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An iterative strategy for Lights Out on Petersen graphs

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We establish some preliminary results for Sutner's σ^+ game, known as Lights Out, played on the generalized Petersen graph $P(n, k)$. While all regular Petersen graphs admit game configurations that are not solvable, we prove that every game on the $P(2n, n)$ graph has a unique solution. Moreover, we introduce a simple iterative strategy for finding the solution to any game on $P(2n, n)$, and generalize its application to a wider class of graphs.

Background

All graphs are assumed to be undirected, without loops or multiple edges. The σ^+ game is a well-known single-player game that can be played on any graph [Sutner 1990]. The handheld game *Lights Out*, released by Tiger Electronics in 1993, was the σ^+ game on a 5×5 grid graph. Since then, the name Lights Out has been widely used and is synonymous with the σ^+ game.

The idea of the game is simple. Each vertex is in one of two states: on or off (think of the vertices as lights). Each vertex also acts as a button. When the player pushes a button, its state toggles, and so do the states of each of its neighbors. Given a graph and an initial configuration of lit vertices, the goal is to turn out all the lights. Several playable versions of the game can be found online [Antonick 2013; Scherphuis 2012; Torrence 2016].

Lights Out has been extensively studied on grid graphs, and a wide range of generalizations have been explored. Our goal here is to present some preliminary results for Lights Out played on the generalized Petersen graphs. In the process, we introduce an iterative strategy that can successfully solve games on a larger class of graphs.

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Definitions

Given a graph G with finite vertex set $V(G)$ and edge set $E(G)$, label the vertices $v_1, v_2, \ldots, v_n$. The *adjacency matrix* $\mathrm{Adj}(G)$ is the $n \times n$ symmetric matrix with a one in position (i, j) if $v_i v_j \in E(G)$, and a zero otherwise. We let I_n denote the $n \times n$ identity matrix.

A *lights out configuration* or *game* $\boldsymbol{g}$ is an $n \times 1$ column vector with entries in the two-element field $\mathbb{F}_2$. If there is a one in position i, we say that v_i is *lit*; if there is a zero we say it is *off*. We let $\mathbf{0}$ and $\mathbf{1}$ denote the all-off and all-on configurations, respectively.

Similarly, a *lights out strategy* $\boldsymbol{s}$ is an $n \times 1$ column vector with entries in $\mathbb{F}_2$. It denotes a collection of buttons to be pushed; if there is a one in position i, we say that vertex v_i is *pushed*. The vector $\mathbf{1}$ represents the strategy where every button is pushed.

If one begins with the all-off configuration and invokes strategy $\boldsymbol{s}$, the resulting configuration is the matrix product $(\mathrm{Adj}(G) + I_n)\boldsymbol{s}$, with arithmetic carried out modulo 2 [Sutner 1990]. Since the matrix $\mathrm{Adj}(G) + I_n$ is so important, we call it the *transition matrix* for the σ^+ game on G, and denote it $A(G)$, or simply A if the underlying graph is understood.

In general, if one begins with configuration $\boldsymbol{g}$ and invokes strategy $\boldsymbol{s}$, the configuration that results is $A\boldsymbol{s} + \boldsymbol{g}$, with arithmetic carried out modulo 2. We say that strategy $\boldsymbol{s}$ *solves* game $\boldsymbol{g}$ if $A\boldsymbol{s} + \boldsymbol{g} = \mathbf{0}$, the all-off configuration. Since we are working modulo 2, this is equivalent to saying $A\boldsymbol{s} = \boldsymbol{g}$. We say configuration $\boldsymbol{g}$ is *solvable* if there exists a strategy that solves it.

For any vector $\boldsymbol{g} \in (\mathbb{F}_2)^n$, the *light number* of $\boldsymbol{g}$ is the sum of the entries in $\boldsymbol{g}$. The *parity of* $\boldsymbol{g}$ is odd or even according to the parity of the light number of $\boldsymbol{g}$.

