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We give a complete characterization of the spectrum of composition operators, induced by an automorphism of the open unit disk, acting on a family of Banach spaces of analytic functions that includes the Bloch space and BMOA. We show that for parabolic and hyperbolic automorphisms the spectrum is the unit circle. For the case of elliptic automorphisms, the spectrum is either the unit circle or a finite cyclic subgroup of the unit circle.

1. Introduction

For an analytic self-map φ of the open unit disk $\mathbb{D}$ and a Banach space X of functions analytic on $\mathbb{D}$, we define the composition operator with symbol φ, denoted C_φ, by the rule $C_\varphi f = f \circ \varphi$ for all $f \in X$. The study of composition operators began formally with Nordgren's paper [1968], where he explored properties of composition operators acting on the Hardy Hilbert space H^2. Since then the study has proved to be an active area of research, most likely due to the fact that the study of such operators lies at the intersection of complex function theory and operator theory.

The spectrum of C_φ has been studied on many classical spaces of analytic functions, such as the Hardy spaces, Bergman spaces, weighted Hardy and Bergman spaces, Besov spaces, and the Dirichlet space. The interested reader is directed to [Cowen and MacCluer 1995] for general references.

The motivation for this paper was to determine the spectrum of a composition operator, induced by a disk automorphism, acting on the Bloch space. The Bloch space is the largest space of analytic functions on $\mathbb{D}$ that is Möbius invariant. This is one reason the Bloch space is a welcoming environment to study composition operators. The techniques developed apply to a larger class of spaces that includes the Bloch space.

The purpose of this paper is to determine the spectrum of C_φ acting on a family of Banach spaces, where φ is a disk automorphism. The spectrum will depend on

the fixed point classification of the automorphisms of $\mathbb{D}$. This is a standard approach to the study of composition operators induced by automorphisms. We show the spectrum of C_φ, acting on a particular family of Banach spaces, induced by a disk automorphism, must be a subset of the unit circle $\partial\mathbb{D}$, and in some instances is the entire unit circle. Finally, we compare these results to particular examples of classical spaces.

2. Preliminaries

2A. *Automorphisms.* The automorphisms of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ are precisely the analytic bijections on $\mathbb{D}$ which have the form

$$\varphi(z) = \lambda \frac{a - z}{1 - \overline{a}z},$$

where λ is a unimodular constant and a is a point in $\mathbb{D}$. These automorphisms form a group under composition denoted by Aut($\mathbb{D}$). Every element of Aut($\mathbb{D}$) has two fixed points (counting multiplicity), and thus can be classified by the location of the fixed points:

elliptic: one fixed point in $\mathbb{D}$ and one in the complement of $\overline{\mathbb{D}}$;

parabolic: one fixed point on the unit circle $\partial\mathbb{D}$ (of multiplicity 2);

hyperbolic: two distinct fixed points on $\partial\mathbb{D}$.

Two disk automorphisms φ and ψ are conformally equivalent if there exists a disk automorphism τ for which $\psi = \tau \circ \varphi \circ \tau^{-1}$. Many properties of automorphisms are preserved under conformal equivalence. The main advantage of conformal equivalence is in the placement of the fixed points. Every elliptic disk automorphism is conformally equivalent to one whose fixed point in $\mathbb{D}$ is the origin.

Lemma 2.1. *Let φ be an elliptic disk automorphism with fixed point a in $\mathbb{D}$. Then φ is conformally equivalent to $\psi(z) = \lambda z$ where $\lambda = \varphi'(a)$.*

Proof. Let τ_a be the involution automorphism which interchanges 0 and a, that is

$$\tau_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

Define $\psi = \tau_a \circ \varphi \circ \tau_a^{-1}$ on $\mathbb{D}$. Since a is a fixed point of φ, ψ fixes the origin, and is a rotation. So there is a unimodular constant λ such that $\psi(z) = \lambda z$. To complete the proof, we will show $\lambda = \varphi'(a)$. Observe $\psi'(z) = \lambda$ for all $z \in \mathbb{D}$. In particular

$$\lambda = \psi'(0) = \tau_a'\big(\varphi(\tau_a(0))\big)\varphi'\big(\tau_a(0)\big)\tau_a'(0) = \varphi'(a)\tau_a'(a)\tau_a'(0) = \varphi'(a).$$

Thus φ is conformally equivalent to the rotation $\psi(z) = \varphi'(a)z$. $\qquad\square$

Every parabolic disk automorphism is conformally equivalent to one whose fixed point (of multiplicity 2) is 1. The following lemma is found as Exercise 2.3.5c of [Cowen and MacCluer 1995], and a complete proof can be found in [Pons 2007].

Lemma 2.2 [Pons 2007, Lemma 4.1.2]. *Let φ be a parabolic disk automorphism. Then φ is conformally equivalent to either*

$$\psi_1(z) = \frac{(1+i)z - 1}{z + i - 1} \quad or \quad \psi_2(z) = \frac{(1-i)z - 1}{z - i - 1}.$$

Every hyperbolic disk automorphism is conformally equivalent to one whose fixed points in $\partial\mathbb{D}$ are ± 1.

Lemma 2.3 [Nordgren 1968, Theorem 6]. *Let φ be a hyperbolic disk automorphism. Then, for some $r \in (0, 1)$, φ is conformally equivalent to*

$$\psi(z) = \frac{z + r}{1 + rz}.$$

2B. *The space of bounded analytic functions.* The set of analytic functions on $\mathbb{D}$ is denoted by $H(\mathbb{D})$. The space of bounded analytic functions on $\mathbb{D}$, denoted $H^\infty = H^\infty(\mathbb{D})$, is a Banach space under the norm

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

The bounded analytic functions on $\mathbb{D}$ is a rich space containing many interesting types of functions, such as polynomials and Blaschke products. In addition, the disk algebra $\mathcal{A}(\mathbb{D})$, the set of analytic functions on $\mathbb{D}$ continuous to $\partial\mathbb{D}$, is a closed subspace of H^∞.

