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The Jones polynomial for knots and links was a breakthrough discovery in the early 1980s. Since then, it's been generalized in many ways; in particular, by considering knots and links which live in thickened surfaces and by allowing arcs between punctures or marked points on the boundary of the surface. One such generalization was recently introduced by Roger and Yang and has connections with hyperbolic geometry. We provide generators and relations for Roger and Yang's Kauffman bracket arc algebra of the torus with one puncture and the sphere with three or fewer punctures.

Roger and Yang's Kauffman bracket arc algebra is a generalization of the well-known Kauffman bracket skein algebra of a surface, whose definition in [Turaev 1988; Przytycki 1991] is based on Kauffman's skein theoretic description of the Jones polynomial for knots and links [Jones 1985; Kauffman 1987]. Later, the skein algebra of a hyperbolic surface was interpreted as a quantization of the surface's Teichmüller space from hyperbolic geometry [Turaev 1991; Bullock et al. 1999; Przytycki and Sikora 2000]. Interest thus grew for the skein algebra, as a construction important in the Jones polynomial skein theory but also deeply related to Teichmüller theory.

Following this body of work on the skein algebra, Roger and Yang introduced a "skein algebra of arcs" to be a skein theory version of Penner's decorated Teichmüller space. Penner [1987] defined the decorated Teichmüller space as an alternate way to describe the hyperbolic structures of a surface using lengths of both simple closed curves and arcs between punctures on the surface (each decorated with a choice of horoball). Roger and Yang defined their arc algebra as a quantization of Penner's decorated Teichmüller space, roughly in the same way that the skein algebra is a quantization of the usual Teichmüller space. The arc algebra includes both simple closed curves and arcs between punctures on the surface. In addition to the two

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usual bracket skein relations for framed links, there are two extra relations for arcs and loops near the punctures.

Understanding the algebraic structure of Roger and Yang’s arc algebra is an important first step to exploring its role as an intermediary between quantum topology and hyperbolic geometry. Here, we seek finite presentations of the arc algebra for some simple surfaces, namely for spheres with three or fewer punctures and for tori with one or no punctures. In the companion paper [Bobb et al. 2016], we show that the arc algebra is finitely generated. Our work is inspired by analogous statements for the skein algebra found in [Bullock and Przytycki 2000; Bullock 1999].

1. The Kauffman bracket arc algebra

Let $F_{g,n}$ denote a compact, orientable surface of genus g with n points p_1, p_2, \dots, p_n removed. The points removed are the *punctures*. Let A be an indeterminate, with formal square roots $A^{\frac{1}{2}}$ and $A^{-\frac{1}{2}}$. In addition, let there be an indeterminate v_i associated to each puncture p_i , and let $R_n = \mathbb{Z}[A^{\pm\frac{1}{2}}][v_1^{\pm 1}, v_2^{\pm 1}, \dots, v_n^{\pm 1}]$ denote the ring of Laurent polynomials in the commuting variables $A^{\frac{1}{2}}$ and v_1, \dots, v_n .

A framed curve in the thickened surface $F_{g,n} \times [0, 1]$ is the union of framed knots and framed arcs that go from puncture to puncture. (See [Roger and Yang 2014] for a precise definition.) Let $\mathcal{G}(F_{g,n})$ be the R_n -module freely generated by the framed curves in $F_{g,n} \times [0, 1]$, up to isotopy, and let $\mathcal{K}(F_{g,n})$ be the submodule generated by terms of the following four forms:

- (1) skein relation: $\left(\text{crossing} - \left(A \left(\text{cup} \right) + A^{-1} \left(\text{cap} \right) \right) \right)$,
- (2) puncture-skein relation on i -th puncture: $v_i \left(\text{puncture with lines} - \left(A^{\frac{1}{2}} \left(\text{puncture with cup} \right) + A^{-\frac{1}{2}} \left(\text{puncture with cap} \right) \right) \right)$,
- (3) framing relation: $\left(\text{loop} - (-A^2 - A^{-2}) \right)$,
- (4) puncture-framing relation: $\left(\text{puncture loop} - (A + A^{-1}) \right)$,

where the diagrams in each form are assumed to be identical outside of the small balls depicted. Let $\mathcal{A}(F_{g,n})$ denote the quotient $\mathcal{G}(F_{g,n})/\mathcal{K}(F_{g,n})$.