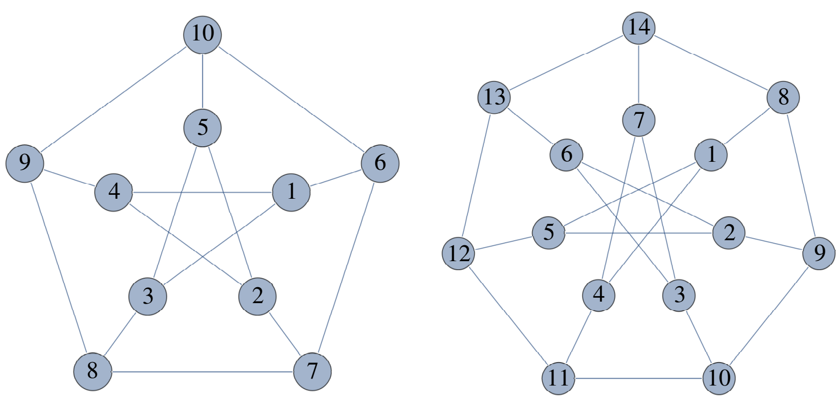

Figure 1. The Petersen graphs $P(5, 2)$ and $P(7, 3)$.

Let n and k be integers with $n \geq 3$ and $1 \leq k < n$. The *generalized Petersen graph* $P(n, k)$ is a graph with $2n$ vertices arranged in two concentric "rings" with n vertices in each ring. We label the vertices on the inner (or lower) ring v_1 to v_n, and on the outer (or upper) ring v_{n+1} to v_{2n}. Each outer-ring vertex v_i is connected to the inner-ring vertex v_{i-n}, and to its two nearest outer-ring neighbors (v_{i-1} and v_{i+1} for $n+1 < i < 2n$, while v_{2n} is connected to v_{2n-1} and v_{n+1}). Each inner-ring vertex v_i is connected to v_{i+k}, where the index $i+k$ is reduced modulo n if $i+k > n$. In other words, vertices on the inner ring are connected to one another by "skipping" k vertices. The classic Petersen graph $P(5, 2)$ is shown on the left in Figure 1, with $P(7, 3)$ beside it.

For a given value of n, one need only consider $k = 1$ through $k = \lfloor n/2 \rfloor$, since the graphs $P(n, k)$ and $P(n, n-k)$ are isomorphic. (The first skips k vertices clockwise, the second skips k vertices counterclockwise.) Also, $P(2n, n)$ is not regular, as the inner vertices have valence 2 (see Figure 2). All other Petersen graphs are 3-regular.

An oscillating strategy

The Petersen graphs have an important property: if one refers to the vertices $\{v_1, \ldots, v_n\}$ as the lower vertices, and $\{v_{n+1}, \ldots, v_{2n}\}$ as the upper vertices, then for each i with $1 \leq i \leq n$, there is an edge $v_i v_{n+i}$, and these are the only edges between a lower vertex and an upper vertex. In other words, the adjacency matrix $\mathrm{Adj}(P(n, k))$ has the block form

$$\begin{pmatrix} C' & I_n \\ I_n & D' \end{pmatrix},$$

where C' is the adjacency matrix for the subgraph induced by the lower vertices, D' is the adjacency matrix for the subgraph induced by the upper vertices, and where the two identity matrices specify the edges $v_i v_{n+i}$ between upper and lower vertices.

We now introduce an iterative "oscillating" lights out strategy that can be applied to any graph whose adjacency matrix has this structure. That is, in this section we suppose that we are given a graph G with $2n$ vertices such that when the vertices of G are appropriately ordered its adjacency matrix has the block form above, with I_n in the lower-left and upper-right corners (and where C' and D' can be any symmetric matrices over $\mathbb{F}_2$ with zeros on their respective main diagonals). We call such a graph *Petersen-like*.

The strategy works as follows: Suppose that $\boldsymbol{g}$ is a game on the Petersen-like graph G. For each lower vertex v_k that is lit, push button v_{n+k}. This will have the effect of turning off all lower vertices. Then, for each upper vertex v_{n+k} that is now lit, push button v_k. This will have the effect of turning off all upper vertices (and possibly lighting or relighting some lower ones). Together, we call these two operations performed in succession "one oscillation". One then repeats the process: After the first oscillation, if any lower vertices are lit, push the corresponding upper vertices. Then if any upper vertices are still lit, push the corresponding lower vertices, and so on.

