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Stability analysis for numerical methods applied to an inner ear model

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Diependaal, Duifhuis, Hoogstraten and Viergever investigated three time-integration methods to solve a simplified one-dimensional model of the human cochlea. Two of these time-integration methods are dealt with in this paper, namely fourth-order Runge–Kutta and modified Sielecki. The stability of these two methods is examined, both theoretically and experimentally. This leads to the conclusion that in the case of the fourth-order Runge–Kutta method, a bigger time step can be used in comparison to the modified Sielecki method. This corresponds with the conclusion drawn in the article by Diependaal, Duifhuis, Hoogstraten and Viergever.

1. Introduction

1.1. ***Motivation.*** Deafness can be caused by a problem with the mechanical part of the human ear, which consists of three parts, namely the outer ear, the middle ear and the inner ear. The inner ear includes the cochlea (the organ of hearing) and the vestibular system (balance). The cochlea converts incoming sounds into electrochemical (nerve) impulses.

Almost always, it is possible to improve the hearing of those who are hearing impaired. Therefore it is important that deafness or hearing impairment is detected as early as possible. To test the functioning of the hearing system, subjective thresholds are determined at standardised frequencies and are related to standardised average thresholds. However, in general these tests cannot be performed on everyone. For example, they cannot be administered to people who are incapable of responding, such as babies and young children. Besides that, this method tests the functioning of the entire hearing system, not only of the cochlea. This leads to a different problem, because to improve the diagnosis of a hearing deficit it would be useful to separate the functioning of the cochlea from the neural processing [van Hengel 1996].

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There is an objective test (diagnosing cochlear dysfunction) which has the potential to detect deafness or hearing impairment as early as possible. In this test otoacoustic emissions play an important role. Otoacoustic emissions are very weak sounds produced by the cochlea, in response to stimulation or spontaneously. These sounds can be measured with a sensitive microphone in the ear canal. These otoacoustic emissions give directly measurable information about the condition of the cochlea, and thus can be used when diagnosing cochlear dysfunction. It is known that subjects with cochlear hearing deficits have emissions that differ from those found in people with normal hearing. Since otoacoustic emissions can be directly linked to cochlear functioning, it is possible for objective tests to be carried out on anyone, including babies and small children. The problem is that it remains difficult to link otoacoustic emission levels to cochlear functioning. A deeper understanding of the generative mechanism(s) is thus required. Since the cochlea is extremely vulnerable and difficult to access, in vivo studies on otoacoustic emissions cannot be performed in humans. However, these studies are performed in animals to help understand the phenomenon. Additionally, cochlea models are used to study otoacoustic emissions [van Hengel 1996].

1.2. ***Early work.*** The model used in this paper is obtained from an internal report by Marc van den Raadt, in which the numerical treatment of motion equations is described in detail and which is partly based on the paper by Diependaal et al. [1987], where they examined three time-integration methods (Heun, fourth-order Runge–Kutta and modified Sielecki) in order to solve their model. They also dealt with the numerical stability of these three methods. The time-integration methods have to be numerically stable and this limits the size of the time step used for a given problem.

1.3. ***What is new in this paper?*** There exist two kinds of stability, analytical and numerical. It is possible that a second-order differential equation is analytically stable (positive damping), but at the same time the used numerical method can be unstable, because too large a step size is used or an improper time-integration method is applied. While most authors examine only the analytical stability, we consider the numerical stability as well and realize that these two kinds of stability are not the same. Diependaal et al. seem to make this distinction between analytical and numerical stability as well. However, their stability analysis is limited to a numerical test (determining the bounds in an experimental way), and they obtain a conservative guess of the step-size limit for each time-integration method by testing different step sizes. Using these numerical tests, Diependaal et al. conclude that in the fourth-order Runge–Kutta method, a bigger time step can be used in comparison with Heun and modified Sielecki methods. In this paper we only examine two time-integration methods, fourth order Runge–Kutta and modified Sielecki, and a real numerical stability analysis is conducted. So, the theoretical bounds for the time steps are derived and verified with a numerical test. As far as the authors know, the method used to investigate the stability of the modified Sielecki method is not known in the literature, and therefore a new contribution of this paper.

The goal is also to derive a method for stability analysis on the model with parameter variations to simulate hearing loss.

1.4. ***Structure of the paper.*** In Section 2, the biological background, the mathematical model and the discretisation of this model are dealt with. The numerical methods used during the project and their properties are examined in Section 3. The numerical experiments are examined in Section 4. Finally, in Section 5 various conclusions are drawn.

2. Problem definition

2.1. ***Biological background.*** The cochlea plays an important part in the processing of incoming sounds. The incoming sound waves behave like pressure waves in the ear. The pressure waves, which reach the eardrum, are transmitted via vibrations of the middle ear ossicles to the oval window at the base of the cochlea. These vibrations move the cochlear fluids, which stimulate tiny hair cells on the cochlear partition. Individual hair cells respond to specific sound frequencies so that, depending on the frequency of the sound, only certain hair cells are stimulated [Robles and Ruggero 2001; Bell 2004].