The following two families of functions will be used in the next section. To prove these functions are in H^∞, we take a geometric approach using conformal mappings of the plane. To this effect let $\mathbb{H}_\ell$ and $\mathbb{H}_r$ denote the open left and right half planes respectively, i.e., $\mathbb{H}_\ell = \{\operatorname{Re} z < 0\}$ and $\mathbb{H}_r = \{\operatorname{Re} z > 0\}$.

Lemma 2.4. *For $s \geq 0$, the function*

$$f_s(z) = \exp\frac{s(z + 1)}{z - 1}$$

is in H^∞.

Proof. If $s = 0$, then f_s is identically 1. So, $f_s(z)$ is in H^∞. Now suppose $s > 0$. The function f_s is comprised of the functions

(1) $z \mapsto (z + 1)/(z - 1)$ (mapping $\mathbb{D}$ onto $\mathbb{H}_\ell$),

(2) $z \mapsto sz$ (mapping $\mathbb{H}_\ell$ onto $\mathbb{H}_\ell$),

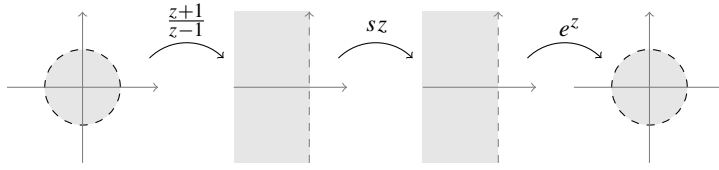(3) $z \mapsto e^z$ (mapping $\mathbb{H}_\ell$ onto $\mathbb{D} \setminus \{0\}$).

Figure 1. Map $f_s(z) = \exp \dfrac{s(z+1)}{z-1}$ for $s > 0$.

So f_s maps $\mathbb{D}$ into $\mathbb{D}$, as in Figure 1, and thus $f_s(z)$ is an element of H^∞.   □

Lemma 2.5. *For real value t, the function*

$$f_t(z) = \left(\frac{1+z}{1-z}\right)^{it}$$

is in H^∞.

Proof. For $t = 0$, f_t is identically 1, and thus is in H^∞. Now suppose $t > 0$. We will rewrite the function f_t as

$$f_t(z) = \exp\left(it \log \frac{1+z}{1-z}\right),$$

where log is the principle branch of the logarithm. Then f_t is comprised of the functions

(1) $z \mapsto (1+z)/(1-z)$ (mapping $\mathbb{D}$ onto $\mathbb{H}_r$),

(2) $z \mapsto \log z$ (mapping H_r onto the horizontal strip $S_h = \{0 < \operatorname{Im} z < 2\pi\}$),

(3) $z \mapsto itz$ (mapping S_h onto the vertical strip $S_v = \{-2\pi < \operatorname{Re} z < 0\}$),

(4) $z \mapsto e^z$ (mapping S_v into $A(e^{-2\pi}, 1) = \{e^{-2\pi} < |z| < 1\}$).

So f_t maps $\mathbb{D}$ into $A(e^{-2\pi}, 1) \subseteq \mathbb{D}$, as depicted in Figure 2. In the case of $t < 0$, the vertical strip S_v becomes $\{0 < \operatorname{Re} z < 2\pi\}$. The map $z \mapsto e^z$ takes S_v into $A(1, e^{2\pi}) \subseteq e^{2\pi}\mathbb{D}$, as depicted in Figure 3. In either case, $f_t(z)$ is an element of H^∞ since $\|f_t\|_\infty < e^{2\pi}$ for all $t \in \mathbb{R}$.   □
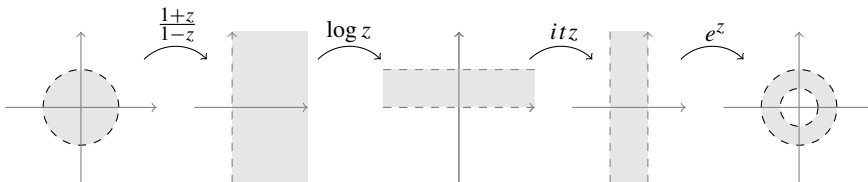

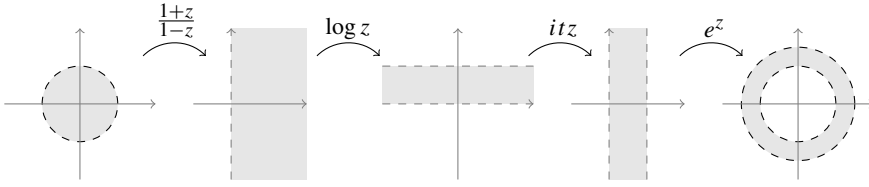**Figure 2.** Map $f_t(z) = \exp\left(it \log \dfrac{1+z}{1-z}\right)$ for $t > 0$.

Figure 3. Map $f_t(z) = \exp\left(it \log \dfrac{1+z}{1-z}\right)$ for $t < 0$.

These functions above, together with the monomials, play such a pivotal role in Section 3 that we denote the union of these functions by $\mathcal{F}$, i.e.,

$$\mathcal{F} = \{f_s : s \geq 0\} \cup \{f_t : t \in \mathbb{R}\} \cup \{z^k : k \in \mathbb{N}\}.$$

2C. *Spectrum of C_φ.* In this section we collect useful results regarding the spectrum of operators on Banach spaces. For a bounded linear operator T on a Banach space X, the spectrum of T is given by

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$$

where I denotes the identity operator on X. The spectrum is a nonempty, closed subset of $\mathbb{C}$. The spectral radius of T is given by

$$\rho(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Due to the fact that the spectrum is closed, we have the spectrum of T is contained in the closed disk centered at the origin of radius $\rho(T)$.

