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Some nonsimple modules for centralizer algebras of the symmetric group

Craig Dodge, Harald Ellers, Yukihide Nakada and Kelly Pohland

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James classified the simple modules over the group algebra $k\Sigma_n$ using modules denoted D^λ, where λ is a partition of n. In particular, he showed that D^λ is simple or zero for every partition λ and, furthermore, that for every simple $k\Sigma_n$-module S there exists a partition λ such that $D^\lambda \cong S$. This paper is an extension of a paper of Dodge and Ellers in which they studied analogous modules $\mathcal{D}^{(\lambda,\mu)}$ over the centralizer algebra $k\Sigma_n^{\Sigma_l}$, where λ is a partition of n and μ a partition of l. For every positive prime p we find counterexamples to their conjecture that the $k\Sigma_n^{\Sigma_l}$-modules $\mathcal{D}^{(\lambda,\mu)}$ are always simple or zero, where k is a field of characteristic p. We also study the relationship between $\mathcal{D}^{(\lambda,\mu)}$ and $\mathrm{Hom}_{k\Sigma_l}(D^\mu, \mathrm{res}^{\Sigma_n}_{\Sigma_l} D^\lambda)$ in special cases.

1. Introduction

Let n be a positive integer and k an algebraically closed field of characteristic p. James [1978] studied simple modules over the group algebra $k\Sigma_n$, where Σ_n is the symmetric group on n letters. He defined for each partition $\lambda \vdash n$ the permutation module M^λ with basis consisting of all λ-tabloids. The *Specht module* S^λ is defined to be the submodule of M^λ generated by polytabloids. The kernel intersection theorem can be used to characterize S^λ as

$$S^\lambda = \bigcap \{\ker\varphi \mid \varphi : M^\lambda \to M^{\lambda'},\ \lambda' \rhd \lambda\},$$

where $\lhd$ is the dominance order on partitions [James 1998, p. 97]. He also defined a bilinear form on M^λ using the set of tabloids as an orthonormal basis and proved in [James 1998, 2.2] using the characterization of S^λ above that

$$S^{\lambda\perp} = \sum \{\mathrm{im}\,\varphi \mid \varphi : M^{\lambda'} \to M^\lambda,\ \lambda' \rhd \lambda\}.$$

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James then defined the module D^λ by

$$D^\lambda = S^\lambda/(S^\lambda \cap S^{\lambda\perp})$$

and proved that D^λ is always zero or simple, that $D^\lambda \neq 0$ if and only if λ is p-regular, and that all simple $k\Sigma_n$-modules occur exactly once as λ runs through all p-regular partitions.

Dodge and Ellers applied similar ideas to study representations of centralizer algebras of the symmetric group. In general, let G be a finite group, let H be a subgroup of G, and let k be an algebraically closed field of characteristic p. The centralizer algebra kG^H is defined by

$$kG^H = \{a \in kG \mid ah = ha,\ \forall h \in H\}.$$

Given a kG-module M and a kH-module N we can construct a kG^H-module in a very natural way. The space

$$\operatorname{Hom}_{kH}(N, \operatorname{res}^G_H M)$$

can be given a natural action by kG^H in the following manner:

$$(a\varphi)(t) = a(\varphi(t))$$

for all $a \in kG^H$, $t \in N$ and $\varphi \in \operatorname{Hom}_{kH}(N, \operatorname{res}^G_H M)$.

Dodge and Ellers [2016] studied the representation theory of $k\Sigma_n{}^{\Sigma_l}$, where Σ_n is the symmetric group on n letters, $l \le n$, and Σ_l is identified with a subgroup of Σ_n permuting the first l letters. Here we review the notation and definitions they used. Let $\mu \vdash l$ and $\lambda \vdash n$. Define a dominance relation on such partition pairs (λ, μ) by

$$(\lambda', \mu') \rhd (\lambda, \mu) \quad \text{if } \lambda' \rhd \lambda \text{ or } (\lambda' = \lambda \text{ and } \mu' \rhd \mu).$$

Define the $k\Sigma_n^{\Sigma_l}$-module

$$\mathcal{M}^{(\lambda,\mu)} = (M^\mu, M^\lambda).$$

This module is designed to be analogous to the permutation modules of the symmetric group. They then define the modules

$$\begin{aligned}
\mathcal{S}^{(\lambda,\mu)} &= \bigcap\{\ker\varphi \mid \varphi : \mathcal{M}^{(\lambda,\mu)} \to \mathcal{M}^{(\lambda',\mu')},\ (\lambda',\mu') \rhd (\lambda,\mu)\},\\
\mathcal{S}^{(\lambda,\mu)\perp} &= \sum\{\operatorname{im}\varphi \mid \varphi : \mathcal{M}^{(\lambda',\mu')} \to \mathcal{M}^{(\lambda,\mu)},\ (\lambda',\mu') \rhd (\lambda,\mu)\},\\
\mathcal{D}^{(\lambda,\mu)} &= \mathcal{S}^{(\lambda,\mu)}/(\mathcal{S}^{(\lambda,\mu)} \cap \mathcal{S}^{(\lambda,\mu)\perp}).
\end{aligned}$$

In the above definitions φ is a $k\Sigma_n^{\Sigma_l}$-module homomorphism. Note that a bilinear form on $\mathcal{M}^{\lambda,\mu}$ has not been defined; the notation for the module $\mathcal{S}^{(\lambda,\mu)\perp}$ was chosen to highlight its similarity to $S^{\lambda\perp}$ in [James 1978]. Paralleling the approach to the representation theory of $k\Sigma_n$ in [James 1978], Dodge and Ellers [2016] proved that if $\lambda \vdash n$ and $\mu \vdash l$, and $l < p$, then $\mathcal{D}^{(\lambda,\mu)}$ is either simple or zero, in agreement with James' result. In addition, they showed that

$$\mathcal{D}^{(\lambda,\mu)} \cong \operatorname{Hom}_{k\Sigma_l}(D^{\mu}, \operatorname{res}_{\Sigma_l}^{\Sigma_n} D^{\lambda})$$

under the same conditions. They conjectured that these facts hold in general when λ and μ are p-regular. In this paper we compute explicit examples to test their conjectures.

For all positive prime p, we explicitly compute the structures of

$$\operatorname{Hom}_{\Sigma_p}(D^{(p)}, \operatorname{res}_{\Sigma_p}^{\Sigma_{p+3}} D^{(p+2,1)})$$

in Sections 3, 4, and 5 and the structures of $\mathcal{D}^{((p+2,1),(p))}$ in Sections 6 and 7. In particular, we show that the space $\operatorname{Hom}_{\Sigma_p}(D^{(p)}, \operatorname{res}_{\Sigma_p}^{\Sigma_{p+3}} D^{(p+2,1)})$ is neither simple nor zero and prove the following characterizations of $\mathcal{D}^{((p+2,1),(p))}$:

Proposition 1.1. *Let k be a field of characteristic p, where $p \neq 3$. Then*

$$\mathcal{D}^{((p+2,1),(p))} \cong \operatorname{Hom}_{k\Sigma_p}(D^{(p)}, \operatorname{res}_{\Sigma_p}^{\Sigma_{p+3}} D^{(p+2,1)})$$

as $k\Sigma_{p+3}^{\Sigma_p}$-modules and therefore $\mathcal{D}^{((p+2,1),(p))}$ is neither simple nor zero.

Proposition 1.2. *Let k be a field of characteristic 3. Then*

$$\mathcal{D}^{((5,1),(3))} \cong \operatorname{Hom}_{k\Sigma_3}(D^{(3)}, \operatorname{res}_{\Sigma_3}^{\Sigma_6} D^{(5,1)})/L$$

as $k\Sigma_6^{\Sigma_3}$-modules, where L is a submodule isomorphic to $\mathcal{M}^{((6),(3))}$. Moreover, $\mathcal{D}^{((5,1),(3))}$ is neither simple nor zero.

Thus neither is simple nor zero for any characteristic p, contrary to the conjectures of Dodge and Ellers. In addition, this shows that the isomorphism conjectured above does not hold in characteristic 3. Finally, in Section 9 we show that in characteristic 2 there is no ordering on pairs of partitions for which the conjectures hold when $n = 5$ and $l = 2$.