The cochlea looks like a coiled tube. Mechanically this tube is divided into two compartments by the cochlear partition, consisting of the basilar membrane and the organ of Corti (the unfolded cochlea is shown in Figure 1). The two compartments are filled with fluid, which will have equivalent mechanical properties to water in this proposed model. The organ of Corti, which is a cellular layer on the basilar membrane, contains the hair cells that start to move when sound waves enter the

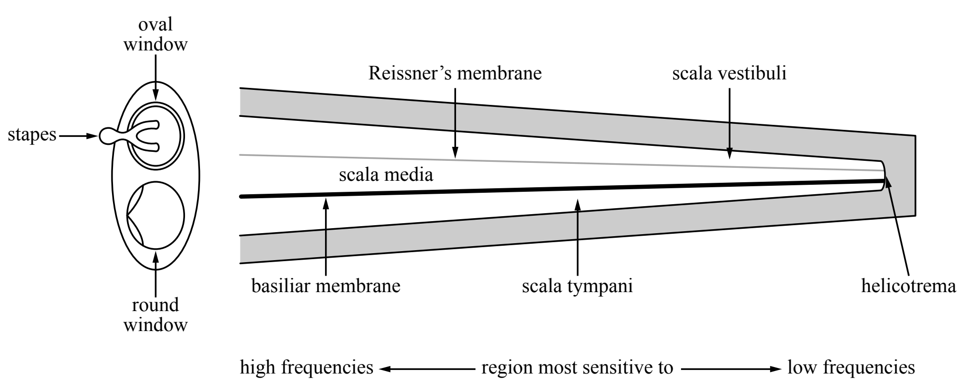

Figure 1. The unfolded cochlea.

ear. The part of the cochlear partition situated at the base will resonate at higher frequencies and the part at the apex will resonate at lower frequencies [Robles and Ruggero 2001; Bell 2004].

2.2. ***Mathematical model.*** In our model, the cochlea will be viewed as a straight cylinder. The cochlea will be divided by the cochlear partition into two fluid channels of the same height and the same width. We will also assume that the basilar membrane has the same width from the base to the apex. It is assumed that the base is located at the left side of the cylinder and the apex at the right side.

The one-dimensional cochlea model [Diependaal et al. 1987] is defined by

$$\frac{\partial^2 p}{\partial x^2}(x,t) - \frac{2\rho}{h}\frac{\partial^2 y}{\partial t^2}(x,t) = 0, \quad 0 \le x \le L, \; t \ge 0, \tag{1}$$

where the transmembrane fluid pressure $p(x,t)$ can be written as

$$p(x,t) = m_s \ddot{y}(x,t) + d_s(x)\dot{y}(x,t) + s_s(x)y(x,t).$$

For simplicity, this pressure is assumed to be equal to zero at the helicotrema (at $x = L$). Note that this assumption is an approximation because in reality there can be a small fluid flow as a result of the remaining pressure difference and some damping that affects the flow. For high frequencies, this is almost negligible, but it can play an important role for frequencies below 1 kHz. Equation (1) describes the movement of a cochlear section, and in this equation ρ stands for the density of the cochlear fluid and h is the height of a scala. Here $y(x,t)$ is the excitation in the oscillators, m_s the specific acoustic mass of the basilar membrane, d_s the specific acoustic damping of the basilar membrane and s_s the specific acoustic stiffness of the basilar membrane. Both d_s and s_s vary with the placement of an oscillator.

2.3. ***Spatial discretisation.*** If we define $G(x,t) = d_s(x)\dot{y}(x,t) + s_s(x)y(x,t)$, then $p(x,t) - G(x,t) = m_s\ddot{y}(x,t)$. The differential equation (1) can be written as

$$-\frac{\partial^2 p}{\partial x^2}(x,t) + \frac{2\rho}{hm}p(x,t) = \frac{2\rho}{hm}G(x,t), \quad 0 \le x \le L, \qquad t \ge 0. \tag{2}$$

This model is used to describe $N+1$ individual cochlear sections. These sections behave as harmonic oscillators. We assume that we have $N+1$ oscillators and therefore we divide the interval $[0, L]$ into $N+1$ equidistant subintervals (with length ΔX). The oscillator at $n = 0$ is part of the middle ear, the oscillator at $n = 1$ has the highest frequency in the cochlea and the oscillator at $n = N$, at the right the helicotrema, has the lowest frequency in the cochlea.

The following approximation of the second partial derivative of $p(x,t)$ [Vuik et al. 2006] is used:

$$\frac{\partial^2 p}{\partial x^2}(x,t) \approx \frac{p(x+\Delta X,t) - 2p(x,t) + p(x-\Delta X,t)}{(\Delta X)^2}.$$