Determining the spectrum of a particular composition operator can be difficult depending on the symbol of the operator and the space on which it is acting. However, the difficulties can be avoided if the operator is similar to a "simpler" operator. Linear operators S and T (not necessarily bounded) on a Banach space X are similar if there exists a bounded linear operator U on X, having bounded inverse, such that $T = USU^{-1}$. If S and T are both bounded operators, then similarity preserves the spectrum.

Theorem 2.6. *Let S and T be bounded operators on a Banach space X. If S and T are similar, then $\sigma(S) = \sigma(T)$.*

Proof. Suppose S and T are similar operators on X. By definition, there exists an invertible, bounded operator U such that $T = USU^{-1}$. Let $\lambda \in \mathbb{C}$ and observe that

$$\begin{aligned}
T - \lambda I &= USU^{-1} - \lambda I \\
&= USU^{-1} - \lambda UU^{-1} \\
&= USU^{-1} - U(\lambda I)U^{-1} \\
&= U(S - \lambda I)U^{-1}.
\end{aligned}$$

Thus, we have that $S - \lambda I$ is not invertible if and only if $T - \lambda I$ is not invertible. Therefore $\sigma(S) = \sigma(T)$. $\qquad\qquad\square$

3. Main results

In this section, we determine the spectrum of C_φ for φ a disk automorphism acting on a particular family of Banach spaces of analytic functions. The spaces we consider will be denoted by $\mathcal{X}$ and have the following properties:

(i) $\mathcal{X}$ contains $\mathcal{F}$,

(ii) for all $\varphi \in \mathrm{Aut}(\mathbb{D})$, C_φ is bounded on $\mathcal{X}$ and $\rho(C_\varphi) = 1$.

The set of automorphisms of $\mathbb{D}$, as seen previously, is a very nice subset of the analytic self-maps of $\mathbb{D}$. By property (ii), every composition operator induced by a disk automorphism is bounded on $\mathcal{X}$. In fact, every such composition operator is invertible. This result, that we prove below, can be viewed as a consequence of Theorem 1.6 of [Cowen and MacCluer 1995].

Proposition 3.1. *Let φ be a disk automorphism and C_φ the induced composition operator on $\mathcal{X}$. Then C_φ is invertible with inverse $C_\varphi^{-1} = C_{\varphi^{-1}}$.*

Proof. Since $\varphi \in \mathrm{Aut}(\mathbb{D})$, φ is invertible, and φ^{-1} is an automorphism. The composition operator $C_{\varphi^{-1}}$ is bounded by property (ii) and

$$C_\varphi(C_{\varphi^{-1}}(f)) = C_\varphi(f \circ \varphi^{-1}) = f \circ \varphi^{-1} \circ \varphi = f,$$
$$C_{\varphi^{-1}}(C_\varphi(f)) = C_{\varphi^{-1}}(f \circ \varphi) = f \circ \varphi \circ \varphi^{-1} = f .$$

Therefore, C_φ is invertible with $C_\varphi^{-1} = C_{\varphi^{-1}}$. $\qquad\qquad\square$

Since the spectral radius of C_φ on $\mathcal{X}$ is 1 for $\varphi \in \mathrm{Aut}(\mathbb{D})$, we see that the search for the spectrum can be restricted to subsets of $\overline{\mathbb{D}}$. However, our search can be refined further to subsets of the unit circle.

Theorem 3.2. *Let φ be a disk automorphism and C_φ the induced composition operator on $\mathcal{X}$. Then $\sigma(C_\varphi) \subseteq \partial\mathbb{D}$.*

Proof. By property (ii) of $\mathcal{X}$, we have $\rho(C_\varphi) = 1$. So, $\sigma(C_\varphi) \subseteq \overline{\mathbb{D}}$. Since, by Proposition 3.1, C_φ is invertible with the inverse $C_\varphi^{-1} = C_{\varphi^{-1}}$, then $0 \notin \sigma(C_\varphi)$. So, the function $f(z) = z^{-1}$ is analytic in some neighborhood of $\sigma(C_\varphi)$. By the Spectral Mapping Theorem (see Theorem 5.14 of [MacCluer 2009]), we have $\sigma(f \circ C_\varphi) = f(\sigma(C_\varphi))$, and so,

$$\sigma(C_{\varphi^{-1}}) = \sigma(C_\varphi^{-1}) = \sigma(C_\varphi)^{-1} = \{\lambda^{-1} : \lambda \in \sigma(C_\varphi)\} .$$

Since $\varphi^{-1} \in \mathrm{Aut}(\mathbb{D})$, $\sigma(C_{\varphi^{-1}}) \subseteq \overline{\mathbb{D}}$. Thus for $\lambda \in \sigma(C_\varphi)$, both λ and λ^{-1} are in $\overline{\mathbb{D}}$. This implies $\lambda \in \partial\mathbb{D}$. So $\sigma(C_\varphi) \subseteq \partial\mathbb{D}$, as desired. $\qquad\qquad\square$

Since the disk automorphisms are classified into three categories, according to fixed points, we will treat each type of automorphism separately. However, the strategy to determine $\sigma(C_\varphi)$ is the same. For a disk automorphism φ, we have shown φ to be conformally equivalent to a particularly "nice" disk automorphism: in the elliptic case a disk automorphism that fixes 0, in the parabolic case a disk automorphism that fixes 1, and in the hyperbolic case a disk automorphism that fixes ± 1. In the next result, we show that conformally equivalent automorphisms induce similar composition operators on $\mathcal{X}$. This result is not unique to the space $\mathcal{X}$, but is true for any space for which automorphisms induce bounded composition operators (see p. 250 of [Cowen and MacCluer 1995]).