The strategy can be expressed explicitly in matrix form. Let U and L be the $2n \times 2n$ matrices, defined in block form as

$$U = \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix},$$

where 0 denotes the $n \times n$ zero matrix. Then for any configuration $\boldsymbol{g}$, the upper vertices that correspond to lit lower vertices are $U\boldsymbol{g}$, so the configuration that results after pushing the upper vertices corresponding to the lit lower vertices is

$$\boldsymbol{g}_1 = AU\boldsymbol{g} + \boldsymbol{g} = (AU + I)\boldsymbol{g}.$$

And pushing the lower vertices corresponding to lit upper vertices in $\boldsymbol{g}_1$ yields the configuration

$$\boldsymbol{g}_2 = AL\boldsymbol{g}_1 + \boldsymbol{g}_1 = (AL + I)\boldsymbol{g}_1 = (AL + I)(AU + I)\boldsymbol{g}.$$

It follows that after performing k successive upper-lower iterations, the final configuration is

$$[(AL + I)(AU + I)]^k \boldsymbol{g}.$$

We call the matrix $[(AL + I)(AU + I)]$ the *oscillating matrix*.

Noting that the lights out transition matrix A has the form

$$A = \begin{pmatrix} C & I \\ I & D \end{pmatrix},$$

where $C = C' + I$ and $D = D' + I$, it is simple to calculate the oscillating matrix:

$$(AL + I) = \begin{pmatrix} C & I \\ I & D \end{pmatrix} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & C \\ 0 & I \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & C \\ 0 & 0 \end{pmatrix}.$$

Similarly,

$$(AU + I) = \begin{pmatrix} C & I \\ I & D \end{pmatrix} \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ D & 0 \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ D & I \end{pmatrix}.$$

The product is

$$(AL + I)(AU + I) = \begin{pmatrix} I & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ D & I \end{pmatrix} = \begin{pmatrix} CD & C \\ 0 & 0 \end{pmatrix}.$$

Theorem 1. *Suppose G is a graph with $2n$ vertices whose lights out transition matrix has the form*

$$A = \begin{pmatrix} C & I_n \\ I_n & D \end{pmatrix}.$$

If the product CD is nilpotent, then A is invertible, and the oscillating strategy will solve any initial configuration on G. Specifically, if $(CD)^m$ is the zero matrix, then at most $m+1$ oscillations are required.

Proof. A straightforward inductive argument shows that for $k \geq 1$,

$$[(AL+I)(AU+I)]^k = \begin{pmatrix} (CD)^k & (CD)^{k-1}C \\ 0 & 0 \end{pmatrix}.$$

So if $(CD)^m = 0$ and if $\boldsymbol{g}$ is any initial configuration, $[(AL+I)(AU+I)]^{m+1}\boldsymbol{g} = \mathbf{0}$, and we see that $m+1$ iterations will suffice to solve $\boldsymbol{g}$.

It remains to show that A is invertible. Since we have shown that the oscillating strategy will solve any configuration $\boldsymbol{g}$, and since there are 2^{2n} possible configurations and the same number of possible strategies, we see that for each game there is precisely one strategy that solves it. So $A\boldsymbol{s} = \boldsymbol{g}$ has a unique solution $\boldsymbol{s}$ for each game $\boldsymbol{g}$. This means that A is invertible. $\square$

Examples

In each example, G is a Petersen-like graph with $2n$ vertices and with lights out transition matrix $A = \left(\begin{smallmatrix} C & I_n \\ I_n & D \end{smallmatrix}\right)$.

Example 2. Suppose n is even, $C = I_n$, and D is the all-ones matrix. Since n is even and we are working over $\mathbb{F}_2$, one has $(CD)^2 = D^2 = 0$ (the zero matrix). In graph theoretic terms, the lower vertices have no edges among them, while the upper vertices induce the complete graph K_n. According to Theorem 1, at most three iterations of the oscillating strategy will solve any game.