Then (2) can be written as a matrix representation $\boldsymbol{Ap}(t) = \boldsymbol{b}(t)$ [Vuik et al. 2006]. Here matrix $\boldsymbol{A}$ is a tridiagonal matrix (a_{ij}), vector $\boldsymbol{p}(t)$ is an unknown vector and the vector $\boldsymbol{b}(t)$ consists of known terms like the stimulus $p_e(t)$ and the vector $\boldsymbol{G}(t)$:

$$a_{ij} = \begin{cases} \left(1+\frac{2m_{c01}}{m_m}\right)/\Delta X & \text{if } i=j=1, \\ \left(2+\frac{2m_c}{m}\right)/\Delta X & \text{if } i=j \text{ and } i \in \{2,3,\dots,N\}, \\ \left(1+\frac{2m_c}{m}+\frac{2m_c}{m_h+2m_c}\right)/\Delta X & \text{if } i=j=N+1, \\ -1/\Delta X & \text{if } j=i+1 \text{ and } i \in \{1,2,\dots,N\}, \\ -1/\Delta X & \text{if } j=i-1 \text{ and } i \in \{2,3,\dots,N+1\}, \end{cases}$$

$$p_i = p(x_i,t) \quad \text{for } i \in \{0,1,\dots,N\},$$

$$b_j = \begin{cases} \frac{2m_{c01}}{m_m}\big(n_t p_e(t) + G(x_{j-1},t)\big)/\Delta X & \text{if } j=1, \\ \frac{2m_c}{m} G(x_{j-1},t)/\Delta X & \text{if } j \in \{2,3,\dots,N+1\}. \end{cases}$$

This almost corresponds with the system derived by Diependaal et al., but the equation associated with the oscillator at $n=0$ (first equation of $\boldsymbol{Ap}(t) = \boldsymbol{b}(t)$) differs. This deviation is a result of the fact that the oscillator at $n=0$ is part of the middle ear. The other equations do correspond with that of Diependaal et al. for equidistant subintervals (ΔX is constant), except the notation deviates:

$$\frac{2m_c}{m} = \frac{2\rho b_{BM}}{m_s S_{sc}}(\Delta X)^2 \quad \text{and} \quad \frac{2m_c}{m_h+2m_c} \approx 0.999999999102402 \approx 1.$$

2.4. ***System of equations for the time.*** Consider the functions $\dot{y}(x,t)$ and $\ddot{y}(x,t)$. These lead to a second-order system of time equations. This second-order system can be transformed into a first-order system if we define $\dot{y}(x,t) = u(x,t)$. Then it holds that $\ddot{y}(x,t) = \dot{u}(x,t)$ [Diependaal et al. 1987] and $\dot{u}(x,t) = \ddot{y}(x,t) = (p(x,t) - G(x,t))/m_s$.

So this system is given by

$$\begin{cases} \dot{y}(x,t) = u(x,t), \\ \dot{u}(x,t) = \big(p(x,t) - G(x,t)\big)/m_s, \end{cases} \qquad \begin{cases} y(x,0) = 0, & 0 \le x \le L, \\ u(x,0) = 0, & t \ge 0. \end{cases}$$

It is transformed into a system consisting of vectors after spatial discretisation has taken place because the vector $\boldsymbol{p}(t)$ can be determined from $\boldsymbol{Ap}(t) = \boldsymbol{b}(t)$.

Consider the system

$$\begin{cases} \dot{\boldsymbol{y}}(t) = \boldsymbol{u}(t), \\ \dot{\boldsymbol{u}}(t) = \boldsymbol{Q} \cdot [\boldsymbol{p}(t) - \boldsymbol{c}(t)], \end{cases} \quad t \ge 0, \qquad \begin{cases} \boldsymbol{y}(0) = \boldsymbol{0}, \\ \boldsymbol{u}(0) = \boldsymbol{0}, \end{cases} \tag{3}$$

where the matrix $\boldsymbol{Q}$ is a diagonal matrix (q_{ii}) and the vector $\boldsymbol{c}(t)$ consists of known terms like the stimulus $p_e(t)$ and the vector $\boldsymbol{G}(t)$:

$$q_{ii} = \begin{cases} 1/m_{sm} & \text{if } i = 1, \\ 1/m_s & \text{if } i \in \{2, 3, \dots, N+1\}, \end{cases}$$

$$c_j = \begin{cases} n_t p_e(t) + G(x_{j-1}, t) & \text{if } j = 1, \\ G(x_{j-1}, t) & \text{if } j \in \{2, 3, \dots, N+1\}. \end{cases}$$

Because of the oscillator in the middle ear, this system deviates a little bit for $n = 0$ (first equation of $\boldsymbol{Q}[\boldsymbol{p}(t) - \boldsymbol{c}(t)]$) with respect to [Diependaal et al. 1987]. However, in this case the influence of the middle ear is taken into account.

The vector $\boldsymbol{p}(t)$ is determined by

$$\boldsymbol{p}(t) = \boldsymbol{A}^{-1}\boldsymbol{b}(t) = \boldsymbol{A}^{-1} \begin{pmatrix} -\frac{2m_{c01}}{m_m} G(x_0, t) \\ -\frac{2m_c}{m} G(x_1, t) \\ \vdots \\ -\frac{2m_c}{m} G(x_N, t) \end{pmatrix} - \boldsymbol{A}^{-1} \begin{pmatrix} -\frac{2m_{c01}}{m_m} n_t p_e(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