2. $\mathcal{M}^{((p+3),\mu)}$ in arbitrary characteristic p

We consider the relationship between the spaces $\operatorname{Hom}_{k\Sigma_p}(D^{(p)}, \operatorname{res}_{\Sigma_p}^{\Sigma_{p+3}} D^{(p+2,1)})$ and $\mathcal{D}^{((p+2,1),(p))}$ when p is a positive prime. Since the pairs of partitions (λ, μ) such that $(\lambda, \mu) \rhd ((p+2, 1), (p))$ are those of the form $((p+3), \mu)$, where $\mu \vdash p$, we first study the modules corresponding to such pairs.

Proposition 2.1. *Let k be a field of characteristic p. Then all modules of the form $\mathcal{M}^{((p+3),\mu)}$, where $\mu \vdash p$, are one-dimensional and mutually isomorphic as $k\Sigma_{p+3}^{\Sigma_p}$-modules.*

Proof. Fix a partition $\mu \vdash p$ and a nonzero tabloid $y_0 \in M^\mu$. From [James 1978, Theorem 13.19] we know that $\mathcal{M}^{((p+3),\mu)}$ is nonzero, so we may choose a nonzero $f \in \mathcal{M}^{((p+3),\mu)}$. Since $f(y_0) \in M^{(p+3)} \cong k$, we have

$$f(y_0) = \sigma f(y_0) = f(\sigma y_0)$$

for all $\sigma \in \Sigma_p$, and since M^μ is a cyclic $k\Sigma_p$-module generated by any nonzero tabloid, it follows that $f(y) = f(y_0)$ for any tabloid $y \in M^\mu$. Thus if $f_0 \in \mathcal{M}^{((p+3),\mu)}$ is defined by $f_0(y_0) = 1$ then $\mathcal{M}^{((p+3),\mu)} = \operatorname{span}\{f_0\}$ as a $k\Sigma_{p+3}^{\Sigma_p}$-module. In particular, it is one-dimensional.

We now describe a generating set for $k\Sigma_{p+3}^{\Sigma_p}$. From [Kleshchev 2005, Proposition 2.1.1] we have

$$k\Sigma_{p+3}^{\Sigma_p} = \langle Z(k\Sigma_p),\ (p{+}1\ \ p{+}2),\ (p{+}1\ \ p{+}2\ \ p{+}3),\ L_{p+1},\ L_{p+2},\ L_{p+3}\rangle,$$

where $Z(k\Sigma_p)$ is the center of $k\Sigma_p$ and L_k is the *Jucys–Murphy element* defined as

$$L_k = \sum_{1 \le m < k} (m\ k).$$

It is well known that $Z(k\Sigma_p)$ is spanned by elements $s_\tau \in k\Sigma_p$ for τ a partition of p, where s_τ denotes the sum of all elements in Σ_p with cycle type corresponding to the partition τ. Let K_τ denote the conjugacy class corresponding to the partition τ. Since any element of Σ_{p+3} acts trivially on the codomain of $\operatorname{Hom}_{k\Sigma_p}(D^\mu, \operatorname{res}_{\Sigma_p}^{\Sigma_{p+3}} D^{(p+3)}) = \mathcal{M}^{((p+3),\mu)}$, we deduce that the action of the module is described by the table

	f_0
s_τ	$\lvert K_\tau \rvert f_0$
$(p{+}1\ \ p{+}2)$	f_0
$(p{+}1\ \ p{+}2\ \ p{+}3)$	f_0
L_{p+1}	0
L_{p+2}	f_0
L_{p+3}	$2f_0$

Since our choice of μ was arbitrary, it follows that all modules of the form $\mathcal{M}^{((p+3),\mu)}$ are mutually isomorphic, as claimed. □

3. $\operatorname{Hom}_{k\Sigma_2}(D^{(2)}, \operatorname{res}_{\Sigma_2}^{\Sigma_5} D^{(4,1)})$ in characteristic 2

Next we determine the structure of $\operatorname{Hom}_{k\Sigma_2}(D^{(2)}, \operatorname{res}_{\Sigma_2}^{\Sigma_5} D^{(4,1)})$. In this and all following sections, when $D^\lambda \cong S^\lambda$ we will identify a coset in D^λ with its corresponding element in S^λ as an abuse of notation. We first note that $D^{(2)}$ is trivial by definition.

We have that $M^{(4,1)}$ is spanned by

$$\left\{ \frac{\overline{2\ 3\ 4\ 5}}{\underline{1}},\ \frac{\overline{1\ 3\ 4\ 5}}{\underline{2}},\ \frac{\overline{1\ 2\ 4\ 5}}{\underline{3}},\ \frac{\overline{1\ 2\ 3\ 5}}{\underline{4}},\ \frac{\overline{1\ 2\ 3\ 4}}{\underline{5}} \right\}.$$

We will denote these tabloids by x_1, x_2, x_3, x_4, x_5, respectively. Since $x_2, \dots, x_5$ correspond to the standard tableau in $M^{(4,1)}$, we know from [James 1978, Theorem 8.4] that the Specht module $S^{(4,1)}$ has basis $\{x_2 - x_1,\ x_3 - x_1,\ x_4 - x_1,\ x_5 - x_1\}$. For simplicity we denote each element in this basis by $c_i = x_i - x_1$ for $2 \le i \le 5$. To compute $S^{(4,1)\perp}$, note that since the map $\mathcal{M}^{((5),(2))} \to \mathcal{M}^{((4,1),(2))}$ defined by $1 \mapsto x_1 + x_2 + x_3 + x_4 + x_5$ is a $k\Sigma_5^{\Sigma_2}$-module homomorphism, it follows that $x_1 + x_2 + x_3 + x_4 + x_5 \in S^{(4,1)\perp}$. Moreover, since $S^{(4,1)}$ is four-dimensional we can conclude from [James 1978, 1.3] that $S^{(4,1)\perp}$ has dimension 1 and hence that $S^{(4,1)\perp}$ has basis $\{x_1 + x_2 + x_3 + x_4 + x_5\}$. Notice that $S^\lambda \cap S^{\lambda\perp} = 0$, so $D^{(4,1)} \cong S^{(4,1)}$. Now, fix $z \in D^{(2)}$ with $z \neq 0$, and let

$$f : D^{(2)} \to \operatorname{res}_{\Sigma_2}^{\Sigma_5} D^{(4,1)}$$

be defined by

$$f(z) = a_2c_2 + a_3c_3 + a_4c_4 + a_5c_5.$$

Observe that since $D^{(2)} \cong k$ and k is a field of characteristic 2, we have $f \in \operatorname{Hom}_{k\Sigma_2}(D^{(2)}, \operatorname{res}_{\Sigma_2}^{\Sigma_5} D^{(4,1)})$ if and only if $[(1) + (12)]f = 0$. Therefore, we need

$$\begin{aligned}
[(1) + (12)]f(z) &= a_2c_2 + a_3c_3 + a_4c_4 + a_5c_5 - a_2c_2 \\
&\qquad\qquad + a_3(c_3 - c_2) + a_4(c_4 - c_2) + a_5(c_5 - c_2) \\
&= -a_3c_2 - a_4c_2 - a_5c_2 \\
&= -(a_3 + a_4 + a_5)c_2 = 0.
\end{aligned}$$

Thus f is a $k\Sigma_2$-module homomorphism exactly when $a_3 + a_4 + a_5 = 0$. Hence f has the form

$$\begin{aligned}
f(z) &= a_2c_2 + a_3c_3 + a_4c_4 + (-a_3 - a_4)c_5 \\
&= a_2c_2 + a_3(c_3 - c_5) + a_4(c_4 - c_5).
\end{aligned}$$

Therefore a basis for $\operatorname{Hom}_{k\Sigma_2}(D^{(2)}, \operatorname{res}_{\Sigma_2}^{\Sigma_5} D^{(4,1)})$ is

$$\alpha(z) = c_2 = x_1 + x_2, \quad \beta(z) = c_3 - c_5 = x_3 - x_5, \quad \gamma(z) = c_4 - c_5 = x_4 - x_5.$$

