

involve

a journal of mathematics

A family of elliptic curves of rank ≥ 4

Farzali Izadi and Kamran Nabardi



A family of elliptic curves of rank ≥ 4

Farzali Izadi and Kamran Nabardi

(Communicated by Ken Ono)

In this paper we consider a family of elliptic curves of the form $y^2 = x^3 - c^2x + a^2$, where (a, b, c) is a primitive Pythagorean triple. First we show that the rank is positive. Then we construct a subfamily with rank ≥ 4 .

1. Introduction

As is well known, an elliptic curve E over a field \mathbb{K} can be explicitly expressed by the generalized Weierstrass equation of the form

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where $a_1, a_2, a_3, a_4, a_6 \in \mathbb{K}$. In this paper we are interested in the case where $\mathbb{K} = \mathbb{Q}$. By the Mordell–Weil theorem [Washington 2008], every elliptic curve over \mathbb{Q} has a commutative group $E(\mathbb{Q})$ which is finitely generated, i.e., $E(\mathbb{Q}) \cong \mathbb{Z}^r \times E(\mathbb{Q})_{\text{tors}}$, where r is a nonnegative integer and $E(\mathbb{Q})_{\text{tors}}$ is the subgroup of elements of finite order in $E(\mathbb{Q})$. This subgroup is called the torsion subgroup of $E(\mathbb{Q})$ and the integer r is called the rank of E and is denoted by $\text{rank } E$.

By Mazur’s theorem [Silverman and Tate 1992], the torsion subgroup $E(\mathbb{Q})_{\text{tors}}$ is one of the following 15 groups: $\mathbb{Z}/n\mathbb{Z}$ with $1 \leq n \leq 10$ or $n = 12$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}$ with $1 \leq m \leq 4$. Besides, it is not known which values of rank r are possible. The folklore conjecture is that a rank can be arbitrarily large, but it seems to be very hard to find examples with large ranks. The current record is an example of an elliptic curve over \mathbb{Q} with rank ≥ 28 , found by Elkies in May 2006 (see [Dujella 2012]). Having classified the torsion part, one is interested in seeing whether or not the rank is unbounded among all the elliptic curves. There is no known guaranteed algorithm to determine the rank and it is not known which integers can occur as ranks.

Specialization is a significant technique for finding a lower bound for the rank of a family of elliptic curves. One can consider an elliptic curve on the rational function field $\mathbb{Q}(T)$ and then obtain elliptic curves over \mathbb{Q} by specializing the variable T to suitable values $t \in \mathbb{Q}$ (see [Silverman 1994, Chapter III, Theorem 11.4] for more

MSC2010: primary 11G05; secondary 14H52, 14G05.

Keywords: elliptic curves, rank, Pythagorean triple.

details). Using this technique, Nagao and Kouya [1994] found curves of rank ≥ 21 , and Fermigier [1996] obtained a curve of rank ≥ 22 .

In order to determine r , one should find the generators of the free part of the Mordell–Weil group. Determining the *associated height matrix* is a useful technique for finding a set of generators. In the following, we briefly describe it.

Let $m/n \in \mathbb{Q}$, where $\gcd(m, n) = 1$. Then the *height* of m/n is defined by

$$h\left(\frac{m}{n}\right) = \log(\max\{|m|, |n|\}).$$

Corresponding to $P = (x, y) \in E(\mathbb{Q})$, we define

$$H(P) = h(x) \quad \text{and} \quad \hat{h}(P) = \frac{1}{2} \lim_{N \rightarrow \infty} \frac{H(2^N \cdot P)}{4^N},$$

where $H(P)$ is called the *canonical height* of $P \in E(\mathbb{Q})$. The Néron–Tate pairing to an elliptic curve is defined by

$$\langle \cdot, \cdot \rangle : E(\mathbb{Q}) \times E(\mathbb{Q}) \rightarrow \mathbb{R}, \quad \langle P, Q \rangle = \hat{h}(P + Q) - \hat{h}(P) - \hat{h}(Q).$$

The *associated height matrix* to $\{P_i\}_{i=1}^r$ is

$$\mathcal{H} := \left(\langle P_i, P_j \rangle \right)_{1 \leq i \leq r, 1 \leq j \leq r}.$$

If $\det \mathcal{H} \neq 0$, then the points $\{P_i\}_{i=1}^r$ are linearly independent and $\text{rank } E(\mathbb{Q}) \geq r$ (see [Silverman 1994, Chapter III] for more details and proofs).

In this work we deal with a family of elliptic curves which are related to the Pythagorean triples and, by using both the specialization and, the associated height matrix techniques, prove the following theorem.