Proposition 3.3. *Let φ and ψ be conformally equivalent disk automorphisms. Then the induced composition operators C_φ and C_ψ on $\mathcal{X}$ are similar.*

Proof. Suppose φ and ψ are conformally equivalent disk automorphisms. Then there exists a disk automorphism τ such that $\psi = \tau \circ \varphi \circ \tau^{-1}$. For $f \in \mathcal{X}$, observe

$$C_\psi f = f \circ (\tau \circ \varphi \circ \tau^{-1}) = ((f \circ \tau) \circ \varphi) \circ \tau^{-1} = (C_{\tau^{-1}} C_\varphi C_\tau) f.$$

Since $C_{\tau^{-1}}$ is bounded and invertible on $\mathcal{X}$ with $C_{\tau^{-1}}^{-1} = C_\tau$, then $C_\psi = C_{\tau^{-1}} C_\varphi C_{\tau^{-1}}^{-1}$. Therefore C_φ and C_ψ are similar. $\qquad\square$

With Proposition 3.3 and Lemmas 2.1, 2.2, and 2.3, it suffices to determine the spectrum of composition operators induced by these "nice" disk automorphisms, since similarity of bounded operators preserves the spectrum.

Theorem 3.4. *Let φ be an elliptic disk automorphism with fixed point a in $\mathbb{D}$. Then the spectrum of C_φ acting on $\mathcal{X}$ is the closure of the positive powers of $\varphi'(a)$. Moreover, this closure is a finite subgroup of the unit circle if $\varphi'(a)^n = 1$ for some natural number n, and is the unit circle otherwise.*

Proof. By Lemma 2.1, φ is conformally equivalent to $\psi(z) = \lambda z$ where $\lambda = \varphi'(a)$. By Proposition 3.3, it suffices to show that $\sigma(C_\psi)$ is the closure of the positive powers of λ. Let $G = \langle \lambda \rangle = \{\lambda^k : k \in \mathbb{N}\}$, which is a subset of $\partial\mathbb{D}$ since $|\lambda| = 1$. For each $k \in \mathbb{N}$, the function $f_k(z) = z^k$ is in $\mathcal{X}$ by property (i), and we have $(C_\psi f_k)(z) = \lambda^k f_k(z)$. Thus λ^k is an eigenvalue of C_ψ corresponding to the eigenfunction f_k. So $G \subseteq \sigma(C_\psi)$, and since the spectrum is closed, we have $\overline{G} \subseteq \sigma(C_\psi) = \sigma(C_\varphi)$. If the order of λ is infinite, then G is dense in $\partial\mathbb{D}$, and so $\overline{G} = \partial\mathbb{D}$.

Now suppose λ has order $m < \infty$. Then $G = \{\lambda^k : k = 1, \ldots, m\}$. So, $\overline{G} = G$. We now wish to show $\sigma(C_\psi) \subseteq \overline{G}$. Since $\sigma(C_\psi) \subseteq \partial\mathbb{D}$ by Theorem 3.2 it suffices to show that if $\mu \in \partial\mathbb{D} \setminus \overline{G}$ then $\mu \notin \sigma(C_\psi)$. Suppose $\mu \in \partial\mathbb{D} \setminus \overline{G}$.

Since $\mu \notin \overline{G}$, it clear that $\mu \notin G$ and $\mu^m \neq 1$. In order to show $\mu \notin \sigma(C_\psi)$, we will show that $C_\psi - \mu I$ is invertible by proving that for every $g \in \mathcal{X}$, there exists a unique $f \in \mathcal{X}$ such that $f \circ \psi - \mu f = g$.

Since the order of λ is m, we have

$$\psi^{(m)}(z) = \underbrace{(\psi \circ \cdots \circ \psi)}_{m-\text{times}}(z) = \lambda^m z = z.$$

By repeated composition with ψ, we obtain the system of linear equations:

$$f \circ \psi - \mu f = g$$
$$f \circ \psi^{(2)} - \mu(f \circ \psi) = g \circ \psi$$
$$\vdots$$
$$f - \mu(f \circ \psi^{(m-1)}) = g \circ \psi^{(m-1)}.$$

This system of linear equations can be expressed as the matrix equation $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} -\mu & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \vdots & & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & 0 & -\mu \end{bmatrix}, \vec{x} = \begin{bmatrix} f \\ f \circ \psi \\ \vdots \\ \vdots \\ f \circ \psi^{(m-2)} \\ f \circ \psi^{(m-1)} \end{bmatrix}, \text{ and } \vec{b} = \begin{bmatrix} g \\ g \circ \psi \\ \vdots \\ \vdots \\ g \circ \psi^{(m-2)} \\ g \circ \psi^{(m-1)} \end{bmatrix}.$$

The determinant of A is $(-1)^m(\mu^m - 1)$, which is not zero since $\mu \notin G$. Thus there is a unique solution for $\vec{x}$. It gives us the unique solution f, which is a finite linear combination of function in $\mathcal{X}$ of the form $g \circ \psi^{(j-1)}$ for $j = 1, \ldots, m$, and thus f is in $\mathcal{X}$. It follows that $C_\psi - \mu I$ is invertible. So, $\mu \notin \sigma(C_\psi)$. Therefore, $\sigma(C_\varphi) = \sigma(C_\psi) \subseteq \overline{G}$.   □

Theorem 3.5. *Let φ be a parabolic disk automorphism. Then the spectrum of C_φ acting on $\mathcal{X}$ is the unit circle.*

Proof. From Lemma 2.2, φ is conformally equivalent to either

$$\psi_1(z) = \frac{(1+i)z - 1}{z + i - 1} \quad \text{or} \quad \psi_2(z) = \frac{(1-i)z - 1}{z - i - 1}.$$

By Theorems 3.2 and 3.3 it suffices to show that $\partial\mathbb{D}$ is a subset of $\sigma(C_{\psi_1})$ and $\sigma(C_{\psi_2})$.