Next we examine how $k\Sigma_5^{\Sigma_2}$ acts on $\{\alpha, \beta, \gamma\}$. As our generators for $k\Sigma_5^{\Sigma_2}$, we will be using the generating set from Proposition 2.1, namely

$$k\Sigma_5^{\Sigma_2} = \langle (1),\ (12),\ (34),\ (345),\ L_3,\ L_4,\ L_5 \rangle.$$

The action of the module is described by the table

	α	β	γ
(12)	α	β	γ
(34)	α	γ	β
(345)	α	$\gamma-\beta$	$-\beta$
L_3	α	α	0
L_4	0	γ	$\alpha+\beta$
L_5	0	$-\alpha-\gamma$	$-\alpha-\beta$

Thus we can see that span$\{\alpha\}$ is a submodule of $\mathrm{Hom}_{k\Sigma_2}(D^{(2)}, \mathrm{res}^{\Sigma_5}_{\Sigma_2} D^{(4,1)})$. Comparing this table with that on page 880 describing $\mathcal{M}^{((5),(2))}$, we see that span$\{\alpha\} \ncong \mathcal{M}^{((5),(2))}$. The quotient by this one-dimensional submodule has basis $\{\bar\beta, \bar\gamma\}$, and the action of the module is described by the table

	$\bar\beta$	$\bar\gamma$
(12)	$\bar\beta$	$\bar\gamma$
(34)	$\bar\gamma$	$\bar\beta$
(345)	$\bar\gamma-\bar\beta$	$-\bar\beta$
L_3	$\bar 0$	$\bar 0$
L_4	$\bar\gamma$	$\bar\beta$
L_5	$-\bar\gamma$	$-\bar\beta$

We will show that this is a simple two-dimensional module. If this is not simple, it must contain a one-dimensional submodule. We leave to the reader the easy confirmation that span$\{\bar\beta\}$ and span$\{\bar\gamma\}$ are not submodules. So suppose $a_0, a_1 \neq 0$ and assume for contradiction that the one-dimensional k-vector space span$\{a_0\bar\beta + a_1\bar\gamma\}$ is a submodule. It follows then that

$$((34)+(345))(a_0\bar\beta + a_1\bar\gamma) \in \mathrm{span}\{a_0\bar\beta + a_1\bar\gamma\},$$

so we have

$$\begin{aligned}((34)+(345))(a_0\bar\beta + a_1\bar\gamma) &= a_0(34)\bar\beta + a_1(34)\bar\gamma + a_0(345)\bar\beta + a_1(345)\bar\gamma\\ &= a_0\bar\gamma + a_1\bar\beta + a_0\bar\gamma - a_0\bar\beta - a_1\bar\beta\\ &= -a_0\bar\beta.\end{aligned}$$

Thus, it must be that $a_0 = 0$, a contradiction. Thus, for all $a_0, a_1 \in k$, we have that span$\{a_0\bar\beta + a_1\bar\gamma\}$ is not a submodule of $\mathrm{Hom}_{k\Sigma_2}(D^{(2)}, \mathrm{res}^{\Sigma_5}_{\Sigma_2} D^{(4,1)})/\,\mathrm{span}\{\alpha\}$, so the quotient is a two-dimensional simple module.

4. $\mathrm{Hom}_{k\Sigma_3}(D^{(3)}, \mathrm{res}^{\Sigma_6}_{\Sigma_3} D^{(5,1)})$ in characteristic 3

Let k be a field of characteristic 3. We now determine the structure of

$$\mathrm{Hom}_{k\Sigma_3}(D^{(3)}, \mathrm{res}^{\Sigma_6}_{\Sigma_3} D^{(5,1)}).$$

Notice that $D^{(3)}$ is trivial. We have that $M^{(5,1)}$ is spanned by

$$\left\{ \frac{\overline{2\ 3\ 4\ 5\ 6}}{\underline{1}}, \frac{\overline{1\ 3\ 4\ 5\ 6}}{\underline{2}}, \frac{\overline{1\ 2\ 4\ 5\ 6}}{\underline{3}}, \frac{\overline{1\ 2\ 3\ 5\ 6}}{\underline{4}}, \frac{\overline{1\ 2\ 3\ 4\ 6}}{\underline{5}}, \frac{\overline{1\ 2\ 3\ 4\ 5}}{\underline{6}} \right\}.$$

We will again denote these standard tabloids by $x_1, x_2, x_3, x_4, x_5, x_6$, respectively. Since $x_2, \dots, x_6$ correspond to the standard tableau in $M^{(5,1)}$, we know from [James 1978, Theorem 8.4] that the Specht module $S^{(5,1)}$ is spanned by $\{x_2 - x_1,\ x_3 - x_1,\ x_4 - x_1,\ x_5 - x_1,\ x_6 - x_1\}$. For simplicity we denote each element in this basis by $c_i = x_i - x_1$ for $2 \le i \le 6$. To compute $S^{(5,1)\perp}$, note that since the map $\mathcal{M}^{((6),(2))} \to \mathcal{M}^{((5,1),(2))}$ defined by $1 \mapsto x_1 + x_2 + x_3 + x_4 + x_5 + x_6$ is a $k\Sigma_6^{\Sigma_3}$-module homomorphism, it follows that $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \in S^{(5,1)\perp}$. Moreover, since $S^{(5,1)}$ is five-dimensional, we can conclude from [James 1978, 1.3] that $S^{(5,1)\perp}$ has dimension 1 and hence that $S^{(5,1)\perp}$ has basis $\{x_1 + x_2 + x_3 + x_4 + x_5 + x_6\}$. From this, it is clear that $S^{(5,1)} \cap S^{(5,1)\perp} = 0$, so $D^{(5,1)} \cong S^{(5,1)}$. We now fix $z \in D^{(3)}$ with $z \neq 0$ and let

$$f : D^{(3)} \to \mathrm{res}^{\Sigma_6}_{\Sigma_3} D^{(5,1)}$$

be defined by

$$f(z) = a_2c_2 + a_3c_3 + a_4c_4 + a_5c_5 + a_6c_6.$$

Since Σ_3 is generated by (12) and (13), we have $f \in \mathrm{Hom}_{k\Sigma_3}(D^{(3)}, \mathrm{res}^{\Sigma_6}_{\Sigma_3} D^{(5,1)})$ exactly when $f(z) = (12)f(z)$ and $f(z) = (13)f(z)$. Thus we must have

$$\begin{aligned}(12)f(z) &= a_2(-c_2) + a_3(c_3 - c_2) + a_4(c_4 - c_2) + a_5(c_5 - c_2) + a_6(c_6 - c_2)\\ &= (-a_2 - a_3 - a_4 - a_5 - a_6)c_2 + a_3c_3 + a_4c_4 + a_5c_5 + a_6c_6,\end{aligned}$$

so $a_2 = -a_2 - a_3 - a_4 - a_5 - a_6$. Similarly,

$$\begin{aligned}(13)f(z) &= a_2(c_2 - c_3) + a_3(-c_3) + a_4(c_4 - c_3) + a_5(c_5 - c_3) + a_6(c_6 - c_3)\\ &= a_2c_2 + (-a_2 - a_3 - a_4 - a_5 - a_6)c_3 + a_4c_4 + a_5c_5 + a_6c_6,\end{aligned}$$

so $a_3 = -a_2 - a_3 - a_4 - a_5 - a_6$. Thus $a_2 = a_3$, and since $a_2 = -a_2 - a_3 - a_4 - a_5 - a_6$ and k has characteristic 3, we get that $0 = a_4 + a_5 + a_6$. Consequently,

$$f(z) = a_2c_2 + a_3c_3 + a_4c_4 + a_5c_5 + a_6c_6 = a_2(c_2 + c_3) + a_4(c_4 - c_6) + a_5(c_5 - c_6).$$

Therefore, we get that $\mathrm{Hom}_{k\Sigma_3}(D^{(3)}, \mathrm{res}^{\Sigma_6}_{\Sigma_3} D^{(5,1)})$ is spanned by $\{\alpha, \beta, \gamma\}$, where

$$\alpha(z) = c_2 + c_3 = x_1 + x_2 + x_3, \quad \beta(z) = c_4 - c_6 = x_4 - x_6, \quad \gamma(z) = c_5 - c_6 = x_5 - x_6.$$