3. Numerical methods

3.1. ***System of first-order differential equations.*** We consider system (3). In the following definition of the vector $\boldsymbol{G}(t)$, it can be seen that the properties of the middle ear are again taken into account for the first oscillator, and therefore it differs from the vector $\boldsymbol{g}(t)$ in the paper of Diependaal et al.:

$$\boldsymbol{G}(t) = \begin{pmatrix} G(x_0, t) \\ G(x_1, t) \\ \vdots \\ G(x_N, t) \end{pmatrix} = \begin{pmatrix} S_{ST} \cdot n_t^2 \cdot Z_a \cdot u(x_0, t) + s_{sm} \cdot y(x_0, t) \\ d_s(x_1) \cdot u(x_1, t) + s_s(x_1) \cdot y(x_1, t) \\ \vdots \\ d_s(x_N) \cdot u(x_N, t) + s_s(x_N) \cdot y(x_N, t) \end{pmatrix} = \boldsymbol{D}\boldsymbol{u}(t) + \boldsymbol{S}\boldsymbol{y}(t).$$

However, for the oscillators in the cochlea, $\boldsymbol{G}(t)$ equals $\boldsymbol{g}(t)$, but again a somewhat different notation is used. Here the matrices $\boldsymbol{D}$ and $\boldsymbol{S}$ are diagonal matrices (d_{ii}) and (s_{ii}) respectively given by

$$d_{ii} = \begin{cases} S_{ST} \cdot n_t^2 \cdot Z_a & \text{if } i = 1, \\ d_s(x_{i-1}) & \text{if } i \in \{2, 3, \dots, N+1\}, \end{cases}$$

$$s_{ii} = \begin{cases} s_{sm} & \text{if } i = 1, \\ s_s(x_{i-1}) & \text{if } i \in \{2, 3, \dots, N+1\}. \end{cases}$$

Consider system (3) on the time interval $[0, T]$. This time interval is divided into M equidistant subintervals $[t_0, t_1], [t_1, t_2], \dots, [t_{M-1}, t_M]$ (with length Δt). After dividing the time interval into subintervals, the following steps must be followed from $j = 1$ to $j = M$ [Diependaal et al. 1987]:

(1) Calculate at time t_{j-1} the vectors $\boldsymbol{c}$ and $\boldsymbol{b}$.

(2) Solve $\boldsymbol{p}$ by using $\boldsymbol{A}\boldsymbol{p}(t) = \boldsymbol{b}(t)$.

(3) Calculate $\boldsymbol{w}[t, \boldsymbol{y}(t), \boldsymbol{u}(t)] = \boldsymbol{Q} \cdot [\boldsymbol{p}(t) - \boldsymbol{c}(t)]$.

(4) Integrate the equations $\dot{\boldsymbol{y}}(t) = \boldsymbol{u}(t)$ and $\dot{\boldsymbol{u}}(t) = \boldsymbol{w}[t, \boldsymbol{y}(t), \boldsymbol{u}(t)]$ from t_{j-1} to t_j.

At step (4), the fourth-order Runge–Kutta and modified Sielecki methods are used.

3.2. ***Fourth-order Runge–Kutta method.*** The fourth-order Runge–Kutta method is given in [Diependaal et al. 1987] by

$$\begin{cases} \boldsymbol{y}(t+\Delta t) = \boldsymbol{y}(t) + \frac{1}{6}[\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4], \\ \boldsymbol{u}(t+\Delta t) = \boldsymbol{u}(t) + \frac{1}{6}[\boldsymbol{l}_1 + 2\boldsymbol{l}_2 + 2\boldsymbol{l}_3 + \boldsymbol{l}_4]. \end{cases}$$

The following four steps are carried out:

(1) Determine the predictors $\boldsymbol{k}_1$ and $\boldsymbol{l}_1$.

(2) Determine the predictors $\boldsymbol{k}_2$ and $\boldsymbol{l}_2$.

(3) Determine the predictors $\boldsymbol{k}_3$ and $\boldsymbol{l}_3$.

(4) Determine the predictors $\boldsymbol{k}_4$ and $\boldsymbol{l}_4$.

At each step the vector $\boldsymbol{p}(t)$ has to be determined from $\boldsymbol{A}\boldsymbol{p}(t) = \boldsymbol{b}(t)$ for the function $\boldsymbol{w}[t, \boldsymbol{y}(t), \boldsymbol{u}(t)] = \boldsymbol{Q} \cdot [\boldsymbol{p}(t) - \boldsymbol{c}(t)]$.

To apply the fourth-order Runge–Kutta method, system (3) must be written as the matrix representation

$$\begin{pmatrix} \dot{y}(x_0,t) \\ \dot{y}(x_1,t) \\ \vdots \\ \dot{y}(x_N,t) \\ \dot{u}(x_0,t) \\ \dot{u}(x_1,t) \\ \vdots \\ \dot{u}(x_N,t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{MS} & \boldsymbol{MD} \end{pmatrix} \begin{pmatrix} y(x_0,t) \\ y(x_1,t) \\ \vdots \\ y(x_N,t) \\ u(x_0,t) \\ u(x_1,t) \\ \vdots \\ u(x_N,t) \end{pmatrix} - \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{N} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ n_t p_e(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4}$$

The matrix $\boldsymbol{O}$ is an $(N+1) \times (N+1)$ null matrix, $\boldsymbol{I}$ is an $(N+1) \times (N+1)$ identity matrix, $\boldsymbol{M}$ is determined by $\boldsymbol{M} = \boldsymbol{Q}[\boldsymbol{A}^{-1}\boldsymbol{R} - \boldsymbol{I}]$ with $\boldsymbol{R}$ an $(N+1)\times(N+1)$ diagonal matrix with

$$r_{ii} = \begin{cases} -\frac{2m_{c01}}{m_m} & \text{if } i = 1, \\ -\frac{2m_c}{m} & \text{if } i \in \{2, 3, \dots, N+1\}, \end{cases}$$

and $\boldsymbol{N}$ is an $(N+1) \times (N+1)$ matrix determined by $\boldsymbol{N} = \boldsymbol{Q}\big[-\frac{2m_{c01}}{m_m} \cdot \boldsymbol{A}^{-1} + \boldsymbol{I}\big]$.