Main Theorem 1.1. *Let (a, b, c) be a primitive Pythagorean triple. Then, there are infinitely many elliptic curves of the form*

$$E : y^2 = x^3 - c^2x + a^2 \tag{1-1}$$

with $\text{rank} \geq 4$.

If (a, b, c) is a primitive Pythagorean triple, then one can easily check that $a = i^2 - j^2$, $b = 2ij$, and $c = i^2 + j^2$, where $\gcd(i, j) = 1$, and i, j have opposite parity. So, we can consider (1-1) as

$$E_{i,j} : y^2 = x^3 - (i^2 + j^2)^2x + (i^2 - j^2)^2. \tag{1-2}$$

It is clear that two points $P_{i,j} = (0, i^2 - j^2)$ and $Q_{i,j} = (i^2 + j^2, i^2 - j^2)$ are on (1-2) and so $\text{rank } E_{i,j} > 0$. In the next section, we construct a subfamily with $\text{rank} \geq 3$.

2. A subfamily with rank ≥ 3

First, we look at (1-2) as a one-parameter family by letting

$$a = t^2 - 1, \quad b = 2t, \quad c = t^2 + 1, \tag{2-1}$$

where $t \in \mathbb{Q}$. Then, instead of (1-2) one can take

$$E_t : y^2 = x^3 - (t^2 + 1)^2x + (t^2 - 1)^2, \quad t \in \mathbb{Q}. \tag{2-2}$$

Lemma 2.1. *There are infinitely many elliptic curves of the form (2-2) with rank ≥ 3 .*

Proof. Clearly we have two points

$$P_t = (0, t^2 - 1), \quad Q_t = (t^2 + 1, t^2 - 1). \tag{2-3}$$

We impose another point in (2-2) with x -coordinate 1. This implies $1 - 4t^2$ is a square, say v^2 . Then $1 - 4t^2 = v^2$ defines a circle of the form $(2t)^2 + v^2 = 1$. Hence

$$t = \frac{\alpha}{\alpha^2 + 1}, \quad v = \frac{\alpha^2 - 1}{\alpha^2 + 1}, \tag{2-4}$$

with $\alpha \in \mathbb{Q}$. Then, instead of (2-2), one can take

$$E_\alpha : y^2 = x^3 - \left(\left(\frac{\alpha}{\alpha^2 + 1} \right)^2 + 1 \right)^2 x + \left(\left(\frac{\alpha}{\alpha^2 + 1} \right)^2 - 1 \right)^2, \tag{2-5}$$

having three points

$$P_\alpha = \left(0, \left(\frac{\alpha}{\alpha^2 + 1} \right)^2 - 1 \right), \quad Q_\alpha = \left(\left(\frac{\alpha}{\alpha^2 + 1} \right)^2 + 1, \left(\frac{\alpha}{\alpha^2 + 1} \right)^2 - 1 \right), \quad R_\alpha = \left(1, \frac{\alpha^2 - 1}{\alpha^2 + 1} \right).$$

When we specialize to $\alpha = 2$, we obtain a set of points $S = \{P_2, Q_2, R_2\} = \left\{ \left(0, \frac{-21}{25} \right), \left(\frac{29}{25}, \frac{-21}{25} \right), \left(1, \frac{3}{5} \right) \right\}$ on

$$E_2 : y^2 = x^3 - \frac{841}{25}x + \frac{44}{25}. \tag{2-6}$$

Using SAGE, one can easily check that the associated height matrix of S has nonzero determinant $\approx 22.879895 \neq 0$ showing that these three points are independent and so $\text{rank } E_2 \geq 3$. The specialization result of Silverman [1994] implies that for all but finitely many rational numbers, $\text{rank } E_\alpha \geq 3$. \square

3. Proof of the main theorem

We impose another point with x -coordinate $-2\alpha/(\alpha^2 + 1)$ in (2-5). Hence we want $1 + 2\alpha/(\alpha^2 + 1)$ to be a square. It suffices that $\alpha^2 + 1$ is a square, say β^2 . Therefore,

$$\alpha = \frac{2m}{1 - m^2}, \quad \beta = \frac{m^2 + 1}{1 - m^2}, \tag{3-1}$$

where $m \in \mathbb{Q}$. From the above expressions, one can transform (2-5) to

$$E_m : y^2 = x^3 - \frac{(m^8 + 8m^6 - 2m^4 + 8m^2 + 1)^2}{(2m^2 + m^4 + 1)^4}x + \frac{(m^8 + 14m^4 + 1)^2}{(2m^2 + m^4 + 1)^4}. \tag{3-2}$$