There is a natural stacking operation for framed curves in the thickened surface $F_{g,n} \times [0, 1]$ which extends to $\mathcal{A}(F_{g,n})$. That is, if $[L_1], [L_2] \in \mathcal{A}(F_{g,n})$ are respectively represented by framed curves L_1, L_2 in $F_{g,n} \times [0, 1]$, the product

$$[L_1] * [L_2] = [L'_1 \cup L'_2] \in \mathcal{A}(F_{g,n})$$

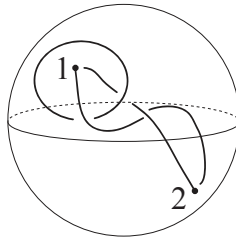


Figure 1. A framed curve with three components on $F_{0,2}$.

is represented by the union of the framed curve $L'_1 \subset F_{g,n} \times [0, \frac{1}{2}]$ (obtained by rescaling L_1 in $F_{g,n} \times [0, 1)$) and the framed curve $L'_2 \subset F_{g,n} \times [\frac{1}{2}, 1]$ (obtained by rescaling L_2 in $F_{g,n} \times [0, 1)$). This stacking operation makes $\mathcal{A}(F_{g,n})$ into an algebra, called the *Kauffman bracket arc algebra* of the surface $F_{g,n}$.

Diagrams in this paper represent projections of framed curves onto $F_{g,n}$ with over- and under-crossing information depicted by breaks in the projection at double-points in the projection, and where the framing of curves is vertical, at right angles to the plane of the paper. Although more than one component of a framed curve can end at any puncture, they must do so at different heights. Diagrams will indicate the order in height of the crossings as necessary. Figure 1 shows a framed curve consisting of three components (two framed arcs and a framed knot) in a sphere with two punctures. No further labeling at punctures is necessary in Figure 1 since arcs intersect each puncture only twice.

Figure 2 shows a product of two framed curves on a twice-punctured torus. The product can be simplified by using a Reidemeister 2 move followed by relation (2) (puncture-skein relation) to “pull off” a pair of strands that meet at a puncture and

$$\begin{aligned}
 & \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right] * \left[\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right] = \left[\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right] = \left[\begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right] \\
 & = v_1^{-1} \left(\begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} \right) \\
 & = v_1^{-1} (A + A^{-1}) \left(\begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} \right)
 \end{aligned}$$

Figure 2. Rewriting a framed curve in $\mathcal{A}(F_{1,2})$.

relation (4) (puncture-framing relation) to “remove” trivial components enclosing a puncture.

The Kauffman bracket skein algebra $\mathcal{S}(F_{g,n})$ defined by Turaev [1988] and Przytycki [1991] is closely related to the arc algebra $\mathcal{A}(F_{g,n})$. Recall that the skein algebra $\mathcal{S}(F_{g,n})$ can be constructed by considering the quotient $\mathcal{G}_0(F_{g,n})/\mathcal{K}_0(F_{g,n})$, where $\mathcal{G}_0(F_{g,n})$ is the $\mathbb{Z}[A, A^{-1}]$ -module generated by the framed links in the thickened surface $F_{g,n} \times [0, 1]$ and $\mathcal{K}_0(F_{g,n})$ is the submodule generated by only relation (1) (the skein relation) and relation (3) (the framing relation) from above. Again, multiplication is induced by the stacking of framed links in the thickened surface. Compared with the skein algebra $\mathcal{S}(F_{g,n})$, the definition of the arc algebra $\mathcal{A}(F_{g,n})$ differs in two ways: in the choice of a larger ring and in the inclusion of two extra relations.