The table describing the action on this basis is

	α	β	γ
$(12)+(13)+(23)$	0	0	0
$(123)+(132)$	2α	2β	2γ
(45)	α	γ	β
(456)	α	$2\beta+\gamma$	2β
L_4	2α	α	0
L_5	0	γ	$\alpha+\beta$
L_6	α	$2\alpha+2\gamma$	$2\alpha+2\beta$

From this table we can deduce that span$\{\alpha\}$ and span$\{\alpha+\beta+\gamma\}$ are submodules of $\mathrm{Hom}_{k\Sigma_3}(D^{(3)}, \mathrm{res}_{\Sigma_3}^{\Sigma_6} D^{(5,1)})$. The table describing the action on span$\{\alpha+\beta+\gamma\}$ is

	$\alpha+\beta+\gamma$
$(12)+(13)+(23)$	0
$(123)+(132)$	$2(\alpha+\beta+\gamma)$
(45)	$\alpha+\beta+\gamma$
(456)	$\alpha+\beta+\gamma$
L_4	0
L_5	$\alpha+\beta+\gamma$
L_6	$2(\alpha+\beta+\gamma)$

Comparing these tables to that on page 880, we see that span$\{\alpha\} \ncong \mathcal{M}^{((6),(3))}$ and span$\{\alpha+\beta+\gamma\} \cong \mathcal{M}^{((6),(3))}$. The corresponding quotient

$$\mathrm{Hom}_{k\Sigma_3}(D^{(3)}, \mathrm{res}_{\Sigma_3}^{\Sigma_6} D^{(5,1)})/(\mathrm{span}\{\alpha\} \oplus \mathrm{span}\{\alpha+\beta+\gamma\})$$

is one-dimensional with basis $\{\bar{\beta}\}$ and the table describing the action on this basis is

	$\bar{\beta}$
$(12)+(13)+(23)$	$\bar{0}$
$(123)+(132)$	$2\bar{\beta}$
(45)	$2\bar{\beta}$
(456)	$\bar{\beta}$
L_4	$\bar{0}$
L_5	$2\bar{\beta}$
L_6	$\bar{\beta}$

Note that $\{\bar{\beta}\}$ is isomorphic to neither span$\{a\}$ nor $\mathcal{M}^{((6),(3))}$.

5. $\mathbf{Hom}_{k\Sigma_p}(D^{(p)}, \mathbf{res}_{\Sigma_p}^{\Sigma_{p+3}} D^{(p+2,1)})$ in characteristic p

Let $p \geq 5$ be prime, and let k be a field of characteristic p. We determine the structure of

$$\mathrm{Hom}_{k\Sigma_p}(D^{(p)}, \mathrm{res}_{\Sigma_p}^{\Sigma_{p+3}} D^{(p+2,1)}).$$

Notice that $D^{(p)}$ is trivial. Using notation similar to that in Sections 3 and 4, $M^{(p+2,1)}$ is spanned by $\{x_1, \ldots, x_{p+3}\}$. From computations entirely analogous to those in characteristics 2 and 3, we know that the Specht module $S^{(p+2,1)}$ has basis $\{c_2, \ldots, c_{p+3}\}$, where $c_i = x_i - x_1$ for $2 \leq i \leq p+3$, and that $S^{(p+2,1)\perp}$ has dimension 1 with basis $\{x_1+x_2+\cdots+x_{p+3}\}$. Consequently $S^{(p+2,1)} \cap S^{(p+2,1)\perp} = 0$ and $D^{(p+2,1)} \cong S^{(p+2,1)}$.

Fix $z \in D^{(p)}$ with $z \neq 0$. Let $f : D^{(p)} \to \mathrm{res}_{\Sigma_p}^{\Sigma_{p+3}} D^{(p+2,1)}$ be defined by $f(z) = a_2c_2 + a_3c_3 + \cdots + a_{p+3}c_{p+3}$. Notice that

$$f \in \mathrm{Hom}_{k\Sigma_p}(D^{(p)}, \mathrm{res}_{\Sigma_p}^{\Sigma_{p+3}} D^{(p+2,1)})$$

if and only if $f(z) = (12)f(z) = (13)f(z) = \cdots = (1\ p)f(z)$ since Σ_p is generated by $(12), \ldots, (1\ p)$. Since

$$(1\ i)f(z) = a_2(c_2 - c_i) + a_3(c_3 - c_i) + \cdots + a_i(-c_i) + \cdots + a_{p+3}(c_{p+3} - c_i)$$

and

$$f(z) = a_2c_2 + a_3c_3 + \cdots + a_{p+3}c_{p+3},$$

it follows that for all $2 \leq i \leq p$ we must have

$$\begin{aligned} a_2(c_2 - c_i) + a_3(c_3 - c_i) + \cdots + a_i(-c_i) + \cdots + a_{p+3}(c_{p+3} - c_i) \\ = a_2c_2 + a_3c_3 + \cdots + a_{p+3}c_{p+3}. \end{aligned}$$

Simplifying, we have

$$a_ic_i = \left(-\sum_{k=2}^{p+3} a_k\right)c_i,$$

so $a_i = -a_2 - a_3 - \cdots - a_{p+3}$. Since this holds for arbitrary $2 \leq i \leq p$, we get that $a_2 = a_3 = \cdots = a_p$. In particular, substituting this into the above equality with $i = 2$ we have

$$a_{p+1} + a_{p+2} + a_{p+3} = -a_2 - a_2 - a_3 - \cdots - a_p = -pa_2 = 0$$

since k has characteristic p. Hence f must have the form

$$\begin{aligned} f(z) &= a_2c_2 + a_3c_3 + \cdots + a_{p+3}c_{p+3} \\ &= a_2(c_2 + c_3 + \cdots + c_p) + a_{p+1}(c_{p+1} - c_{p+3}) + a_{p+2}(c_{p+2} - c_{p+3}). \end{aligned}$$

From this, we can see that a basis for $\operatorname{Hom}_{k\Sigma_p}(D^{(p)}, \operatorname{res}^{\Sigma_{p+3}}_{\Sigma_p} D^{(p+2,1)})$ is $\{\alpha, \beta, \gamma\}$, where

$$\begin{aligned} \alpha(z) &= c_2 + \cdots + c_p = x_1 + x_2 + \cdots + x_p, \\ \beta(z) &= c_{p+1} - c_{p+3} = x_{p+1} - x_{p+3}, \\ \gamma(z) &= c_{p+2} - c_{p+3} = x_{p+2} - x_{p+3}. \end{aligned}$$

Recall from Proposition 2.1 that for a partition τ, we let s_τ denote the sum of all elements in Σ_p with cycle type corresponding to τ and let K_τ denote the conjugacy class corresponding to τ. Notice that since each element of Σ_p permutes $\{1, \ldots, p\}$, we can conclude that $\sigma\alpha = \alpha$, $\sigma\beta = \beta$, and $\sigma\gamma = \gamma$ for any $\sigma \in \Sigma_p$. From this we can derive the action of $k\Sigma_{p+1}^{\Sigma_p}$ on this basis, and the table describing this is

	α	β	γ
s_τ	$\lvert K_\tau\rvert\alpha$	$\lvert K_\tau\rvert\beta$	$\lvert K_\tau\rvert\gamma$
$(p{+}1\ \ p{+}2)$	α	γ	β
$(p{+}1\ \ p{+}2\ \ p{+}3)$	α	$\gamma-\beta$	$-\beta$
L_{p+1}	$-\alpha$	α	0
L_{p+2}	0	γ	$\alpha+\beta$
L_{p+3}	α	$-\alpha-\gamma$	$-\alpha-\beta$

Notice that $\operatorname{span}\{\alpha\}$ is a submodule. Comparing its action to the action described in the table on page 880 we see that $\operatorname{span}\{\alpha\} \ncong \mathcal{M}^{((p+3),(p))}$. The table describing the action on the corresponding quotient module is

	$\bar{\beta}$	$\bar{\gamma}$
s_τ	$\lvert K_\tau\rvert\bar{\beta}$	$\lvert K_\tau\rvert\bar{\gamma}$
$(p{+}1\ \ p{+}2)$	$\bar{\gamma}$	$\bar{\beta}$
$(p{+}1\ \ p{+}2\ \ p{+}3)$	$\bar{\gamma}-\bar{\beta}$	$-\bar{\beta}$
L_{p+1}	$\bar{0}$	$\bar{0}$
L_{p+2}	$\bar{\gamma}$	$\bar{\beta}$
L_{p+3}	$-\bar{\gamma}$	$-\bar{\beta}$