Note that playing Lights Out on K_n itself is quite boring. Pushing any single vertex changes the state of *every* vertex. Hence the only solvable games are $\mathbf{0}$ and $\mathbf{1}$. It is interesting that adding a "dead end" edge to each vertex of K_n transforms it into a graph where every game is solvable, and where the oscillating strategy can solve any game.

Example 3. Suppose m is a natural number and $n = 2m$. Let

$$C = \begin{pmatrix} I_m & I_m \\ I_m & I_m \end{pmatrix}, \quad D = \begin{pmatrix} T & S \\ S & T \end{pmatrix},$$

where S and T may be any $m \times m$ symmetric matrices over $\mathbb{F}_2$ such that T has ones along its main diagonal. Then in block form, each of the four blocks of CD

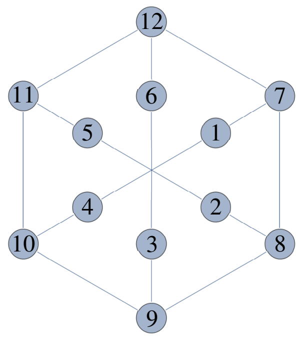

Figure 2. The Petersen graph $P(6, 3)$.

is $S+T$, and since we are working over $\mathbb{F}_2$, we know $(CD)^2$ is the zero matrix. According to Theorem 1, at most three iterations suffice to solve any game.

If $T = I_m$ and $S = 0$, then G is isomorphic to m copies of the path graph P_4. In this setting, the upper vertices have valence 1, and the lower vertices have valence 2.

If T is the $m \times m$ tridiagonal matrix, and S is the $m \times m$ matrix with ones in the upper-right and lower-left corners only, then G is the Petersen graph $P(2m, m)$. See Figure 2.

Corollary 4. *For every natural number m, the transition matrix for the Petersen graph $P(2m, m)$ is invertible, and at most three iterations of the oscillating strategy suffice to solve any game.* □

Example 5. Suppose n is even. Let $C = (c_{i,j})$ be the $n \times n$ matrix where $c_{1,1} = 1$, all other entries in the first row and column are zero, and if $i > 1$ and $j > 1$, then $c_{i,j} = 1$. In graph theoretic terms, the lower vertices induce the graph that is the union of the isolated vertex v_1 with the complete graph on $v_2, \ldots, v_n$. Let D be the $n \times n$ matrix with ones in the first row and column, ones on the main diagonal, and zeros elsewhere. In graph theoretic terms, the upper vertices induce the star graph $K_{1,n-1}$ with hub at v_{n+1}. It is a simple matter to check that $C = C^2$, so C is not nilpotent. Also, $D^n = D^2$ is the matrix obtained from D by swapping all ones with zeros and zeros with ones, so D is not nilpotent. Yet one readily verifies that $(CD)^2$ is the zero matrix, so Theorem 1 applies: every game on G is solvable, and at most three iterations of the oscillating strategy suffice to solve any game.

Example 6. Suppose n is even. Let $C = (c_{i,j})$ be the $n \times n$ matrix where $c_{1,1} = c_{2,2} = 1$, all other entries in the first two rows and columns are zero, and for either $i > 2$ or $j > 2$, we have $c_{i,j} = 1$. In graph theoretic terms, the lower vertices induce the graph that is the union of isolated vertices v_1 and v_2 with the complete graph on $v_3, \ldots, v_n$. Let D be the $n \times n$ matrix with ones in the first row and column, ones on the main diagonal, and zeros elsewhere (exactly the same as it was in

Example 5). In graph theoretic terms, the upper vertices induce the star graph $K_{1,n-1}$ with hub at v_{n+1}. It is a simple matter to check that $C^n = C^2$ has precisely two ones, in positions $(1, 1)$ and $(2, 2)$, so C is not nilpotent. Also, $D^n = D^2$ is the matrix obtained from D by swapping all ones with zeros and zeros with ones, so D is not nilpotent. Yet one readily verifies that $(CD)^3$ is the zero matrix, so Theorem 1 applies: every game on G is solvable, and at most four iterations of the oscillating strategy suffice to solve any game.