Lemma 1.1. *There exists a well-defined nontrivial algebra homomorphism*

$$\psi : \mathcal{S}(F_{g,n}) \rightarrow \mathcal{A}(F_{g,n})$$

so that $\psi([K]) = [K]$ for a framed link K in $F_{g,n}$.

Proof. Consider the map $i : \mathcal{G}_0(F_{g,n}) \rightarrow R_n \otimes \mathcal{G}_0(F_{g,n})$ with $i(x) = 1 \otimes x$ that changes the scalars, and the map $j : R_n \otimes \mathcal{G}_0(F_{g,n}) \rightarrow \mathcal{G}(F_{g,n})$ with $j(p \otimes x) = p \cdot x$ that includes the framed links into the framed curves. Let $\hat{\psi} = j \circ i$. Notice that $\hat{\psi}(\mathcal{K}_0(F_{g,n})) \subseteq \mathcal{K}(F_{g,n})$. Thus $\hat{\psi} : \mathcal{G}_0(F_{g,n}) \rightarrow \mathcal{G}(F_{g,n})$ descends to a map $\psi : \mathcal{G}_0(F_{g,n})/\mathcal{K}_0(F_{g,n}) \rightarrow \mathcal{G}(F_{g,n})/\mathcal{K}(F_{g,n})$. □

We are interested in the image of ψ . In certain small cases, it generates $\mathcal{A}(F_{g,n})$, a fact we will exploit later on page 697.

2. Generators and relations for the arc algebra

A general strategy for finding generating sets for $\mathcal{A}(F_{g,n})$ is to rewrite framed curves using ones with fewer crossings. We say that a framed knot is a *simple knot* if it allows a projection onto $F_{g,n} \times \{0\}$ without any crossings and it does not bound a disk containing one or no punctures. A framed arc is a *simple arc* if its endpoints are at distinct punctures and it allows a projection without any crossings. A *simple curve* is either a simple knot or a simple arc.

Lemma 2.1. *If a set of elements $\{x_1, x_2, \dots, x_m\}$ generates the simple curves then it generates all of $\mathcal{A}(F_{g,n})$.*

Proof. Suppose we have a basis element $[L] \in \mathcal{A}(F_{g,n})$ represented by a framed curve $L \subset F_{g,n} \times [0, 1]$. By application of the skein relation and the puncture-skein relation, $[L]$ may be written as a linear combination of skeins represented by framed curves each of which has no crossings and intersects a puncture at most once. In particular, the connected components of each framed curve can be isotoped to be at

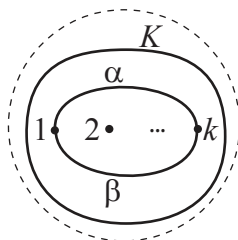


Figure 3. A neighborhood of K in $F_{0,n}$.

different heights, so $[L]$ is a linear combination of products of simple knots, simple arcs, and possibly some loops that bound disks containing one or no punctures. Those latter loops may be removed by application of the framing and puncture relations. Thus $[L]$ is a linear combination of simple curves. Since $\{x_1, x_2, \dots, x_m\}$ generate the simple curves, $[L]$ is also in the set generated by $\{x_1, x_2, \dots, x_m\}$. \square

Remark. Observe that if $(A^2 - 1)$ is invertible, then the puncture-skein relation implies that

$$\text{Diagram} = v_i A^{\frac{1}{2}} (A^2 - 1)^{-1} \left(- \text{Diagram} + A \text{Diagram} \right)$$

and

$$\text{Diagram} = v_i A^{-\frac{1}{2}} (A^{-2} - 1)^{-1} \left(- \text{Diagram} + A^{-1} \text{Diagram} \right).$$

So when $A^2 - 1$ is invertible, if a set of elements generates only the simple arcs, then it generates all of $\mathcal{A}(F_{g,n})$ by Lemma 2.1. However, in the following examples, we will work under the most general set-up, and we will *not* assume that $A^2 - 1$ is invertible.

Arc algebra of punctured spheres. We begin by a refinement of Lemma 2.1 in the case of punctured spheres, $F_{0,n}$ with $n \geq 2$.