We now show that this quotient is simple. Since the quotient is two-dimensional, we can show that it is simple by showing that there are no one-dimensional submodules. We leave it to the reader to confirm that $\operatorname{span}\{\bar{\beta}\}$ and $\operatorname{span}\{\bar{\gamma}\}$ are not submodules. So let $a_0, a_1 \neq 0$ and suppose for contradiction that $\operatorname{span}\{a_0\bar{\beta} + a_1\bar{\gamma}\}$ is a submodule. Then

$$\begin{aligned} \big((p{+}1\ \ p{+}2) + (p{+}1\ \ p{+}2\ \ p{+}3)\big)(a_0\bar{\beta} + a_1\bar{\gamma}) &= a_0\bar{\gamma} + a_1\bar{\beta} + a_0(\bar{\gamma} - \bar{\beta}) + a_1(-\bar{\beta}) \\ &= 2a_0\bar{\gamma} - a_0\bar{\beta}, \end{aligned}$$

so for some $c \in k$, we have $ca_0 = -a_0$ and $ca_1 = 2a_0$. Thus, $c = -1$ and $a_1 = -2a_0$. Similarly,

$$\big((p{+}1\ \ p{+}2) - (p{+}1\ \ p{+}2\ \ p{+}3)\big)(a_0\bar{\beta} + a_1\bar{\gamma}) = a_0\bar{\gamma} + a_1\bar{\beta} - a_0(\bar{\gamma} - \bar{\beta}) - a_1(-\bar{\beta}) \\ = (a_0 + 2a_1)\bar{\beta},$$

so $a_0 + 2a_1 = 0$ since $a_1 \neq 0$. Thus, $a_0 = -2a_1$, and since $a_1 = -2a_0$, we must have $a_1 = 4a_1$. For char $k \neq 3$ this is a contradiction. Hence, span$\{a_0\bar{\beta} + a_1\bar{\gamma}\}$ is not a submodule for all $a_0, a_1 \in k$ and the quotient is a two-dimensional simple module.

6. $\mathcal{D}^{((p+2,1),(p))}$ in characteristic $p \neq 3$

In this section we compute the structure of $\mathcal{D}^{((p+2,1),(p))}$ over a field of characteristic p when $p \neq 3$ and prove Proposition 1.1. To compute the structure of $\mathcal{D}^{((p+2,1),(p))}$ we will need the following lemma.

Lemma 6.1. *Let A be a finite-dimensional k-algebra, let $S_1, \dots, S_n$ be simple A-modules, and suppose K and L are A-modules with L having no S_i as a composition factor and K having every S_i as a composition factor. Let $\varphi : K \to L$ be an A-module homomorphism, and let M be minimal among submodules of K having every S_i as a composition factor. Then $M \subseteq \ker\varphi$.*

Proof. Suppose, for contradiction, that $M \not\subseteq \ker\varphi$. Then the inclusion $M \supset \ker\varphi \cap M$ is strict. Refine the filtration $M \supset (\ker\varphi \cap M) \supseteq 0$ into a composition series. Since M is minimal among submodules of K having every S_i as a composition factor, they cannot all belong to the composition series of $\ker\varphi \cap M$. Thus S_1, without loss of generality, is a composition factor of $M/(\ker\varphi \cap M)$. But

$$M/(\ker\varphi \cap M) \cong \varphi(M) \subseteq L,$$

so S_1 is a composition factor of L, a contradiction. □

The remainder of this section will be devoted to the proof of Proposition 1.1.

Suppose k has characteristic $p \neq 3$. We first compute a basis for $\mathcal{M}^{((p+2,1),(p))}$. For each $1 \le i \le p+3$, let t_i be the $(p+2, 1)$-tableau with i in the second row, and let $x_i = \{t_i\}$. Then $\{x_1, \dots, x_{p+3}\}$ forms a basis for $M^{(p+2,1)}$.

Let $0 \neq z \in M^{(p)}$ and let $f : M^{(p)} \to \operatorname{res}^{\Sigma_{p+3}}_{\Sigma_p} M^{(p+2,1)}$ be defined by

$$f(z) = \sum_{n=1}^{p+3} a_n x_n.$$

Since the transpositions $(1\ \ i)$ for $1 \le i \le p$ generate the group Σ_p, for f to be a Σ_p-homomorphism it is sufficient that $[(1) - (1\ \ i)]f = 0$ for all $2 \le i \le p$. Fix one such i. Then

$$\begin{aligned}[(1)-(1\ \ i)]f(z) &= \left(\sum_{n=1}^{p+3} a_n x_n\right) - \left(a_i x_1 + a_1 x_i + \sum_{n\neq 1,i} a_n x_n\right) \\ &= a_1x_1 + a_ix_i - a_ix_1 - a_1x_i \\ &= (a_1-a_i)(x_1-x_i).\end{aligned}$$

Thus we must have $a_1 = a_i$. Since this must be true for all $2 \le i \le p$, we deduce that $\mathcal{M}^{((p+2,1),(p))}$ has a basis $\{\alpha, \beta'_{p+1}, \beta'_{p+2}, \beta'_{p+3}\}$, where

$$\begin{aligned} \alpha(z) &= x_1 + \cdots + x_p, & \beta'_{p+2}(z) &= x_{p+2}, \\ \beta'_{p+1}(z) &= x_{p+1}, & \beta'_{p+3}(z) &= x_{p+3}. \end{aligned}$$

From this it is easy to check that

$$\begin{aligned} \alpha(z) &= x_1 + \cdots + x_p, & \beta_{p+2}(z) &= x_{p+2} - x_{p+3}, \\ \beta_{p+1}(z) &= x_{p+1} - x_{p+3}, & \beta_{p+3}(z) &= x_1 + \cdots + x_{p+3} \end{aligned}$$

is also a basis for $\mathcal{M}^{((p+2,1),(p))}$. The set $\{\alpha, \beta_{p+1}, \beta_{p+2}\}$ can be identified with the basis of $\mathrm{Hom}_{k\Sigma_p}(D^{(p)}, \mathrm{res}^{\Sigma_{p+3}}_{\Sigma_p} D^{(p+2,1)})$ found in Section 5, so we can deduce that

$$N = \mathrm{span}\{\alpha, \beta_{p+1}, \beta_{p+2}\}$$

is a subspace of $\mathcal{M}^{((p+2,1),(p))}$ isomorphic to $\mathrm{Hom}_{k\Sigma_p}(D^{(p)}, \mathrm{res}^{\Sigma_{p+3}}_{\Sigma_p} D^{(p+2,1)})$. Furthermore, the table describing the action on β_{p+3} is

	β_{p+3}
s_τ	$\lvert K_\tau\rvert \beta_{p+3}$
$(p+1\ \ p+2)$	β_{p+3}
$(p+1\ \ p+2\ \ p+3)$	β_{p+3}
L_{p+1}	0
L_{p+2}	β_{p+3}
L_{p+3}	$2\beta_{p+3}$

so $K = \mathrm{span}\{\beta_{p+3}\}$ is a submodule of $\mathcal{M}^{((p+2,1),(p))}$, and comparing this table to that on page 880 we see that it is isomorphic to $\mathcal{M}^{((p+3),(p))}$. Hence we have the direct sum decomposition

$$\mathcal{M}^{((p+2,1),(p))} = N \oplus K.$$

We now compute $\mathcal{D}^{((p+2,1),(p))}$. Since we know from Section 5 that the composition factors of $\mathrm{Hom}_{k\Sigma_p}(D^{(p)}, D^{(p+2,1)})$ consist of simple modules not isomorphic to $\mathcal{M}^{((p+3),(p))}$, it follows from Lemma 6.1 that $N \subseteq \ker\varphi$ for every