First suppose φ is conformally equivalent to ψ_1. Consider the function

$$f_s(z) = \exp \frac{s(z + 1)}{z - 1}$$

for $s \geq 0$. By property (i), f_s is in $\mathcal{X}$. Observe

$$(C_{\psi_1} f_s)(z) = f_s(\psi_1(z)) = f_s\left(\frac{(1+i)z - 1}{z + i - 1}\right)$$

$$= \exp \frac{s\left(\frac{(1+i)z-1}{z+i-1} + 1\right)}{\frac{(1+i)z-1}{z+i-1} - 1} = \exp \frac{s((1+i)z - 1 + z + i - 1)}{(1+i)z - 1 - z - i + 1}$$

$$= \exp \frac{s((2+i)z + i - 2)}{i(z - 1)} = \exp \frac{s((1-2i)z + 1 + 2i)}{z - 1}$$

$$= \exp\left(\frac{s(z+1)}{z-1} - 2is\right) = e^{i(-2s)} f_s(z).$$

So, f_s is an eigenfunction of C_{ψ_1} for $s \geq 0$. Then, $\partial\mathbb{D} = \{e^{i(-2s)} : s \geq 0\}$ is a subset of $\sigma(C_{\psi_1})$. If φ is conformally equivalent to ψ_2, then by a similar calculation, we have

$$(C_{\psi_2} f_s)(z) = e^{2is} f_s(z),$$

and so $\partial\mathbb{D} = \{e^{2is} : s \geq 0\}$ is a subset of $\sigma(C_{\psi_2})$. Therefore, $\sigma(C_\varphi) = \partial\mathbb{D}$, as desired. $\square$

Theorem 3.6. *Let φ be a hyperbolic disk automorphism. Then the spectrum of C_φ acting on $\mathcal{X}$ is the unit circle.*

Proof. From Lemma 2.3, φ is conformally equivalent to $\psi(z) = (z+r)/(1+rz)$ for some $r \in (0, 1)$. By Theorems 3.2 and 3.3 it suffices to show that $\partial\mathbb{D} \subseteq \sigma(C_\psi)$. Consider the function

$$f_t(z) = \left(\frac{1+z}{1-z}\right)^{it}$$

for $t \in \mathbb{R}$. By property (i), f_t is in $\mathcal{X}$. Observe

$$(C_\psi f_t)(z) = f_t(\psi(z)) = f_t\left(\frac{z+r}{1+rz}\right)$$

$$= \left(\frac{1 + \frac{z+r}{1+rz}}{1 - \frac{z+r}{1+rz}}\right)^{it} = \left(\frac{1 + rz + z + r}{1 + rz - z - r}\right)^{it}$$

$$= \left(\frac{(r+1)z + (r+1)}{(r-1)z - (r-1)}\right)^{it}$$

$$= \left(\frac{r+1}{r-1}\right)^{it} f_t(z).$$

So, f_t is an eigenfunction of C_ψ for t real. Then $\partial\mathbb{D} = \{((r+1)/(r-1))^{it} : 0 < r < 1, t \in \mathbb{R}\}$ is a subset of $\sigma(C_\psi) = \sigma(C_\varphi)$. Therefore $\sigma(C_\varphi) = \partial\mathbb{D}$, as desired. $\square$

4. Examples and comparisons

In this section we first consider examples of spaces that satisfy the properties of $\mathcal{X}$. For these spaces, our results characterize the spectrum of composition operators induced by disk automorphisms. Lastly, we consider spaces that do not satisfy the properties of $\mathcal{X}$ but for which the spectrum of composition operators induced by automorphisms is known. We will compare the spectra for those spaces with the characterization for $\mathcal{X}$.

4A. *Examples.* First, we will discuss examples of spaces that satisfy the properties of $\mathcal{X}$.

4A1. *Bounded analytic functions.* The property (i) of $\mathcal{X}$ is satisfied by H^∞ by Lemmas 2.4 and 2.5. In fact, on H^∞, any analytic self-map of $\mathbb{D}$ induces a bounded composition operator C_φ such that $\|C_\varphi\| = 1$. Equality is achieved since H^∞ contains the constant function 1. The spectral radius formula (see Theorem 5.15 of [MacCluer 2009]) then implies that $\rho(C_\varphi) = 1$. Thus, property (ii) is satisfied. Thus H^∞ belongs to the family of Banach spaces of analytic functions $\mathcal{X}$.

4A2. *Bloch space.* The Bloch space on $\mathbb{D}$, denoted $\mathcal{B} = \mathcal{B}(\mathbb{D})$, is the space of analytic functions on $\mathbb{D}$ such that $\beta_f = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty$. The quantity β_f is a seminorm, called the Bloch seminorm. The Bloch space is a Banach space under the norm

$$\|f\|_\mathcal{B} = |f(0)| + \beta_f.$$

It is well-known that $\mathcal{B}$ is a Banach space of analytic functions that contains H^∞, and thus satisfies property (i) of $\mathcal{X}$. In fact, every analytic self-map of $\mathbb{D}$ induces a bounded composition operator on $\mathcal{B}$ (see [Arazy et al. 1985, p. 126]). Donaway, in his Ph.D. thesis, proved the spectral radius of every composition operator induced by an analytic function on $\mathbb{D}$, and in particular the disk automorphisms, is 1, by [Donaway 1999, Corollary 3.9]. So the Bloch space satisfies all the properties of $\mathcal{X}$.

