bigl((x_1+x_2)^2x_1x_2\bigr)+\tfrac{1}{6}\bigl((x_1+x_2)^2q_1+(-3x_1^2-x_1x_2)q_2+2q_1q_2\bigr),$$

$$s_2s_1s_2s_1:\ \tfrac{1}{2}x_1^4+\tfrac{1}{2}\bigl((-2x_1^2-3x_1x_2)q_1-2q_1q_2-2q_1^2\bigr),$$

$$s_2s_1s_2:\ \tfrac{1}{2}\bigl((x_1+x_2)x_1x_2\bigr)+\tfrac{1}{2}\bigl((x_1+x_2)q_1-x_1q_2\bigr),$$

$$s_1s_2s_1:\ \tfrac{1}{2}x_1^3+\tfrac{1}{2}\bigl((-2x_1-x_2)q_1\bigr),$$

$$s_1s_2:\ x_1x_2+q_1,$$

$$s_2s_1:\ x_1^2-q_1,$$

$$s_2:\ x_1+x_2,$$

$$s_1:\ x_1,$$

$$\mathrm{id}:\ 1.$$

Appendix: Table of curve neighborhood calculations

This appendix contains the curve neighborhoods for all the Weyl group elements. In order to list them as concisely as possible, we need to define

- $\ell, m = 0, 1, 2, 3, \ldots,$
- $N, M = 1, 2, 3, \ldots,$
- $N', M' = 2, 3, 4, \ldots.$

If $w \in W$ then $\Gamma_{(0,0)}(w) = w$, so we won't include that condition in the table.

id	s_1	s_2
$\Gamma_{(N,0)}(\mathrm{id})=s_1$ $\Gamma_{(0,N)}(\mathrm{id})=s_2$ $\Gamma_{(N,1)}(\mathrm{id})=s_1s_2s_1$ $\Gamma_{(1,N')}(\mathrm{id})=s_2s_1s_2s_1s_2$ $\Gamma_{(N',M')}(\mathrm{id})=w_0$	$\Gamma_{(N,0)}(s_1)=s_1$ $\Gamma_{(0,N)}(s_1)=s_1s_2$ $\Gamma_{(N,1)}(s_1)=s_1s_2s_1$ $\Gamma_{(N,N')}(s_1)=w_0$	$\Gamma_{(N,0)}(s_2)=s_2s_1$ $\Gamma_{(0,N)}(s_2)=s_2$ $\Gamma_{(N,1)}(s_2)=s_2s_1s_2s_1$ $\Gamma_{(1,N')}(s_2)=s_2s_1s_2s_1s_2$ $\Gamma_{(N',M')}(s_2)=w_0$
s_1s_2	s_2s_1	$s_1s_2s_1$
$\Gamma_{(N,0)}(s_1s_2)=s_1s_2s_1$ $\Gamma_{(0,N)}(s_1s_2)=s_1s_2$ $\Gamma_{(N,1)}(s_1s_2)=s_1s_2s_1s_2s_1$ $\Gamma_{(N,N')}(s_1s_2)=w_0$	$\Gamma_{(N,0)}(s_2s_1)=s_2s_1$ $\Gamma_{(0,N)}(s_2s_1)=s_2s_1s_2$ $\Gamma_{(N,1)}(s_2s_1)=s_2s_1s_2s_1$ $\Gamma_{(N,N')}(s_2s_1)=w_0$	$\Gamma_{(N,0)}(s_1s_2s_1)=s_1s_2s_1$ $\Gamma_{(0,N)}(s_1s_2s_1)=s_1s_2s_1s_2$ $\Gamma_{(N,1)}(s_1s_2s_1)=s_1s_2s_1s_2s_1$ $\Gamma_{(N,N')}(s_1s_2s_1)=w_0$
$s_2s_1s_2$	$s_1s_2s_1s_2$	$s_2s_1s_2s_1$
$\Gamma_{(N,0)}(s_2s_1s_2)=s_2s_1s_2s_1$ $\Gamma_{(0,N)}(s_2s_1s_2)=s_2s_1s_2$ $\Gamma_{(N,M)}(s_2s_1s_2)=w_0$	$\Gamma_{(N,0)}(s_1s_2s_1s_2)=s_1s_2s_1s_2s_1$ $\Gamma_{(0,N)}(s_1s_2s_1s_2)=s_1s_2s_1s_2$ $\Gamma_{(N,M)}(s_1s_2s_1s_2)=w_0$	$\Gamma_{(N,0)}(s_2s_1s_2s_1)=s_2s_1s_2s_1$ $\Gamma_{(0,N)}(s_2s_1s_2s_1)=s_2s_1s_2s_1s_2$ $\Gamma_{(N,M)}(s_2s_1s_2s_1)=w_0$
$s_1s_2s_1s_2s_1$	$s_2s_1s_2s_1s_2$	w_0
$\Gamma_{(N,0)}(s_1s_2s_1s_2s_1)=s_1s_2s_1s_2s_1$ $\Gamma_{(\ell,N)}(s_1s_2s_1s_2s_1)=w_0$	$\Gamma_{(0,N)}(s_2s_1s_2s_1s_2)=s_2s_1s_2s_1s_2$ $\Gamma_{(N,\ell)}(s_2s_1s_2s_1s_2)=w_0$	$\Gamma_{(\ell,m)}(w_0)=w_0$

Table 6. The curve neighborhoods for every degree at every $w \in W$.

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rae1313@vt.edu *Department of Physics, Virginia Tech University, 306 Robeson Hall, Blacksburg, VA 24061, United States*

mel3jp@virginia.edu *Department of Mathematics, University of Virginia, 141 Cabell Drive, Kerchof Hall, Charlottesville, VA 22904, United States*

lmihalce@math.vt.edu *Department of Mathematics, Virginia Tech University, 460 McBryde Hall, Blacksburg, VA 24060, United States*

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