$\varphi : \mathcal{M}^{((p+2,1),(p))} \to \mathcal{M}^{((p+3),(p))}$, so that $N \subseteq \mathcal{S}^{((p+2,1),(p))}$. The reverse inclusion follows from the fact that N is the kernel of the projection of $\mathcal{M}^{((p+2,1),(p))}$ onto $K \cong \mathcal{M}^{((p+3),(p))}$. Hence

$$\mathcal{S}^{((p+2,1),(p))} = N.$$

We can deduce that $K \subseteq \mathcal{S}^{((p+2,1),(p))\perp}$ since K is the image of the map

$$\mathcal{M}^{((p+3),(p))} \to \mathcal{M}^{((p+2,1),(p))}$$

consisting of the isomorphism to K followed by injection. For the reverse inclusion, let $\varphi : \mathcal{M}^{((p+3),(p))} \to \mathcal{M}^{((p+2,1),(p))}$ be nonzero. Since $\operatorname{im}\varphi \cong \mathcal{M}^{((p+3),(p))}$ by Schur's lemma and K is the only composition factor of $\mathcal{M}^{((p+2,1),(p))}$ isomorphic to $\mathcal{M}^{((p+3),(p))}$, we must have $\operatorname{im}\varphi = K$. Consequently $K \subseteq \mathcal{S}^{((p+2,1),(p))\perp}$ by definition. Thus

$$\mathcal{S}^{((p+2,1),(p))\perp} = K.$$

Since $K \cap N = \{0\}$, we have

$$\mathcal{D}^{((p+2,1),(p))} = \mathcal{S}^{((p+2,1),(p))}/\{0\} \cong N \cong \operatorname{Hom}_{k\Sigma_p}(D^{(p)}, \operatorname{res}^{\Sigma_{p+3}}_{\Sigma_p} D^{(p+2,1)})$$

as claimed. We showed in Sections 3 and 5 that $\operatorname{Hom}_{k\Sigma_p}(D^{(p)}, \operatorname{res}^{\Sigma_{p+3}}_{\Sigma_p} D^{(p+2,1)})$ was neither simple nor zero for $p \neq 3$, and so the same must be true of $\mathcal{D}^{((p+2,1),(p))}$.

7. $\mathcal{D}^{((5,1),(3))}$ in characteristic 3

In this section we compute the structure of $\mathcal{D}^{((5,1),(3))}$ over a field of characteristic 3 and prove Proposition 1.2. This module has a structure different from the analogous modules $\mathcal{D}^{((p+2,1),(p))}$ in other characteristics because the spanning set $\{\alpha, \beta_{p+1}, \beta_{p+2}, \beta_{p+3}\}$ in $\mathcal{M}^{((p+2,1),(p))}$ fails to be linearly independent in characteristic 3. The remainder of this section will be devoted to the proof of Proposition 1.2.

The method used in the proof of Proposition 1.1 to find a basis for $\mathcal{M}^{((p+2,1),(p))}$ works when $p = 3$, so we have a basis

$$\alpha(z) = x_1 + x_2 + x_3, \quad \beta'_4(z) = x_4, \quad \beta'_5(z) = x_5, \quad \beta'_6(z) = x_6$$

for $\mathcal{M}^{((5,1),(3))}$. However, since

$$\begin{aligned}(x_1 + x_2 + x_3) + (x_4 - x_6) + (x_5 - x_6) &= x_1 + x_2 + x_3 + x_4 + x_5 - 2x_6 \\ &= x_1 + x_2 + x_3 + x_4 + x_5 + x_6\end{aligned}$$

in characteristic 3, the set $\{\alpha, \beta_{p+1}, \beta_{p+2}, \beta_{p+3}\}$ used for the characteristic $p \neq 3$ case in Section 6 fails to be independent. Thus we use the basis

$$\alpha(z) = x_1 + x_2 + x_3, \quad \beta_4(z) = x_4 - x_6, \quad \beta_5(z) = x_5 - x_6, \quad \gamma_6(z) = x_6.$$

The set $\{\alpha, \beta_4, \beta_5\}$ can be identified with the basis of $\mathrm{Hom}_{k\Sigma_3}(D^{(3)}, \mathrm{res}^{\Sigma_6}_{\Sigma_3} D^{(5,1)})$ found in Section 4. Thus we can deduce that $N = \mathrm{span}\{\alpha, \beta_4, \beta_5\}$ is a submodule of $\mathcal{M}^{((5,1),(3))}$ isomorphic to $\mathrm{Hom}_{k\Sigma_3}(D^{(3)}, \mathrm{res}^{\Sigma_6}_{\Sigma_3} D^{(5,1)})$. The corresponding quotient has basis $\{\overline{\gamma_6}\}$ and the table describing the action on this basis is

	$\overline{\gamma_6}$
$(12)+(13)+(23)$	0
$(123)+(132)$	$2\overline{\gamma_6}$
(45)	$\overline{\gamma_6}$
(456)	$\overline{\gamma_6}$
L_4	0
L_5	$\overline{\gamma_6}$
L_6	$2\overline{\gamma_6}$

Comparing this table with that on page 880 we see that

$$\mathrm{span}\{\overline{\beta_6}\} \cong \mathcal{M}^{((6),(3))}.$$

We now compute $\mathcal{D}^{((5,1),(3))}$. Recall that $\mathrm{Hom}_{k\Sigma_3}(D^{(3)}, \mathrm{res}^{\Sigma_6}_{\Sigma_3} D^{(5,1)})$ has two composition factors S_1 and S_2 not isomorphic to $\mathcal{M}^{((6),(3))}$, so that the same is true of N. Since N is the kernel of the projection

$$\mathcal{M}^{((5,1),(3))} \to \mathcal{M}^{((5,1),(3))}/N \cong \mathcal{M}^{((6),(3))},$$

we have $\mathcal{S}^{((5,1),(3))} \subseteq N$. To show the reverse inclusion, fix a homomorphism

$$\varphi : \mathcal{M}^{((5,1),(3))} \to \mathcal{M}^{((6),\mu)},$$

where $\mu \vdash 3$; by Proposition 2.1 we know that $\mathcal{M}^{((6),\mu)} \cong \mathrm{res}^{k\Sigma_6}_{k\Sigma_6^{\Sigma_3}} k$. Suppose $\varphi(\beta_4') = a$. Then

$$\begin{aligned}
\varphi(\alpha) &= \varphi(L_4 x) = L_4 a = 0,\\
\varphi(\beta_4) &= \varphi((1-(46))\beta_4') = (1-(46))a = 0,\\
\varphi(\beta_5) &= \varphi(((45)-(46))\beta_4') = ((45)-(46))a = 0,
\end{aligned}$$

so $\varphi(N) = 0$. Thus $N \subseteq \ker\varphi$, and since our choice of φ was arbitrary, it follows that $N \subseteq \mathcal{S}^{((5,1),(3))}$. Consequently

$$\mathcal{S}^{((5,1),(3))} = N.$$

From Section 4 we know that $\mathrm{Hom}_{k\Sigma_3}(D^{(3)}, \mathrm{res}^{\Sigma_6}_{\Sigma_3} D^{(5,1)})$ has a submodule L isomorphic to $\mathcal{M}^{((6),(3))}$. Since

$$N \cong \mathrm{Hom}_{k\Sigma_3}(D^{(3)}, \mathrm{res}^{\Sigma_6}_{\Sigma_3} D^{(5,1)}),$$

it follows that N also has a corresponding submodule K isomorphic to $\mathcal{M}^{((6),(3))}$. We can deduce that $K \subseteq \mathcal{S}^{((5,1),(3))\perp}$ since K is the image of the map

$$\mathcal{M}^{((6),(3))} \to \mathcal{M}^{((5,1),(3))}$$

consisting of the isomorphism to K followed by injection. Since the image of any homomorphism $\mathcal{M}^{((6),(3))} \to \mathcal{M}^{((5,1),(3))}$ must be isomorphic to $\mathcal{M}^{((6),(3))}$ and the only composition factor of N isomorphic to $\mathcal{M}^{((6),(3))}$ is K, it follows that

$$\mathcal{S}^{((5,1),(3))} \cap \mathcal{S}^{((5,1),(3))\perp} = K.$$

Thus

$$\mathcal{D}^{((5,1),(3))} = N/K \cong \operatorname{Hom}_{k\Sigma_3}(D^{(3)}, \operatorname{res}^{\Sigma_6}_{\Sigma_3} D^{(5,1)})/L$$

as claimed.