4A3. *Analytic functions of bounded mean oscillation.* The space of analytic functions on $\mathbb{D}$ with bounded mean oscillation on $\partial\mathbb{D}$, denoted BMOA, is defined to be the set of functions in $H(\mathbb{D})$ such that

$$\|f\|_* = \sup_{z \in \mathbb{D}} \|f \circ \tau_a - f(z)\|_{H^2} < \infty,$$

where H^2 is defined in Section 4B1. The space BMOA is a Banach space under the norm

$$\|f\|_{\mathrm{BMOA}} = |f(0)| + \|f\|_*.$$

It is well-known that BMOA is a Banach space of analytic functions, a subspace of the Bloch space, and contains H^∞ as a subspace since $\|f\|_{\mathrm{BMOA}} \le 3\|f\|_\infty$.

Thus property (i) is satisfied by BMOA. The following result shows property (ii) is satisfied by BMOA also.

Theorem 4.1. *Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ acting on BMOA is bounded and $\rho(C_\varphi) = 1$.*

Proof. As a result of the Littlewood subordination principle (see Theorem 1.7 of [Duren 1970]), every analytic self-map φ of $\mathbb{D}$ induces a bounded composition operator on BMOA.

To compute the spectral radius of C_φ acting on BMOA, we first estimate the norm. By Corollary 2.2 of [Laitila 2009] there is a constant $M > 0$, independent of φ, such that

$$\|C_\varphi\| \le M\left(\sup_{a \in \mathbb{D}}\|\tau_{\varphi(a)} \circ \varphi \circ \tau_a\|_{H^2} + \log\frac{2}{1 - |\varphi(0)|^2}\right). \tag{4-1}$$

Since the function $\tau_{\varphi(a)} \circ \varphi \circ \tau_a$ is a composition of self-maps of the disk, the first term on the right is bounded above by 1. Also,

$$\frac{1}{1 - |\varphi(0)|^2} \le \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \le \frac{2}{1 - |\varphi(0)|}$$

and hence

$$\log\left(\frac{2}{1 - |\varphi(0)|^2}\right) \le \log\left(\frac{4}{1 - |\varphi(0)|}\right) \le 2\log 2 - \log(1 - |\varphi(0)|).$$

Applying these estimates to Equation (4-1), we have

$$\|C_\varphi\| \le M(1 + 2\log 2) - M\log(1 - |\varphi(0)|).$$

This immediately implies that

$$\|C_{\varphi_n}\| \le M(1 + 2\log 2) - M\log(1 - |\varphi_n(0)|)$$

and it follows that $\rho(C_\varphi) = 1$ for all bounded composition operators acting on BMOA by Theorem 3.7 of [Donaway 1999]. $\qquad\square$

Thus BMOA satisfies all the properties of $\mathcal{X}$.

4B. *Comparisons.* We now investigate spaces that do not satisfy the properties of $\mathcal{X}$. We compare the spectrum of induced composition operators on these spaces with those on $\mathcal{X}$.

4B1. *Hardy spaces.* For $1 \le p < \infty$, the Hardy space, denoted $H^p = H^p(\mathbb{D})$, is the space of analytic functions on $\mathbb{D}$ such that

$$\|f\|_{H^p}^p = \sup_{0 < r < 1}\int_{\mathbb{D}}|f(re^{i\theta})|^p\frac{d\theta}{2\pi} < \infty.$$

Under this norm, the Hardy spaces are Banach spaces and for $p = 2$ it is a Hilbert space.

It is well known that H^p is a Banach space of analytic functions that contains H^∞ as a subspace. For the cases of an elliptic or parabolic automorphism φ, it is the case that $\rho(C_\varphi) = 1$ and the spectrum of C_φ on H^p is the same as for C_φ acting on $\mathcal{X}$ (see Theorem 3.9 of [Cowen and MacCluer 1995]). However, it is not the case that the spectral radius is 1 for every composition operator induced by an automorphism. In fact, if φ is hyperbolic, then $\rho(C_\varphi) = \varphi'(a)^{-1/p}$ where a is the Denjoy–Wolff point of φ (see Theorem 3.9 of [Cowen and MacCluer 1995]). In this situation, $\varphi'(a) < 1$ thus making $\rho(C_\varphi) > 1$. In turn, the spectrum is the annulus $\varphi'(a)^{1/p} \le |z| \le \varphi'(a)^{-1/p}$ (see Theorem 4.9 of [Hyvärinen et al. 2013]).

4B2. *Weighted Bergman spaces.* For $1 \le p < \infty$ and $\alpha > -1$, the standard weighted Bergman space, denoted $A_\alpha^p = A_\alpha^p(\mathbb{D})$, is the space of analytic functions on $\mathbb{D}$ such that

$$\|f\|_{A_\alpha^p}^p = \int_{\mathbb{D}} \left(1 - |z|^2\right)^\alpha |f(z)|^p \, dA(z) < \infty,$$

where $dA(z)$ is the normalized Lebesgue area measure on $\mathbb{D}$. The weighted Bergman spaces are Banach spaces under the norm $\|\cdot\|_{A_\alpha^p}$.