These examples should make clear that Theorem 1 is widely applicable. In particular, the last two examples show that for every even n there are multiple Petersen-like graphs for which neither C nor D is nilpotent, but CD is.

A restriction

In each example from the previous section, G is a Petersen-like graph where Theorem 1 applies: G has σ^+ transition matrix $A = \left(\begin{smallmatrix} C & I_n \\ I_n & D \end{smallmatrix}\right)$, and the product CD is nilpotent. Observe that in each example, C and D are $n \times n$ matrices where n is even. This is no accident.

Theorem 7. *Suppose C and D are symmetric $n \times n$ matrices over a field of characteristic* 2, *with ones along their main diagonals. If CD is nilpotent*, *then n is even.*

Before proving this, we present a lemma:

Lemma 8. *Suppose $C = (c_{i,j})$ and $D = (d_{i,j})$ are symmetric $n \times n$ matrices over a field of characteristic* 2, *with ones along their main diagonals. Then the trace* $\operatorname{tr}(CD) = 0$ *if and only if n is even.*

Proof of Lemma 8. The k-th diagonal entry of CD is

$$c_{k,1}d_{1,k} + c_{k,2}d_{2,k} + \cdots + c_{k,n}d_{n,k} = c_{1,k}d_{1,k} + c_{2,k}d_{2,k} + \cdots + c_{n,k}d_{n,k}$$

since C is symmetric. The expression on the right is the sum of the entries in the k-th column of the Hadamard product $C \circ D$. Summing over all columns k, we see that $\operatorname{tr}(CD)$ is the sum of *all* entries of $C \circ D$. But $C \circ D$ is symmetric, since C and D are, and has ones on its main diagonal since each of C and D do. Therefore, the nondiagonal entries appear in pairs (and so cancel modulo 2), and the sum of the diagonal entries is n. So over a field of characteristic 2, we have $\operatorname{tr}(CD) \equiv n \pmod 2$. □

Proof of Theorem 7. We prove the contrapositive. Suppose n is odd. The characteristic polynomial of an $n \times n$ matrix M over a field of characteristic 2 has $\operatorname{tr}(M) \pmod 2$ as the coefficient to the term with power $n - 1$. Since n is odd, the lemma says that $\operatorname{tr}(CD)$ is nonzero. Therefore the characteristic polynomial of CD has a nonzero coefficient for the term with power $n - 1$. This means that CD has a nonzero eigenvalue, and therefore cannot be nilpotent. □

Results for Petersen graphs

The transition matrix $A(G)$ governs the behavior of the σ^+ game on G, and its nullity is particularly important. If G has n vertices, there are 2^n configurations on G. If $A(G)$ has nullity k, then there are 2^k nullspace vectors in $(\mathbb{F}_2)^n$. This means there are 2^{n-k} solvable games on G, and every solvable game has 2^k distinct strategies that solve it: if strategy $\boldsymbol{s}$ solves game $\boldsymbol{g}$, so does $\boldsymbol{s}+\boldsymbol{n} \in (\mathbb{F}_2)^n$ for each nullspace vector $\boldsymbol{n}$. If $A(G)$ is nonsingular, then every game is solvable and has a unique solution.

Our first goal is to determine which Petersen graphs have nonsingular transition matrices.

Lemma 9. *If G is a graph where every vertex is odd-valent, then the σ^+ transition matrix $A(G)$ is singular. In particular, the all-on strategy $\mathbf{1}$ is in the nullspace of $A(G)$.*

Proof. Suppose every vertex in G is odd-valent. Consider the strategy $\mathbf{1}$, where every button is pushed once. Let v be a button. Since v gets pushed, it changes state once on that account. But v has odd valence, so in addition it will change state an odd number of times (once for each button adjacent to it). So ultimately v changes state an even number of times, and hence is left unchanged. Therefore $\mathbf{1}$ is in the nullspace of $A(G)$, so $A(G)$ must be singular. □