Proposition 2.2. *If a set of elements $\{x_1, x_2, \dots, x_n\}$ of the arc algebra $\mathcal{A}(F_{0,n})$ generates the simple arcs, then it generates the entire algebra.*

Proof. By Lemma 2.1, it suffices to show that any simple knot can be rewritten in terms of simple arcs. Given a simple knot $K \subseteq F_{0,n} \times [0, 1]$, notice that it has a projection which separates $F_{0,n}$ into two punctured disks. Let D be the punctured disk bounded by K with the smaller number of punctures, say p_1, \dots, p_k .

Since K is a simple knot, $k \geq 2$. There exist two disjoint simple arcs from p_1 to p_k such that the union of their projections onto $F_{0,n}$ encloses the remaining punctures p_2, \dots, p_{k-1} . Let α and β be the skeins represented by these two arcs, respectively. See Figure 3.

Consider the product $\alpha * \beta \in \mathcal{A}(F_{0,n})$ and apply the puncture-skein relation twice:

$$\begin{aligned} \alpha * \beta &= \text{Diagram of a disk with } k \text{ punctures } 1, 2, \dots, k. \text{ An arc } \alpha \text{ connects } 1 \text{ and } 2, \text{ and an arc } \beta \text{ connects } 1 \text{ and } k. \\ &= \frac{1}{v_1 v_k} \left(A \cdot \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + A^{-1} \cdot \text{Diagram 4} \right). \end{aligned}$$

The diagrams in the sum are:
 1. A disk with punctures 1, 2, ..., k. An arc connects 1 and 2.
 2. A disk with punctures 1, 2, ..., k. An arc connects 1 and k.
 3. A disk with punctures 1, 2, ..., k. An arc connects 1 and 2, and another arc connects 1 and k.
 4. A disk with punctures 1, 2, ..., k. An arc connects 1 and 2.

Thus $[K]$ can be rewritten as a linear combination involving the product of two simple arcs (α and β) and three knots bounding disks with strictly fewer punctures. Notice also that a knot bounding a disk with one or no punctures can be removed using the puncture relation or the framing relation, respectively. Thus by induction, we are done. □

Sphere with two punctures.

Theorem 2.3. $\mathcal{A}(F_{0,2}) = R_2\langle \alpha \mid \alpha^2 = -v_1^{-1}v_2^{-1}(A - A^{-1})^2 \rangle$, where α is represented by a simple arc between the two punctures of $F_{0,2}$.

Proof. On $F_{0,2}$, any simple arc must start at one puncture and end at the other without intersecting itself. Up to isotopy, there is only one such arc, and let α be the skein represented by that arc. By Proposition 2.2, α generates the algebra. Note that in the arc algebra,

$$\begin{aligned} \alpha^2 &= \text{Diagram of a disk with two punctures } 1, 2. \text{ Two arcs connect } 1 \text{ and } 2. \\ &= \frac{1}{v_1 v_2} \left(A \cdot \text{Diagram 1} + 2 \cdot \text{Diagram 2} + A^{-1} \cdot \text{Diagram 3} \right) \\ &= v_1^{-1}v_2^{-1}((A + A^{-1})(A + A^{-1}) + 2(-A^2 - A^{-2})) \\ &= -v_1^{-1}v_2^{-1}(A - A^{-1})^2. \end{aligned}$$

The diagrams in the sum are:
 1. A disk with punctures 1, 2. An arc connects 1 and 2.
 2. A disk with punctures 1, 2. Two arcs connect 1 and 2.
 3. A disk with punctures 1, 2. An arc connects 1 and 2.

In particular, this shows that α^2 is not linearly independent from 1 and α . As α is the only generator, this is the only relation of $\mathcal{A}(F_{0,2})$. □

Sphere with three punctures. To determine the arc algebra $\mathcal{A}(F_{0,3})$, we will need the following lemma from algebra.