It is well known that A_α^p is a Banach space of analytic functions that contains H^∞ as a subspace. For the cases of an elliptic or parabolic automorphism φ, it is the case that $\rho(C_\varphi) = 1$ and the spectrum of C_φ on A_α^p is the same as for C_φ acting on $\mathcal{X}$ (see Lemma 4.2 and Theorem 4.14 of [Hyvärinen et al. 2013]). However, as was the case for the Hardy spaces, it is not the case that the spectral radius is 1 for every composition operator induced by an automorphism. In fact, if φ is hyperbolic, then

$$\rho(C_\varphi) = \max\left\{\frac{1}{\varphi'(a)^s}, \frac{1}{\varphi'(b)^s}\right\}$$

where $s = (\alpha + 2)/p$, a is the Denjoy–Wolff point and b is the other fixed point of φ (see Theorem 4.6 of [Hyvärinen et al. 2013]). In turn, the spectrum contains the annulus

$$\min\left\{\frac{1}{\varphi'(a)^s}, \frac{1}{\varphi'(b)^s}\right\} \le |z| \le \max\left\{\frac{1}{\varphi'(a)^s}, \frac{1}{\varphi'(b)^s}\right\}$$

(see Corollary 4.7 of [Hyvärinen et al. 2013]).

4B3. *Weighted Banach spaces.* For $0 < p < \infty$, the standard weighted Banach space on $\mathbb{D}$, denoted $H_p^\infty = H_p^\infty(\mathbb{D})$, is the space of analytic functions on $\mathbb{D}$ such that

$$\|f\|_{H_p^\infty} = \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^p |f(z)| < \infty,$$

The weighted Banach spaces are, not surprisingly, Banach spaces under $\|\cdot\|_{H_p^\infty}$.

It is well known that H_p^∞ is a Banach space of analytic functions that contain H^∞ as a subspace. For the cases of an elliptic or parabolic automorphism φ, it is the case that $\rho(C_\varphi) = 1$ and the spectrum of C_φ on H_p^∞ is the same as for C_φ acting on $\mathcal{X}$ (see Lemma 4.2 and Theorem 4.14 of [Hyvärinen et al. 2013]). However, as was the case for the Hardy spaces, it is not the case that the spectral radius is 1 for every composition operator induced by an automorphism. In fact, if φ is hyperbolic, then

$$\rho(C_\varphi) = \max\left\{\frac{1}{\varphi'(a)^s}, \frac{1}{\varphi'(b)^s}\right\}$$

where a is the Denjoy–Wolff point and b is the other fixed point of φ. In turn, the spectrum contains the annulus

$$\min\left\{\frac{1}{\varphi'(a)^p}, \frac{1}{\varphi'(b)^p}\right\} \le |z| \le \max\left\{\frac{1}{\varphi'(a)^p}, \frac{1}{\varphi'(b)^p}\right\}$$

(see Theorem 4.6 and Corollary 4.7 of [Hyvärinen et al. 2013]).

4B4. *Dirichlet space.* The Dirichlet space on $\mathbb{D}$, denoted $\mathcal{D}$, is the space of analytic functions on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty$$

where dA denotes the normalized Lebesgue area measure on $\mathbb{D}$. Under the norm

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z)$$

the Dirichlet space has a Hilbert space structure. Although not every analytic self-map of $\mathbb{D}$ induce bounded composition operators on $\mathcal{D}$, univalent maps, and thus the automorphisms, of $\mathbb{D}$ do.

Independently, Donaway [1999, Corollary 3.11] and Martín and Vukotić [2005, Theorem 7] showed that composition operators on $\mathcal{D}$ induced by univalent self-maps of $\mathbb{D}$, and thus the automorphisms, have spectral radius 1. However, by direct calculation, one can see that the functions in $\mathcal{F}$ are not contained in the Dirichlet space; for the case of f_s this is shown in [Pons 2010] (see p. 455). Despite $\mathcal{D}$ not satisfying all the properties of $\mathcal{X}$, the spectrum of automorphism induced composition operators on $\mathcal{D}$ are precisely the same as those on $\mathcal{X}$.

To overcome the lack of eigenfunctions, the authors in [Higdon 1997] and [Gallardo-Gutiérrez and Montes-Rodríguez 2003] used two new approaches. In [Higdon 1997], the author produces approximate eigenfunctions and in [Gallardo-Gutiérrez and Montes-Rodríguez 2003] unitary similarity is the key tool.

Remark 4.2. For all of the spaces discussed in Sections 4A and 4B (and those discussed in the next section), the spectrum of C_φ when φ is elliptic will be the

same as that for C_φ acting on $\mathcal{X}$. This is due to the fact that the eigenfunctions are the monomials, which are contained in all of these spaces.

5. Open questions

We end this paper with open questions which were inspired while developing the examples and comparisons in Sections 4A and 4B.

5A. *The little Bloch space.* While the Bloch space contains the polynomials, they are not dense in $\mathcal{B}$. The closure of the polynomials with respect to $\|\cdot\|_{\mathcal{B}}$ is called the little Bloch space, denoted $\mathcal{B}_0 = \mathcal{B}_0(\mathbb{D})$. More formally, the little Bloch space consists of the functions $f \in \mathcal{B}$ such that

$$\lim_{|z| \to 1} \left(1 - |z|^2\right)\left|f'(z)\right| = 0.$$

From Theorem 12 of [Arazy et al. 1985], bounded composition operators on $\mathcal{B}_0$ are induced exactly by functions *in* $\mathcal{B}_0$, which include the automorphisms. Donaway also proved the spectral radius of every bounded composition operator on $\mathcal{B}_0$ is 1. Thus property (ii) is satisfied by $\mathcal{B}_0$. However, the following result shows that $\mathcal{F}$ is not contained in $\mathcal{B}_0$, and thus property (i) of $\mathcal{X}$ is not satisfied.