3.3. ***The properties of the fourth-order Runge–Kutta method.*** The Runge–Kutta method we are using is a fourth-order method, which means that the total error is of the fourth-order (with respect to the time step). This is an explicit method and it is conditionally stable, which means that it is stable for a time step below a certain bound.

In order to study the stability when using the time-integration fourth-order Runge–Kutta method, the amplification factor associated with this method can be used and it is given by

$$Q(h\lambda) = 1 + h\lambda + \tfrac{1}{2}(h\lambda)^2 + \tfrac{1}{6}(h\lambda)^3 + \tfrac{1}{24}(h\lambda)^4,$$

where h represents the time step used and λ is known from the test equation $y' = \lambda y$. A numerical time-integration method is termed stable if and only if $|Q(h\lambda)| \leq 1$ for a given time step h [Vuik et al. 2006].

The amplification factor $Q(h\lambda)$ given above is the amplification factor in the scalar case, but here the method is used on a matrix representation (4). Now this numerical time-integration method is termed stable if and only if for every eigenvalue μ of the matrix $\left(\begin{smallmatrix} \boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{MS} & \boldsymbol{MD} \end{smallmatrix}\right)$, it holds that $|Q(h\mu)| \leq 1$ for a given time step h [Vuik et al. 2006].

3.4. ***The modified Sielecki method.*** The modified Sielecki method is given in [Diependaal et al. 1987] by

$$\begin{cases} \boldsymbol{u}(t+\Delta t) = \boldsymbol{u}(t) + \Delta t \cdot \boldsymbol{w}[t, \boldsymbol{y}(t), \boldsymbol{u}(t)], \\ \boldsymbol{y}(t+\Delta t) = \boldsymbol{y}(t) + \Delta t \cdot \boldsymbol{u}(t+\Delta t), \end{cases}$$

where the function $\boldsymbol{w}[t, \boldsymbol{y}(t), \boldsymbol{u}(t)]$ is defined as $\boldsymbol{w}[t, \boldsymbol{y}(t), \boldsymbol{u}(t)] = \boldsymbol{Q} \cdot [\boldsymbol{p}(t) - \boldsymbol{c}(t)]$, which can also be represented as

$$\boldsymbol{w}[t, \boldsymbol{y}(t), \boldsymbol{u}(t)] = \boldsymbol{MDu}(t) + \boldsymbol{MSy}(t) - \boldsymbol{N} \begin{pmatrix} n_t p_e(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

3.5. ***The stability of the modified Sielecki method.*** The amplification factor of the modified Sielecki numerical time-integration method is not known or given as far as the authors know. This method is both an implicit and an explicit method and this makes it harder to derive an amplification factor.

To examine the stability of the modified Sielecki method, the scalar-linear case is first considered. For the (scalar) system

$$\begin{cases} \dot{\zeta}(t) = \nu(t), \\ \dot{\nu}(t) = \omega[t, \zeta(t), \nu(t)], \end{cases} \qquad \begin{cases} \zeta(0) = \zeta_0, \\ \nu(0) = \nu_0, \end{cases}$$

the modified Sielecki method is given by

$$\begin{cases} \nu(t+\Delta t) = \nu(t) + \Delta t \cdot \omega[t, \zeta(t), \nu(t)], \\ \zeta(t+\Delta t) = \zeta(t) + \Delta t \cdot \nu(t+\Delta t). \end{cases}$$

In the scalar-linear case the function $\omega[t, \zeta(t), \nu(t)]$ is given by $\omega[t, \zeta(t), \nu(t)] = \lambda \cdot \zeta(t) + \mu \cdot \nu(t) + c$ with $\lambda, \mu \leq 0$ and a constant $c \in \mathbb{R}$. To examine the stability of this method in the scalar-linear case, the function $\omega[t, \zeta(t), \nu(t)] = \lambda \cdot \zeta(t) + \mu \cdot \nu(t)$ can be considered because the constant c does not affect the stability.