Lemma 2.4. *Let A and B be R -algebras. Suppose x_1, x_2, \dots, x_n are elements of the algebra A and ρ is some algebra homomorphism of A to the algebra B . If the elements $\rho(x_1), \rho(x_2), \dots, \rho(x_n)$ are linearly independent in B , then x_1, x_2, \dots, x_n are linearly independent in A .*

Proof. We prove the contrapositive. Suppose x_1, x_2, \dots, x_n are linearly dependent in A . Then there exist coefficients $k_1, k_2, \dots, k_n \in R$ so that

$$k_1x_1 + k_2x_2 + \dots + k_nx_n = 0$$

and at least one k_i is nonzero. Since ρ is an R -algebra homomorphism,

$$k_1\rho(x_1) + k_2\rho(x_2) + \dots + k_n\rho(x_n) = 0.$$

So $\rho(x_1), \rho(x_2), \dots, \rho(x_n)$ are linearly dependent in B as well. □

Theorem 2.5.

$$\mathcal{A}(F_{0,3}) = R_3\langle \alpha_1, \alpha_2, \alpha_3 \mid \alpha_i\alpha_{i+1} = \alpha_{i+1}\alpha_i = v_{i+2}^{-1}\delta\alpha_{i+2}, v_{i+1}v_{i+2}\alpha_i^2 = \delta^2 \rangle,$$

where α_i is represented by the simple arc connecting the punctures p_{i+1} to p_{i+2} in the thrice-punctured sphere $F_{0,3}$, with $i = 1, 2, 3$ and indices interpreted modulo 3, and where $\delta = (A^{\frac{1}{2}} + A^{-\frac{1}{2}})$.

Proof. Since the only simple arcs in $F_{0,3}$ are those connecting distinct punctures, it follows that $\alpha_1, \alpha_2,$ and α_3 generate the arc algebra $K[F_{0,3}]$. Observe that

$$\begin{aligned} \alpha_i^2 &= \text{Diagram: a dashed circle with punctures } i, i+1, i+2 \text{ and an arc from } i+1 \text{ to } i+2. \\ &= v_{i+1}^{-1}v_{i+2}^{-1} \left(A \text{Diagram: } i \text{ to } i+1 \text{ arc} + \text{Diagram: } i \text{ to } i+2 \text{ arc} + \text{Diagram: } i \text{ to } i \text{ loop} + A^{-1} \text{Diagram: } i+1 \text{ to } i+2 \text{ arc} \right) \\ &= v_{i+1}^{-1}v_{i+2}^{-1} (A(A + A^{-1}) + (-A^2 - A^{-2}) + (A + A^{-1}) + A^{-1}(A + A^{-1})) \\ &= v_{i+1}^{-1}v_{i+2}^{-1} (A^{\frac{1}{2}} + A^{-\frac{1}{2}})^2 \\ &= v_{i+1}^{-1}v_{i+2}^{-1} \delta^2. \end{aligned}$$

Also,

$$\begin{aligned} \alpha_i * \alpha_{i+1} &= \text{Diagram: } i \text{ to } i+1 \text{ arc} = v_{i+2}^{-1} \left(A^{\frac{1}{2}} \text{Diagram: } i \text{ to } i+1 \text{ arc} + A^{-\frac{1}{2}} \text{Diagram: } i \text{ to } i+2 \text{ arc} \right) \\ &= v_{i+2}^{-1} (A^{\frac{1}{2}} + A^{-\frac{1}{2}}) \alpha_{i+2} \\ &= v_{i+2}^{-1} \delta \alpha_{i+2} \end{aligned}$$

and similarly

$$\begin{aligned} \alpha_{i+1} * \alpha_i &= \text{diagram} = v_{i+2}^{-1} \left(A^{\frac{1}{2}} \text{diagram}_1 + A^{-\frac{1}{2}} \text{diagram}_2 \right) \\ &= v_{i+2}^{-1} (A^{\frac{1}{2}} + A^{-\frac{1}{2}}) \alpha_{i+2} \\ &= v_{i+2}^{-1} \delta \alpha_{i+2}. \end{aligned}$$