Corollary 10. *If G is a graph where every vertex is odd-valent, and $\boldsymbol{s}$ is a strategy that solves the σ^+ game $\boldsymbol{g}$, then the complementary strategy $\mathbf{1}-\boldsymbol{s}$ also solves game $\boldsymbol{g}$.*

Proof. Suppose $\boldsymbol{s}$ solves game $\boldsymbol{g}$. This means $A\boldsymbol{s}=\boldsymbol{g}$, where $A=A(G)$ is the transition matrix for G. Since we are working modulo 2, we know $\mathbf{1}-\boldsymbol{s}=\mathbf{1}+\boldsymbol{s}$, and we have

$$A(\mathbf{1}-\boldsymbol{s}) = A\mathbf{1}+A\boldsymbol{s} = A\boldsymbol{s} = \boldsymbol{g},$$

since $A\mathbf{1}=\mathbf{0}$ by Lemma 9. So strategy $\mathbf{1}-\boldsymbol{s}$ also solves game $\boldsymbol{g}$. □

We now focus on Petersen graphs. Corollary 4 tells us that the transition matrix for $P(2n,n)$ is invertible, and that the oscillating strategy can be used to solve any game on this graph. Note that for graphs of this type, the lower vertices have valence 2, while the upper vertices have valence 3.

Theorem 11. *The generalized Petersen graph $P(n,k)$ has nonsingular σ^+ transition matrix if and only if $n=2k$.*

Proof. If $n\neq 2k$, then $P(n,k)$ is 3-regular, so by Lemma 9 its transition matrix is singular. If $n=2k$, then Corollary 4 guarantees that the transition matrix is nonsingular. □

While we have shown that the nullity of A is strictly positive for the regular Petersen graphs, determining its precise value is a subtle business (see Table 1). Both Sutner [1988], and Anderson and Feil [1998] pointed out a similar situation for grid graphs, and much work was done subsequently to make sense of it [Barua and Ramakrishnan 1996; Goldwasser et al. 1997; Sutner 2000].

Lemma 12. *Let* $\boldsymbol{g}$ *and* $\boldsymbol{h}$ *be vectors in* $(\mathbb{F}_2)^n$. *Then* $\boldsymbol{g}+\boldsymbol{h} \in (\mathbb{F}_2)^n$ *has even parity if and only if* $\boldsymbol{g}$ *and* $\boldsymbol{h}$ *have the same parity.*

Proof. Let m and n be the light numbers of $\boldsymbol{g}$ and $\boldsymbol{h}$. Let k be the number of coordinate positions where $\boldsymbol{g}$ and $\boldsymbol{h}$ both have the value 1. Then the light number of $\boldsymbol{g}+\boldsymbol{h}$ is

$$(m-k)+(n-k) = m+n-2k \equiv m+n \pmod 2.$$

But $m+n$ is even if and only if m and n have the same parity. □

Lemma 13. *If* G *is a graph where every vertex is odd-valent, and* $\boldsymbol{g}$ *is a solvable game, then* $\boldsymbol{g}$ *is even.*

Proof. Let $A = A(G)$ be the transition matrix for G. Observe that A has an even number of ones in each column, since $A = \mathrm{Adj}(G) + I$, and the adjacency matrix for G has an odd number of ones in each column and zeros on the diagonal. Now for any strategy $\boldsymbol{s}$, the vector $A\boldsymbol{s}$ is the simply the sum of those columns of A where $\boldsymbol{s}$ has a one. It follows from Lemma 12 that any such sum has even parity. So if $A\boldsymbol{s} = \boldsymbol{g}$, it must be the case that $\boldsymbol{g}$ has even parity. □

Corollary 14. *If* G *is a graph where every vertex is odd-valent, and the nullity of* $A(G)$ *is* 1, *then the solvable games are precisely the games with an even number of lights lit. Every solvable game has precisely two solutions, and they are complements of each other.*