Theorem 5.1. *The functions f_s and f_t, for $s > 0$ and $t \neq 0$, are not contained in the little Bloch space.*

Proof. Consider the function

$$f_t(z) = \exp\left(it \log \frac{1+z}{1-z}\right).$$

We show that this function is not in $\mathcal{B}_0$ for $t \in \mathbb{R} \setminus \{0\}$. Taking the derivative,

$$f_t'(z) = f_t(z)\left(it\frac{1-z}{1+z}\right)\frac{2}{(1-z)^2} = f_t(z)\frac{2it}{(1-z)(1+z)}.$$

For $t > 0$, $|f_t(z)| \geq e^{-2\pi}$ and, for $t < 0$, $|f_t(z)| \geq 1$. In either case, there is a constant $C > 0$ such that $|f_t(z)| \geq C$ for all $t \in \mathbb{R} \setminus \{0\}$ and all $z \in \mathbb{D}$. Hence

$$|f_t'(z)| \geq \frac{2C|t|}{|z-1||z+1|}.$$

To show that $f_t \notin \mathcal{B}_0$, we need to show that

$$\lim_{|z| \to 1} \left(1 - |z|^2\right)|f_t'(z)| \neq 0.$$

To see this, first observe that

$$\lim_{|z| \to 1} \left(1 - |z|^2\right)|f_t'(z)| \geq \lim_{|z| \to 1} \left(1 - |z|^2\right)\frac{2C|t|}{|z-1||z+1|}$$

by our estimate from above. If we now take a radial path to 1, that is, we set $z = r$ and let $r \uparrow 1$, we have

$$\lim_{r \to 1^-} (1 - r^2) \frac{2C|t|}{(1-r)(1+r)} = 2C|t| > 0$$

when $t \neq 0$. Thus

$$\lim_{|z| \to 1} (1 - |z|^2) \frac{2Ct}{|z-1||z+1|} \neq 0$$

for $t \neq 0$, and hence f_t is not in $\mathcal{B}_0$.

Next consider the function

$$f_s(z) = \exp \frac{s(z+1)}{z-1}.$$

We will show that this function is not in $\mathcal{B}_0$ for $s > 0$. First observe that

$$f_s'(z) = \left(\exp \frac{s(z+1)}{z-1} \right) \frac{-2s}{(z-1)^2}$$

and thus we aim to show that

$$\lim_{|z| \to 1} (1 - |z|^2)|f_s'(z)| = \lim_{|z| \to 1} (1 - |z|^2) \left| \exp(\frac{s(z+1)}{z-1}) \right| \frac{2s}{|1-z|^2} \neq 0.$$

Fix $x_0 < 0$ and consider the sequence $\{z_n\}$ defined by

$$z_n = \frac{x_0 + in + 1}{x_0 + in - 1}.$$

Since $x_0 < 0$, this sequence is contained in the unit disk and $\{z_n\} \to 1$ as $n \to \infty$. To obtain our conclusion, we show

$$\lim_{n \to \infty} (1 - |z_n|^2) \left| \exp \frac{s(z_n + 1)}{z_n - 1} \right| \frac{2s}{|1 - z_n|^2} \neq 0.$$

First observe that the map $\psi(z) = (z+1)/(z-1)$ is its own inverse and hence $\psi(z_n) = x_0 + in$ for each $n \in \mathbb{N}$. Thus

$$\left| \exp \frac{s(z_n + 1)}{z_n - 1} \right| = \left| \exp(sx_0 + isn) \right| = e^{sx_0} > 0.$$

Substituting,

$$\lim_{n \to \infty} (1 - |z_n|^2) \left| \exp \frac{s(z_n + 1)}{z_n - 1} \right| \frac{2s}{|1 - z_n|^2} = \lim_{n \to \infty} e^{sx_0} (1 - |z_n|^2) \frac{2s}{|1 - z_n|^2}.$$

Next,

$$1 - |z_n|^2 = \frac{-4x_0}{(x_0 - 1)^2 + n^2}$$

and

$$|1 - z_n|^2 = \frac{4}{(x_0 - 1)^2 + n^2}.$$

Thus

$$\lim_{n \to \infty} e^{sx_0}\left(1 - |z_n|^2\right)\frac{2s}{|1 - z_n|^2} = \lim_{n \to \infty} e^{sx_0}\left(\frac{-4x_0}{(x_0 - 1)^2 + n^2}\right)\left(\frac{s\left((x_0 - 1)^2 + n^2\right)}{2}\right)$$
$$= \lim_{n \to \infty} (-2sx_0)e^{sx_0} > 0$$

and hence f_s is not in $\mathcal{B}_0$ for $s > 0$. □

For the little Bloch space, we leave the reader with the following question.

Question 1. For φ a parabolic or hyperbolic automorphism, what is the spectrum of C_φ on the little Bloch space?

5B. *Analytic functions of vanishing mean oscillation.* Like the Bloch space, the polynomials are contained in BMOA, but they are not dense in BMOA. We denote by VMOA the closure of the polynomials in $\|\cdot\|_{\text{BMOA}}$. VMOA is the space of analytic functions with vanishing mean oscillation on $\partial\mathbb{D}$, formally defined as the functions $f \in$ BMOA such that

$$\lim_{|a| \to 1} \left\| f \circ \tau_a - f(a) \right\|_{H^2} = 0.$$

By Corollary 4.2 of [Laitila 2009], C_φ is bounded on VMOA if and only if $\varphi \in$ VMOA. So every automorphism induces a bounded composition operator on VMOA. By the same argument as in Section 4A3, the spectral radius of C_φ induced by a disk automorphism is 1. Thus property (ii) of $\mathcal{X}$ is satisfied. Since VMOA is a subspace of the little Bloch space (see [Gallardo-Gutiérrez et al. 2013]), it follows that VMOA does not satisfy property (i), a corollary of Theorem 5.1.

Corollary 5.2. *The functions f_s and f_t, for $s > 0$ and $t \neq 0$, are not contained in* VMOA.

For VMOA, we leave the reader with the following question.

Question 2. For φ parabolic or hyperbolic automorphism, what is the spectrum of C_φ on VMOA?

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