Consider the (scalar-linear) system

$$\begin{cases} \nu(t+\Delta t) = \nu(t) + \Delta t \cdot \lambda \cdot \zeta(t) + \Delta t \cdot \mu \cdot \nu(t), \\ \zeta(t+\Delta t) = \zeta(t) + \Delta t \cdot \nu(t+\Delta t). \end{cases} \tag{5}$$

After substituting $\nu(t+\Delta t) = \nu(t) + \Delta t \cdot \lambda \cdot \zeta(t) + \Delta t \cdot \mu \cdot \nu(t)$ in $\zeta(t+\Delta t) = \zeta(t) + \Delta t \cdot \nu(t+\Delta t)$, the system (5) can be represented by the matrix representation $\boldsymbol{a}(t+\Delta t) = \boldsymbol{T} \cdot \boldsymbol{a}(t)$. Here

$$\boldsymbol{a}(t+\Delta t) = \begin{pmatrix} \nu(t+\Delta t) \\ \zeta(t+\Delta t) \end{pmatrix}, \quad \boldsymbol{a}(t) = \begin{pmatrix} \nu(t) \\ \zeta(t) \end{pmatrix}, \quad \boldsymbol{T} = \begin{pmatrix} 1+\Delta t \cdot \mu & \Delta t \cdot \lambda \\ \Delta t + (\Delta t)^2 \cdot \mu & 1 + (\Delta t)^2 \cdot \lambda \end{pmatrix}.$$

For a multiplicative norm, it holds for $\boldsymbol{a}_{n+1} = \boldsymbol{T} \cdot \boldsymbol{a}_n$ with $\boldsymbol{a}_n$ the numerical solution that

$$\|\boldsymbol{a}_{n+1}\| = \|\boldsymbol{T} \cdot \boldsymbol{a}_n\| = \|\boldsymbol{T}^n \cdot \boldsymbol{a}_1\| \overset{\text{multiplicativity}}{\leq} \|\boldsymbol{T}^n\| \cdot \|\boldsymbol{a}_1\|.$$

Furthermore the following result is known [Golub and Van Loan 1996]:

$$\boldsymbol{T}^n \to 0 \quad \text{as} \;\; \rho(\boldsymbol{T}) < 1 \;\; \text{with} \;\; \rho(\boldsymbol{T}) = \max\{|\kappa| : \kappa \text{ is an eigenvalue of } \boldsymbol{T}\}.$$

The modified Sielecki numerical time-integration method is termed stable if and only if $|\kappa| \leq 1$ for every eigenvalue κ of $\boldsymbol{T}$ for a given time step Δt, because then the inequality $\rho(\boldsymbol{T}) < 1$ is satisfied.

The same principle (as in the scalar-linear case) can be used for

$$\begin{cases} \boldsymbol{u}(t+\Delta t) = \boldsymbol{u}(t) + \Delta t \cdot \boldsymbol{w}[t, \boldsymbol{y}(t), \boldsymbol{u}(t)], \\ \boldsymbol{y}(t+\Delta t) = \boldsymbol{y}(t) + \Delta t \cdot \boldsymbol{u}(t+\Delta t), \end{cases}$$

where the function $\boldsymbol{w}[t, \boldsymbol{y}(t), \boldsymbol{u}(t)] = \boldsymbol{MDu}(t) + \boldsymbol{MSy}(t)$ is considered to examine the stability of the modified Sielecki method. So consider

$$\begin{cases} \boldsymbol{u}(t+\Delta t) = \boldsymbol{u}(t) + \Delta t \cdot \boldsymbol{MSy}(t) + \Delta t \cdot \boldsymbol{MDu}(t), \\ \boldsymbol{y}(t+\Delta t) = \boldsymbol{y}(t) + \Delta t \cdot \boldsymbol{u}(t+\Delta t). \end{cases} \tag{6}$$

After substituting $\boldsymbol{u}(t+\Delta t)=\boldsymbol{u}(t)+\Delta t\cdot\boldsymbol{MSy}(t)+\Delta t\cdot\boldsymbol{MDu}(t)$ in $\boldsymbol{y}(t+\Delta t)=\boldsymbol{y}(t)+\Delta t\cdot\boldsymbol{u}(t+\Delta t)$, the system (6) can be written as

$$\begin{pmatrix}\boldsymbol{u}(t+\Delta t)\\ \boldsymbol{y}(t+\Delta t)\end{pmatrix}=\begin{pmatrix}\boldsymbol{I}+\Delta t\cdot\boldsymbol{MD} & \Delta t\cdot\boldsymbol{MS}\\ \Delta t\cdot\boldsymbol{I}+(\Delta t)^2\cdot\boldsymbol{MD} & \boldsymbol{I}+(\Delta t)^2\cdot\boldsymbol{MS}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{u}(t)\\ \boldsymbol{y}(t)\end{pmatrix}. \tag{7}$$

The modified Sielecki method is stable if and only if $|\kappa|\le 1$ holds for every eigenvalue κ of the matrix

$$\begin{pmatrix}\boldsymbol{I}+\Delta t\cdot\boldsymbol{MD} & \Delta t\cdot\boldsymbol{MS}\\ \Delta t\cdot\boldsymbol{I}+(\Delta t)^2\cdot\boldsymbol{MD} & \boldsymbol{I}+(\Delta t)^2\cdot\boldsymbol{MS}\end{pmatrix}$$

for a given time step Δt.

4. Numerical experiments

4.1. ***Problem.*** The one-dimensional cochlea model is given by (1) and this equation can be written as a matrix representation (spatial discretisation) $\boldsymbol{Ap}(t)=\boldsymbol{b}(t)$ [Vuik et al. 2006], which is used to determine the vector $\boldsymbol{p}(t)$. The system of first-order time equations (3) can be solved by determining the vector $\boldsymbol{p}(t)$ and using a numerical time-integration method. In this paper the fourth-order Runge–Kutta and modified Sielecki numerical time-integration methods are dealt with.