Proof. On any graph, precisely half of all possible games have an even light number. If the nullity of the transition matrix for any graph is 1, precisely half of all games are solvable (and every solvable game has two solutions). But Lemma 13 says that on an odd-valent graph, every solvable game has an even light number, hence these two sets (solvable games and games with an even light number) must agree. If strategy $\boldsymbol{s}$ solves game $\boldsymbol{g}$, Corollary 10 guarantees the complement $\mathbf{1}-\boldsymbol{s}$ is the other solution. □

Table 1 shows that among the various Petersen graphs, there are many examples satisfying the assumptions of Corollary 14 (e.g., $P(n, 1)$ when n is odd). Other questions and conjectures are suggested by the table. For example, it seems natural to conjecture that the parity of n matches the parity of the nullity of $A(P(n, k))$ for all k.

$n \downarrow \quad k \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	1																			
4	4	0																		
5	1	5																		
6	2	2	0																	
7	1	1	1																	
8	4	2	4	0																
9	1	1	1	1																
10	2	6	6	2	0															
11	1	1	1	1	1															
12	4	2	4	2	4	0														
13	1	1	1	1	1	1														
14	2	2	2	2	2	2	0													
15	1	5	5	9	1	1	5													
16	4	2	4	2	4	2	4	0												
17	1	1	1	1	9	1	9	1												
18	2	2	2	2	2	2	2	2	0											
19	1	1	1	1	1	1	1	1	1											
20	4	6	8	2	4	2	8	6	4	0										
21	1	1	1	1	1	1	1	1	1	1										
22	2	2	2	2	2	2	2	2	2	2	0									
23	1	1	1	1	1	1	1	1	1	1	1									
24	4	2	4	2	4	2	4	2	4	2	4	0								
25	1	5	5	1	1	1	5	5	1	1	1	5								
26	2	2	2	2	2	2	2	2	2	2	2	2	0							
27	1	1	1	1	1	1	1	1	1	1	1	1	1							
28	4	2	4	2	4	2	4	2	4	2	4	2	4	0						
29	1	1	1	1	1	1	1	1	1	1	1	1	1	1						
30	2	6	6	10	2	2	6	6	2	2	10	6	6	2	0					
31	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1					
32	4	2	4	2	4	2	4	2	4	2	4	2	4	2	4	0				
33	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1				
34	2	2	2	2	10	2	10	2	2	10	2	10	2	2	2	2	0			
35	1	5	5	1	1	1	5	5	1	1	1	5	5	1	1	1	5			
36	4	2	4	2	4	2	4	2	4	2	4	2	4	2	4	2	4	0		
37	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		
38	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	0	
39	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
40	4	6	8	2	4	2	8	6	4	2	4	6	8	2	4	2	8	6	4	0

Table 1. Nullity of $A(P(n,k))$ for $n \leq 40$.

Graph	Oscillating strategy
$P(4, 1)$	L
$P(5, 2)$	L
$P(6, 1)$	LUL
$P(6, 2)$	UL
$P(9, 3)$	LUL
$P(12, 1)$	$LULUL$
$P(12, 3)$	$LULULULU$
$P(12, 5)$	$LULUL$
$P(18, 3)$	$LULUL$
$P(18, 6)$	$LULU$
$P(36, 3)$	$LULULULUL$
$P(36, 15)$	$LULULULUL$

Table 2. Regular Petersen graphs $P(n, k)$ with $n \leq 72$ where an oscillating strategy suffices to solve any solvable game. Read each strategy from left to right. For example, UL means first push the upper buttons (opposite lit lower buttons), then push lower buttons (opposite lit upper buttons).

The oscillating strategy on other Petersen graphs

Even when $n \neq 2k$, the oscillating strategy, repeated a finite number of times, will suffice to solve all solvable games on certain Petersen graphs. The proof for each such result is much like the proof of Theorem 1, but matters can be a bit more subtle since the oscillating matrix need not be zero; it need only be the case that every solvable game is in its nullspace.

Table 2 shows the regular Petersen graphs with $n \leq 72$ (and $k < n/2$) for which an oscillating strategy suffices to solve every solvable game. A minimal oscillation sequence is given for each such graph. Playable versions of these games can be found online at [Antonick 2013; Torrence 2016].

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