8. $\mathcal{M}^{(\lambda,\mu)}$ for $\lambda \vdash 5$, $\mu \vdash 2$

The above computations show that in every positive characteristic there are pairs of partitions (λ, μ) for which $\mathcal{D}^{(\lambda,\mu)}$ is neither simple nor zero, as conjectured in [Dodge and Ellers 2016]. However, it may be the case that this may be fixed by choosing a different ordering on pairs of partitions; that is, it may be the case that there exists a different ordering on pairs of partitions for which $\mathcal{D}^{(\lambda,\mu)}$ is always simple or zero. In this section we use the computer algebra system Magma [Bosma et al. 1997] to generate the structure of the $k\Sigma_5^{\Sigma_2}$-module $M^{(\lambda,\mu)}$ when $\lambda \vdash 5$ and $\mu \vdash 2$, and in the next section use this information to show that there does not exist any such ordering in characteristic 2.

We will treat the cases when $\mu = (2)$ and $\mu = (1^2)$ separately.

Case 1: $\mu = (1^2)$. Since $M^{(1^2)} \cong k\Sigma_2$ as $k\Sigma_2$-modules, we have

$$\mathcal{M}^{(\lambda,1^2)} = \operatorname{Hom}_{k\Sigma_2}(k\Sigma_2, \operatorname{res}^{\Sigma_5}_{\Sigma_2} M^\lambda) \cong \operatorname{res}^{\Sigma_5}_{\Sigma_2} M^\lambda$$

so we may compute in M^λ. This can be defined in Magma as a $k\Sigma_5$-module through the command

```
K := PermutationModule(Sym(5), YoungSubgroup(λ : Full := 5), GF(2));
```

However, we wish to define $\mathcal{M}^{(\lambda,1^2)}$ as a $k\Sigma_5^{\Sigma_2}$-module. To do this we will find the matrices of the action of the generators of $k\Sigma_5^{\Sigma_2}$ on the basis of M^λ, and then create a module over the matrix algebra that they generate.

Given an $x \in k\Sigma_5^{\Sigma_2}$ we may find the matrix of x acting on the basis of M^λ through the function

```
mapmatrix := func<x | Matrix(GF(2), Dimension(K), Dimension(K),
  [(VectorSpace(GF(2), Dimension(K)) ! (K.i * x)) : i in {1..Dimension(K)}])>;
```

This function simply creates the matrix of x in the natural way. Magma has a default basis for K, namely the elements $K.i$ for $1 \leq i \leq \dim K$. Thus, for the i-th basis vector $K.i$ of K, we find $K.i * x$ in terms of the basis of K and set it as the i-th row of the matrix.

We will be using the generating set for $k\Sigma_5^{\Sigma_2}$ given in Section 3, namely

$$k\Sigma_5^{\Sigma_2} = \langle (12),\ (34),\ (345),\ L_3,\ L_4,\ L_5 \rangle.$$

Using the function *mapmatrix* we can create the matrix algebra generated by the matrices of the actions of these generators through the command

```
A := MatrixAlgebra<GF(2), Dimension(K) |
  mapmatrix((Sym(5) ! (1,2))),
  mapmatrix((Sym(5) ! (3,4))),
  mapmatrix((Sym(5) ! (3,4,5))),
  mapmatrix((Sym(5) ! (1,3))) + mapmatrix((Sym(5) ! (2,3))),
  mapmatrix((Sym(5) ! (1,4))) +
    mapmatrix((Sym(5) ! (2,4))) + mapmatrix((Sym(5) ! (3,4))),
  mapmatrix((Sym(5) ! (1,5))) + mapmatrix((Sym(5) ! (2,5))) +
    mapmatrix((Sym(5) ! (3,5))) + mapmatrix((Sym(5) ! (4,5))) >;
```

We can then generate $\mathcal{M}^{(\lambda,1^2)}$ as a $k\Sigma_5^{\Sigma_2}$-module through the command

```
M := RModule(A);
```

and find its constituents with multiplicities via

```
ConstituentsWithMultiplicities(M);
```

Case 2: $\mu = (2)$. We first find a basis for $\mathcal{M}^{(\lambda,(2))}$.

Proposition 8.1. *Suppose k is a field of characteristic 2, let $\lambda \vdash 5$, and fix a nonzero $z \in M^{(2)} \cong k$. Then the functions defined by*

$$f_x(z) = \begin{cases} x + (12)x & \text{if } x \neq (12)x, \\ x & \text{if } x = (12)x, \end{cases}$$

where x is a λ-tabloid, constitute a basis for $\mathcal{M}^{(\lambda,(2))}$.

Proof. The independence of the functions f_x follows immediately from the independence of the tableau in M^λ. Fix a nonzero $z \in M^{(2)} \cong k$ and let

$$f : M^{(2)} \to \operatorname{res}_{\Sigma_2}^{\Sigma_5} M^\lambda$$

be defined by

$$f(z) = \sum_{x \text{ a } \lambda\text{-tabloid}} a_x x.$$

To have $f \in \mathcal{M}^{(\lambda,(2))}$ it is necessary and sufficient that $[(1)-(12)]f(z)=0$. Thus we need

$$0=[(1)-(12)]f(z)=\sum_{x \text{ a } \lambda\text{-tabloid}} a_x x-\sum_{x \text{ a } \lambda\text{-tabloid}} a_x(12)x$$
$$=\sum_{x \text{ a } \lambda\text{-tabloid}} a_x x-\sum_{x \text{ a } \lambda\text{-tabloid}} a_{(12)x}x=\sum_{x \text{ a } \lambda\text{-tabloid}} (a_x-a_{(12)x})x.$$

Thus we must have $a_x=a_{(12)x}$ for all x. This means that $f(z)$ is a linear combination of the functions $f_x(z)$, as needed. □

As before, we generate $K=M^\lambda$ as a permutation module over $k\Sigma_5$. To find a basis for $\mathcal{M}^{(\lambda,(2))}$ we first create a list consisting of sums of elements which are mapped to each other via the transposition (12). We accomplish this through the procedure below:

```
BasisSet := [];
BasisGen := procedure(~BasisSet, K)
  for i in {1..Dimension(K)} do
      if K.i + K.i * (Sym(5) ! (1,2)) eq Zero(K) then
              Append(~BasisSet, K.i);
      elif K.i + K.i * (Sym(5) ! (1,2)) in BasisSet then
              print "Skip";
      else
              Append(~BasisSet, K.i + K.i * (Sym(5) ! (1,2)));
      end if;
    end for;
end procedure;
BasisGen(~BasisSet, K);
```

For every basis element $K.i$ of K, we add $K.i((1)+(1,2))$ to the list BasisGen of basis elements if $K.i((1)+(1,2))$ is nonzero and $K.i$ if it is zero. This constitutes a basis for $\mathcal{M}^{(\lambda,(2))}$ by Proposition 8.1. The elif statement excludes duplicate basis elements.

Having created a list of basis elements for $\mathcal{M}^{(\lambda,(2))}$, we create the space spanned by them as a subspace of the vector space of appropriate dimension. We can do this through

```
W := sub<VectorSpace(GF(2), Dimension(K)) | [Eltseq(s) : s in BasisSet]>;
```

The Eltseq command coerces each basis element into a tuple so that it can be embedded into the vector space.