4.2. ***The stability of the two numerical time-integration methods.*** For the fourth-order Runge–Kutta method, system (3) is written as the matrix representation (4).

The fourth-order Runge–Kutta method is termed stable if and only if for every eigenvalue μ_i of the matrix

$$\boldsymbol{E}=\begin{pmatrix}\boldsymbol{O} & \boldsymbol{I}\\ \boldsymbol{MS} & \boldsymbol{MD}\end{pmatrix},$$

it holds that $|Q(\Delta t\mu_i)|\le 1$ for a given time step Δt [Vuik et al. 2006], where the amplification factor Q is given by

$$Q(\Delta t\mu_i)=1+\Delta t\mu_i+\tfrac{1}{2}(\Delta t\mu_i)^2+\tfrac{1}{6}(\Delta t\mu_i)^3+\tfrac{1}{24}(\Delta t\mu_i)^4.$$

For the modified Sielecki method, $\boldsymbol{w}[t,\boldsymbol{y}(t),\boldsymbol{u}(t)]=\boldsymbol{MDu}(t)+\boldsymbol{MSy}(t)$ is considered to examine the stability. Consider the system (6) and write this as the matrix representation given in (7).

The modified Sielecki method will be stable if and only if $|\kappa_j|\le 1$ holds for every eigenvalue κ_j of the matrix

$$\boldsymbol{F}=\begin{pmatrix}\boldsymbol{I}+\Delta t\cdot\boldsymbol{MD} & \Delta t\cdot\boldsymbol{MS}\\ \Delta t\cdot\boldsymbol{I}+(\Delta t)^2\cdot\boldsymbol{MD} & \boldsymbol{I}+(\Delta t)^2\cdot\boldsymbol{MS}\end{pmatrix}$$

for a given time step Δt.

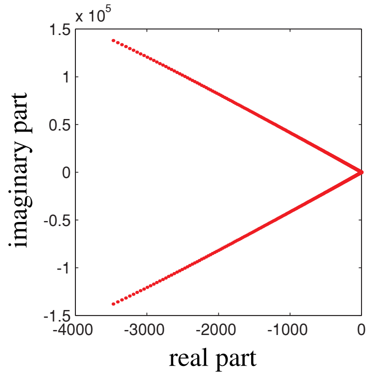

Figure 2. The plot of the eigenvalues of $\boldsymbol{E}$.

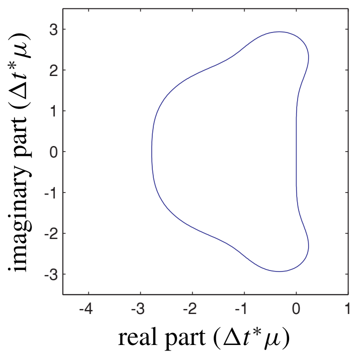

Figure 3. The stability locus of the fourth-order Runge–Kutta method.

4.3. ***Numerical experiments to determine a restriction on the time step*** $\boldsymbol{\Delta t}$**.** To perform these numerical experiments, Matlab is used.

4.3.1. *Fourth-order Runge–Kutta.* The eigenvalues μ_i of the matrix $\boldsymbol{E}$ can be calculated and plotted (see Figure 2).

For the fourth-order Runge–Kutta method to be stable for a given Δt, the inequality

$$|Q(\Delta t\mu_i)| = \left|1 + \Delta t\mu_i + \tfrac{1}{2}(\Delta t\mu_i)^2 + \tfrac{1}{6}(\Delta t\mu_i)^3 + \tfrac{1}{24}(\Delta t\mu_i)^4\right| \le 1$$

has to be satisfied for all μ_i $(i = 1, \ldots, 2N+2)$. In other words, all eigenvalues μ_i multiplied by a time step Δt must lie within the range of the stability locus of the fourth-order Runge–Kutta method. Figure 3 shows this stability locus.

This consideration determines a restriction on the time step Δt. The result of this numerical experiment is the following:

- For $\Delta t = 2.08 \cdot 10^{-5}s$, it holds that $|Q(\Delta t\mu_i)| \le 1$ for all eigenvalues μ_i, and thus this numerical scheme is stable.

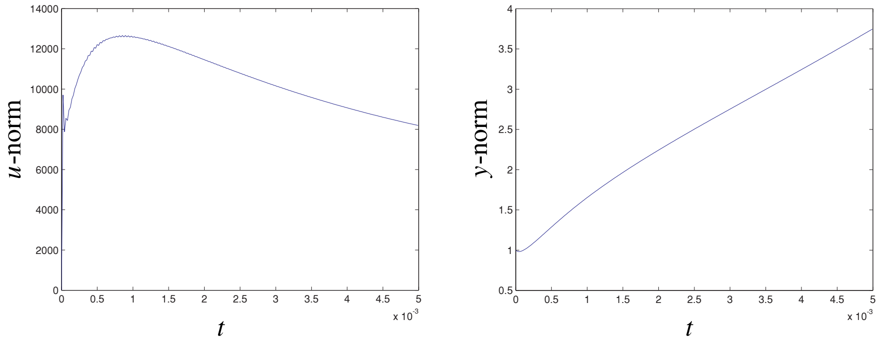

Figure 4. The norms of $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$, divided by $\sqrt{N+1}$, with $\Delta t = 2.08 \cdot 10^{-5}s$.