So we get the four points

$$P_m = (0, \gamma), \quad Q_m = \left(\frac{m^8 + 8m^6 - 2m^4 + 8m^2 + 1}{(m^2 + 1)^4}, \gamma \right),$$

$$R_m = \left(1, \frac{(m^2 - 2m - 1)(m^2 + 2m - 1)}{(m^2 + 1)^2} \right), \quad S_m = \left(\frac{4m(m^2 - 1)}{m^4 + 2m^2 + 1}, \frac{(m^2 - 2m - 1)}{m^2 + 1} \gamma \right),$$

where

$$\gamma = \frac{(m^4 - 2m^3 + 2m^2 + 2m + 1)(m^4 + 2m^3 + 2m^2 - 2m + 1)}{(m^2 + 1)^4}.$$

By specialization to $m = 2$ in (3-2), we have

$$E_2 : y^2 = x^3 - \frac{591361}{390625}x + \frac{231361}{390625}, \quad (3-3)$$

and the set of points $S = \{P_2, Q_2, R_2, S_2\} = \left\{ \left(0, \frac{481}{625}\right), \left(\frac{769}{625}, \frac{481}{625}\right), \left(1, \frac{7}{5}\right), \left(\frac{24}{25}, \frac{481}{3125}\right) \right\}$ on it. The associated height matrix of these four points has nonzero determinant $\approx 722.7181 \neq 0$ showing that these points are independent and so $\text{rank } E_2 \geq 4$. However, by using SAGE we see that $\text{rank } E_2 = 5$. Again, by specialization, we can conclude that for all but finitely many elliptic curves of the form (3-2), we have $\text{rank} \geq 4$.

Acknowledgements

The authors express their hearty thanks to the anonymous referee for a careful reading of the paper and for many comments and remarks which improved its quality.

References

- [Dujella 2012] A. Dujella, "High rank elliptic curves with prescribed torsion", 2012, available at <https://web.math.pmf.unizg.hr/~duje/tors/tors.html>.
- [Fermigier 1996] S. Fermigier, "Construction of high-rank elliptic curves over \mathbb{Q} and $\mathbb{Q}(t)$ with non-trivial 2-torsion (extended abstract)", pp. 115–120 in *Algorithmic number theory* (Talence, 1996), edited by H. Cohen, Lecture Notes in Computer Science **1122**, Springer, Berlin, 1996. MR 1446503 Zbl 0890.11020
- [Nagao and Kouya 1994] K.-I. Nagao and T. Kouya, "An example of elliptic curve over \mathbb{Q} with rank ≥ 21 ", *Proc. Japan Acad. Ser. A Math. Sci.* **70**:4 (1994), 104–105. MR 1276883 Zbl 0832.14022
- [Silverman 1994] J. H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics **151**, Springer, New York, 1994. MR 1312368 Zbl 0911.14015
- [Silverman and Tate 1992] J. H. Silverman and J. Tate, *Rational points on elliptic curves*, Springer, New York, NY, 1992. MR 1171452 Zbl 0752.14034
- [Washington 2008] L. C. Washington, *Elliptic curves: number theory and cryptography*, 2nd ed., Chapman and Hall, Boca Raton, FL, 2008. MR 2404461 Zbl 1200.11043