Although our basis vectors are now elements of a vector space and not a permutation module, we can still act on them by elements of $k\Sigma_5$ by coercing vectors in W back into M^λ. We exploit this property to find the matrix of the action of generators of $k\Sigma_5^{\Sigma_2}$ on $\mathcal{M}^{(\lambda,(2))}$ as follows:

```
A := MatrixAlgebra<GF(2), Dimension(W) |
Matrix(GF(2), Dimension(W), Dimension(W), [Coordinates(W, W ! (
    (K ! Basis(W)[i]) * (Sym(5) ! (1,2))))
    : i in {1..Dimension(W)}]),
Matrix(GF(2), Dimension(W), Dimension(W), [Coordinates(W, W ! (
    (K ! Basis(W)[i]) * (Sym(5) ! (3,4))))
    : i in {1..Dimension(W)}]),
Matrix(GF(2), Dimension(W), Dimension(W), [Coordinates(W, W ! (
    (K ! Basis(W)[i]) * (Sym(5) ! (3,4,5))))
    : i in {1..Dimension(W)}]),
Matrix(GF(2), Dimension(W), Dimension(W), [Coordinates(W, W ! (
    (K ! Basis(W)[i]) * (Sym(5) ! (1,3)) +
    (K ! Basis(W)[i]) * (Sym(5) ! (2,3))))
    : i in {1..Dimension(W)}]),
Matrix(GF(2), Dimension(W), Dimension(W), [Coordinates(W, W ! (
    (K ! Basis(W)[i]) * (Sym(5) ! (1,4)) +
    (K ! Basis(W)[i]) * (Sym(5) ! (2,4)) +
    (K ! Basis(W)[i]) * (Sym(5) ! (3,4))))
    : i in {1..Dimension(W)}]) >;
Matrix(GF(2), Dimension(W), Dimension(W), [Coordinates(W, W ! (
    (K ! Basis(W)[i]) * (Sym(5) ! (1,5)) +
    (K ! Basis(W)[i]) * (Sym(5) ! (2,5)) +
    (K ! Basis(W)[i]) * (Sym(5) ! (3,5)) +
    (K ! Basis(W)[i]) * (Sym(5) ! (4,5))))
    : i in {1..Dimension(W)}]) >;
```

The principle is identical to the algorithm used in the case $\mu = (1^2)$. The only difference is that we are working in the intermediary vector space W rather than directly in M^λ.

Having generated the algebra, we can define the desired module and find its constituents with multiplicities as before.

9. Alternative partial orders

The structures of $\mathcal{M}^{(\lambda,\mu)}$ when $\lambda \vdash 5$ and $\mu \vdash 2$ are compiled in the Appendix. The key piece of information we will use is that $\mathcal{M}^{(\lambda,\mu)}$ has at least three composition factors, except when $(\lambda, \mu) = ((5), (2))$ or $(\lambda, \mu) = ((5), (1^2))$, in which case we have $\mathcal{M}^{((5),(2))} \cong \mathcal{M}^{((5),(1^2))}$ and both are one-dimensional. In particular, when

(λ, μ) is neither $((5), (2))$ nor $((5), (1^2))$, we know that $\mathcal{M}^{(\lambda,\mu)}$ has two composition factors nonisomorphic to $\mathcal{M}^{((5),(2))}$.

Using this fact, we prove the following:

Proposition 9.1. *In characteristic* 2, *there exists no ordering on pairs of partitions* (λ, μ) *for which* $\mathcal{D}^{(\lambda,\mu)}$ *is always simple or zero.*

Proof. Let $\rhd$ be an arbitrary total order on pairs of partitions and let (λ_0, μ_0) be the most dominant partition such that (λ_0, μ_0) is not $((5),(2))$ or $((5),(1^2))$. If (λ_0, μ_0) is the most dominant partition then by definition $\mathcal{D}^{(\lambda_0,\mu_0)} \cong \mathcal{S}^{(\lambda_0,\mu_0)} = \mathcal{M}^{(\lambda_0,\mu_0)}$, so $\mathcal{D}^{(\lambda,\mu)}$ is neither simple nor zero. Otherwise (λ_0, μ_0) is dominated by $((5),(2))$ or $((5),(1^2))$ or both. Then since $\mathcal{M}^{(\lambda_0,\mu_0)}$ has two composition factors not isomorphic to $\mathcal{M}^{((5),(2))} \cong \mathcal{M}^{((5),(1^2))}$, it follows from [Dodge and Ellers 2016, 1.2] that $\mathcal{D}^{(\lambda_0,\mu_0)}$ has two composition factors not isomorphic to $\mathcal{M}^{((5),(2))} \cong \mathcal{M}^{((5),(1^2))}$. In particular it is neither simple nor zero, as claimed. □

10. Concluding remarks

In Sections 6 and 7 we showed that the conjecture that $\mathcal{D}^{(\lambda,\mu)}$ is always simple or zero fails in every positive characteristic p, while Section 9 shows that in general a different choice of partial orders will not correct the conjecture. However, in every example computed in this paper $\mathcal{D}^{(\lambda,\mu)}$ has had at most two composition factors, and they have always been distinct. This suggests that there may still be a bound on the composition length of $\mathcal{D}^{(\lambda,\mu)}$, even if it is not one as conjectured by Dodge and Ellers.

In [Danz et al. 2013], Danz, Ellers, and Murray answered in the negative the question of whether the FG^H-module $\mathrm{Hom}_{FH}(S, \mathrm{res}^G_H\, T)$ is always simple or zero for G a finite group and H a subgroup, F a field of positive characteristic, S a simple FH-module, and T a simple FG-module. However, it was still open whether there were counterexamples when FG and FH were symmetric group algebras. Our computations in Sections 3, 4, and 5 provided examples of spaces of the form $\mathrm{Hom}_{k\Sigma_l}(D^\mu, \mathrm{res}^{\Sigma_n}_{\Sigma_l}\, D^\lambda)$ which were neither simple nor zero, answering this question in the negative as well. The space described in Section 4 has also provided a counterexample to the conjecture that $\mathcal{D}^{(\lambda,\mu)} \cong \mathrm{Hom}_{k\Sigma_l}(D^\mu, \mathrm{res}^{\Sigma_n}_{\Sigma_l}\, D^\lambda)$ when $\mu \vdash l$ and $\lambda \vdash n$, as demonstrated in Section 7. However, unlike the conjecture on the simplicity of $\mathcal{D}^{(\lambda,\mu)}$, we have only been able to provide a counterexample in characteristic 3: the computations in Section 6 are in agreement with the conjecture. Although we have shown that isomorphism cannot hold in general, it may be the case that $\mathcal{D}^{(\lambda,\mu)}$ is always isomorphic to a quotient of $\mathrm{Hom}_{k\Sigma_l}(D^\mu, \mathrm{res}^{\Sigma_n}_{\Sigma_l}\, D^\lambda)$.

Finally, Dodge and Ellers [2016] established that every simple $k\Sigma_n^{\Sigma_l}$-module appears as a composition factor of some $\mathcal{D}^{(\lambda,\mu)}$. Though we have shown that those simple modules are not the modules $\mathcal{D}^{(\lambda,\mu)}$ themselves, our calculations may give hints as to how the simple modules appear as composition factors of the $\mathcal{D}^{(\lambda,\mu)}$. In particular, in our calculations the modules $\mathcal{D}^{(\lambda,\mu)}$ always have a simple head. Thus it is possible that the simple modules appear as simple heads of the $\mathcal{D}^{(\lambda,\mu)}$, in the same way that the simple $k\Sigma_n$-modules D^λ appear as the simple heads of the Specht module S^λ when λ is p-regular.

Appendix: $\mathcal{M}^{(\lambda,\mu)}$ when $\lambda \vdash 5$ and $\mu \vdash 2$

$\mathcal{M}^{(\lambda,\mu)}$	d	Multiplicity
$\mathcal{M}^{((5),(2))}$	1	1
$\mathcal{M}^{((5),(1^2))}$	1	1
$\mathcal{M}^{((4,1),(2))}$	1	1
	1	1
	2	2
$\mathcal{M}^{((4,1),(1^2))}$	1	2
	1	1
	2	1
$\mathcal{M}^{((3,1,1),(2))}$	1	2
	1	3
	2	2
	2	2
$\mathcal{M}^{((3,1,1),(1^2))}$	1	4
	1	4
	2	2
	2	4
$\mathcal{M}^{((2,2,1),(2))}$	1	2
	1	4
	2	4
	2	2

$\mathcal{M}^{(\lambda,\mu)}$	d	Multiplicity
$\mathcal{M}^{((2,2,1),(1^2))}$	1	4
	1	6
	2	8
	2	2
$\mathcal{M}^{((2,1,1,1),(2))}$	1	4
	1	7
	2	3
	2	8
$\mathcal{M}^{((2,1,1,1),(1^2))}$	1	8
	1	12
	2	4
	2	16
$\mathcal{M}^{((1^5),(2))}$	1	12
	1	8
	2	16
	2	4
$\mathcal{M}^{((1^5),(1^2))}$	1	16
	1	24
	2	8
	2	32

Table 1. The constituents of $\mathcal{M}^{(\lambda,\mu)}$ are modules of dimension d (given in the middle column) over GF(2) with corresponding multiplicities given in the third column.

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