We next show that these are the only relations. Notice that the relations above imply that any product $\alpha_i * \alpha_j$ can be rewritten as either a scalar multiple when $i = j$ or as a multiple of the remaining α_k for $k \neq i, j$. Thus any word in $\alpha_1, \alpha_2,$ and α_3 can be rewritten in terms of a scalar multiple of one or zero generators. So any other relation among the generators $\alpha_1, \alpha_2,$ and α_3 can be expressed in the form

$$k_0 + k_1 \alpha_1 + k_2 \alpha_2 + k_3 \alpha_3 = 0,$$

where $k_i \in \mathbb{R}_3$. We will show that $1, \alpha_1, \alpha_2, \alpha_3$ are linearly independent, so that the $k_i = 0$.

Recall that a left regular representation of a group is the linear representation provided by multiplication of group elements on the left. Based on the similarity of the algebra elements $1, \alpha_1, \alpha_2, \alpha_3$ from $\mathcal{A}(F_{0,3})$ with the group elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$, we define a left regular representation ρ for $\mathcal{A}(F_{0,3})$ by

$$\begin{aligned} \rho(1) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, & \rho(\alpha_1) &= \begin{bmatrix} 0 & v_2^{-1} v_3^{-1} \delta^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2^{-1} \delta \\ 0 & 0 & v_3^{-1} \delta & 0 \end{bmatrix}, \\ \rho(\alpha_2) &= \begin{bmatrix} 0 & 0 & v_1^{-1} v_3^{-1} \delta^2 & 0 \\ 0 & 0 & 0 & v_1^{-1} \delta \\ 1 & 0 & 0 & 0 \\ 0 & v_3^{-1} \delta & 0 & 0 \end{bmatrix}, & \rho(\alpha_3) &= \begin{bmatrix} 0 & 0 & 0 & v_1^{-1} v_2^{-1} \delta^2 \\ 0 & 0 & v_1^{-1} \delta & 0 \\ 0 & v_2^{-1} \delta & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Note that the coefficients from each column are exactly those given by the equations describing left multiplication by α_i . In particular, they are the coefficients in the equations $\alpha_i * 1 = \alpha_i, \alpha_i * \alpha_i = v_{i+1}^{-1} v_{i+2}^{-1} \delta^2,$ and $\alpha_i * \alpha_{i+1} = \alpha_{i+1} * \alpha_i = v_{i+2}^{-1} \delta \alpha_{i+2}$ for all i . The matrices $\rho(1), \rho(\alpha_1), \rho(\alpha_2)$ and $\rho(\alpha_3)$ are clearly linearly independent, as can be determined by looking at their first columns. Thus by Lemma 2.4, we have that $1, \alpha_1, \alpha_2, \alpha_3$ are linearly independent. Hence there are no more relations to be found in $K(F_{0,3})$. This also shows that $\{\alpha_1, \alpha_2, \alpha_3\}$ is a minimal set of generators. \square

Surfaces with zero or one punctures. Recall that $R = \mathbb{Z}[A, A^{-1}]$ is a subring of $R_n = \mathbb{Z}[A^{\frac{1}{2}}, A^{-\frac{1}{2}}][v_1^{\pm 1}, v_2^{\pm 1}, \dots, v_n^{\pm 1}]$, and from Lemma 1.1, there exists an algebra homomorphism ψ , which maps the R -algebra $\mathcal{S}(F_{g,n})$ to the R_n -algebra $\mathcal{A}(F_{g,n})$.

First observe that when $n = 0$, the relations in $\mathcal{S}(F_{g,0})$ are exactly those in $\mathcal{A}(F_{g,0})$. That is, $R_0 \otimes \mathcal{K}_0(F_{g,0}) \cong \mathcal{K}(F_{g,0})$. Moreover, the map ψ from the proof of Lemma 1.1 is injective when $n = 0$ and acts as the identity on simple knots. Since all simple curves are simple knots in this case and the image of ψ contains all simple knots, the image of ψ generates all of the arc algebra $\mathcal{A}(F_{g,0})$ by Lemma 2.1. Thus $R_0 \otimes \mathcal{S}(F_{g,0}) \cong \mathcal{A}(F_{g,0})$, and any presentation of $\mathcal{S}(F_{g,0})$ provides a presentation of $\mathcal{A}(F_{g,0})$.