Received: 2014-05-07

Revised: 2015-09-30

Accepted: 2015-10-01

f.izadi@urmia.ac.ir

Department of Mathematics, Urmia University, Urmia, Iran

nabardi@azaruniv.edu

Department of Mathematics,
Azarbaijan Shahid Madani University, Tabriz, Iran

involve

msp.org/involve

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

MANAGING EDITOR

Kenneth S. Berenhaut Wake Forest University, USA

BOARD OF EDITORS

Colin Adams	Williams College, USA	Suzanne Lenhart	University of Tennessee, USA
John V. Baxley	Wake Forest University, NC, USA	Chi-Kwong Li	College of William and Mary, USA
Arthur T. Benjamin	Harvey Mudd College, USA	Robert B. Lund	Clemson University, USA
Martin Bohner	Missouri U of Science and Technology, USA	Gaven J. Martin	Massey University, New Zealand
Nigel Boston	University of Wisconsin, USA	Mary Meyer	Colorado State University, USA
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA	Emil Minchev	Ruse, Bulgaria
Pietro Cerone	La Trobe University, Australia	Frank Morgan	Williams College, USA
Scott Chapman	Sam Houston State University, USA	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran
Joshua N. Cooper	University of South Carolina, USA	Zuhair Nashed	University of Central Florida, USA
Jem N. Corcoran	University of Colorado, USA	Ken Ono	Emory University, USA
Toka Diagana	Howard University, USA	Timothy E. O'Brien	Loyola University Chicago, USA
Michael Dorff	Brigham Young University, USA	Joseph O'Rourke	Smith College, USA
Sever S. Dragomir	Victoria University, Australia	Yuval Peres	Microsoft Research, USA
Behrouz Emamizadeh	The Petroleum Institute, UAE	Y.-F. S. Pétermann	Université de Genève, Switzerland
Joel Foisy	SUNY Potsdam, USA	Robert J. Plemmons	Wake Forest University, USA
Errin W. Fulp	Wake Forest University, USA	Carl B. Pomerance	Dartmouth College, USA
Joseph Gallian	University of Minnesota Duluth, USA	Vadim Ponomarenko	San Diego State University, USA
Stephan R. Garcia	Pomona College, USA	Bjorn Poonen	UC Berkeley, USA
Anant Godbole	East Tennessee State University, USA	James Propp	U Mass Lowell, USA
Ron Gould	Emory University, USA	József H. Przytycki	George Washington University, USA
Andrew Granville	Université Montréal, Canada	Richard Rebarber	University of Nebraska, USA
Jerold Griggs	University of South Carolina, USA	Robert W. Robinson	University of Georgia, USA
Sat Gupta	U of North Carolina, Greensboro, USA	Filip Saidak	U of North Carolina, Greensboro, USA
Jim Haglund	University of Pennsylvania, USA	James A. Sellers	Penn State University, USA
Johnny Henderson	Baylor University, USA	Andrew J. Sterge	Honorary Editor
Jim Hoste	Pitzer College, USA	Ann Trenk	Wellesley College, USA
Natalia Hritonenko	Prairie View A&M University, USA	Ravi Vakil	Stanford University, USA
Glenn H. Hurlbert	Arizona State University, USA	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy
Charles R. Johnson	College of William and Mary, USA	Ram U. Verma	University of Toledo, USA
K. B. Kulasekera	Clemson University, USA	John C. Wierman	Johns Hopkins University, USA
Gerry Ladas	University of Rhode Island, USA	Michael E. Zieve	University of Michigan, USA

PRODUCTION

Silvio Levy, Scientific Editor

Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2016 is US \$160/year for the electronic version, and \$215/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2016 Mathematical Sciences Publishers

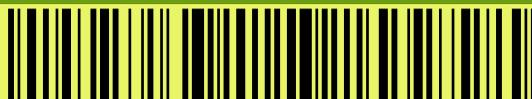
involve

2016

vol. 9

no. 5

An iterative strategy for Lights Out on Petersen graphs BRUCE TORRENCE AND ROBERT TORRENCE	721
A family of elliptic curves of rank ≥ 4 FARZALI IZADI AND KAMRAN NABARDI	733
Splitting techniques and Betti numbers of secant powers REZA AKHTAR, BRITTANY BURNS, HALEY DOHRMANN, HANNAH HOGANSON, OLA SOBIESKA AND ZEROTTI WOODS	737
Convergence of sequences of polygons ERIC HINTIKKA AND XINGPING SUN	751
On the Chermak–Delgado lattices of split metacyclic p -groups ERIN BRUSH, JILL DIETZ, KENDRA JOHNSON-TESCH AND BRIANNE POWER	765
The left greedy Lie algebra basis and star graphs BENJAMIN WALTER AND AMINREZA SHIRI	783
Note on superpatterns DANIEL GRAY AND HUA WANG	797
Lifting representations of finite reductive groups: a character relation JEFFREY D. ADLER, MICHAEL CASSEL, JOSHUA M. LANSKY, EMMA MORGAN AND YIFEI ZHAO	805
Spectrum of a composition operator with automorphic symbol ROBERT F. ALLEN, THONG M. LE AND MATTHEW A. PONS	813
On nonabelian representations of twist knots JAMES C. DEAN AND ANH T. TRAN	831
Envelope curves and equidistant sets MARK HUIBREGTSE AND ADAM WINCHELL	839
New examples of Brunnian theta graphs BYOUNGWOOK JANG, ANNA KRONAEUR, PRATAP LUITEL, DANIEL MEDICI, SCOTT A. TAYLOR AND ALEXANDER ZUPAN	857
Some nonsimple modules for centralizer algebras of the symmetric group CRAIG DODGE, HARALD ELLERS, YUKIHIDE NAKADA AND KELLY POHLAND	877
Acknowledgement	899



1944-4176(2016)9:5;1-0