When $n = 1$, again there are no simple arcs, so that the image of ψ generates all of the arc algebra $\mathcal{A}(F_{g,1})$. So any set generating $\mathcal{S}(F_{g,1})$ also generates $\mathcal{A}(F_{g,1})$. However, the map ψ is no longer injective. Specifically, the relations for the Kauffman skein algebra and the relations for the Kauffman arc algebra will differ; the puncture-framing relation from the Kauffman arc algebra is not a relation in the Kauffman skein algebra. However, this is the only difference. Hence $R_1 \otimes \mathcal{S}(F_{g,1}) / \mathcal{K}_{\text{pfr}}(F_{g,1}) \cong \mathcal{A}(F_{g,1})$, where $\mathcal{K}_{\text{pfr}}(F_{g,1})$ is the submodule generated by only the puncture-framing relation. In summary, the generators of $\mathcal{A}(F_{g,1})$ are generators of $\mathcal{S}(F_{g,1})$, but the relations of $\mathcal{A}(F_{g,1})$ are relations of $\mathcal{S}(F_{g,1})$ along with one corresponding to the puncture-framing relation.

Torus with zero or one punctures. As an example, let us examine the cases of the closed torus and the torus with one puncture. From [Bullock and Przytycki 2000], we have that the Kauffman skein algebras $\mathcal{S}(F_{1,0})$ and $\mathcal{S}(F_{1,1})$ are both generated as $\mathbb{Z}[A, A^{-1}]$ -modules by three simple closed curves $\gamma_1, \gamma_2, \gamma_3$ such that γ_1 and γ_2 intersect once and γ_3 is one of two curves that meet both γ_1 and γ_2 once. Moreover, if ∂ represents a small loop around the puncture of $F_{1,1}$, then

$$\partial = A\gamma_1\gamma_2\gamma_3 - A^2\gamma_1^2 - A^{-2}\gamma_2^2 - A^2\gamma_3^2 + A^2 + A^{-2}. \tag{1}$$

In the skein algebra $\mathcal{S}(F_{1,0})$, we have $\partial = -A^2 - A^{-2}$. Up to a change in scalars from R to R_0 , a presentation of the arc algebra $\mathcal{A}(F_{1,0})$ is the same as the presentation of the skein algebra $\mathcal{S}(F_{1,0})$. That is,

$$\begin{aligned} \mathcal{A}(F_{1,0}) = R_0 \langle \gamma_1, \gamma_2, \gamma_3 \mid A\gamma_i\gamma_{i+1} - A^{-1}\gamma_{i+1}\gamma_i = (A^2 - A^{-2})\gamma_{i+1} \text{ and} \\ -A^2 - A^{-2} = A\gamma_1\gamma_2\gamma_3 - A^2\gamma_1^2 - A^{-2}\gamma_2^2 - A^2\gamma_3^2 + A^2 + A^{-2} \rangle, \end{aligned}$$

where the indices are interpreted modulo 3. On the other hand, in the once-punctured torus, we have $\partial = A + A^{-1}$ in the arc algebra. Thus

$$\begin{aligned} \mathcal{A}(F_{1,1}) = R_1 \langle \gamma_1, \gamma_2, \gamma_3 \mid A\gamma_i\gamma_{i+1} - A^{-1}\gamma_{i+1}\gamma_i = (A^2 - A^{-2})\gamma_{i+1} \text{ and} \\ A + A^{-1} = A\gamma_1\gamma_2\gamma_3 - A^2\gamma_1^2 - A^{-2}\gamma_2^2 - A^2\gamma_3^2 + A^2 + A^{-2} \rangle. \end{aligned}$$

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
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