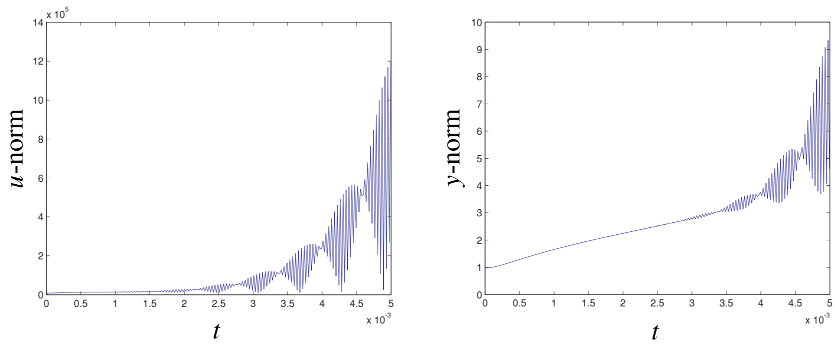

Figure 5. The norms of $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$, divided by $\sqrt{N+1}$, with $\Delta t = 2.09 \cdot 10^{-5}s$.

- For $\Delta t = 2.09 \cdot 10^{-5}$ s it holds that $|Q(\Delta t \mu_i)| > 1$ for two eigenvalues, and thus this numerical scheme is no longer stable.

The determined time steps ($\Delta t = 2.08 \cdot 10^{-5}s$ and $\Delta t = 2.09 \cdot 10^{-5}s$) can also be tested on the fourth-order Runge–Kutta numerical time-integration method. The initial conditions $\boldsymbol{u}(0) = \mathbf{1}$, $\boldsymbol{y}(0) = \mathbf{1}$ are used instead of $\boldsymbol{u}(0) = \mathbf{0}$, $\boldsymbol{y}(0) = \mathbf{0}$. A small change of the initial conditions causes a perturbation.

If this perturbation is bounded, then the fourth-order Runge–Kutta method is stable. The expectation is that the perturbation is bounded by using the time step $\Delta t = 2.08 \cdot 10^{-5}s$ and unbounded by using the time step $\Delta t = 2.09 \cdot 10^{-5}s$. To investigate this, the norms of the vectors $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$ divided by $\sqrt{N+1}$ are calculated and plotted (see Figure 4 for time step $\Delta t = 2.08 \cdot 10^{-5}s$ and Figure 5 for time step $\Delta t = 2.09 \cdot 10^{-5}s$).

From Figure 4, it can be seen that the perturbation remains bounded. For $\Delta t = 2.08 \cdot 10^{-5}s$, the numerical scheme is indeed stable.

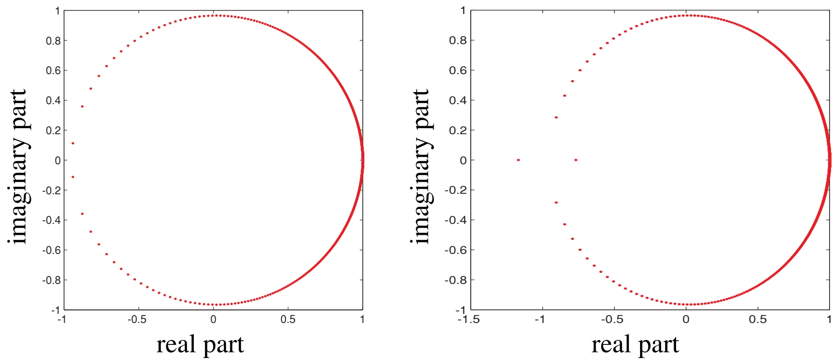

Figure 6. The eigenvalues of $\boldsymbol{F}$ with $\Delta t = 1.41 \cdot 10^{-5} s$ (left) and $\Delta t = 1.42 \cdot 10^{-5} s$ (right).

From Figure 5, it can be concluded that the perturbation is unbounded. For $\Delta t = 2.09 \cdot 10^{-5} s$, it holds that the numerical scheme is unstable.

4.3.2. *Modified Sielecki.* By trying different values for the time step Δt, we can determine for which time step the modified Sielecki numerical time-integration method is stable or unstable. It can be seen that for the time step $\Delta t = 1.41 \cdot 10^{-5} s$, the inequality $|\kappa_j| < 1$ holds for all eigenvalues κ_j of the matrix $\boldsymbol{F}$ (see Figure 6). It can also be seen that for the time step $\Delta t = 1.42 \cdot 10^{-5} s$, the inequality $|\kappa_j| > 1$ holds for one eigenvalue (see Figure 6).

The determined time steps ($\Delta t = 1.41 \cdot 10^{-5} s$ and $\Delta t = 1.42 \cdot 10^{-5} s$) can also be tested on the modified Sielecki numerical time-integration method. The initial conditions $\boldsymbol{u}(0) = \mathbf{1},\ \boldsymbol{y}(0) = \mathbf{1}$ instead of $\boldsymbol{u}(0) = \mathbf{0},\ \boldsymbol{y}(0) = \mathbf{0}$ are used, causing a perturbation.

If this perturbation is bounded, then the modified Sielecki method is stable. The expectation is that the perturbation is bounded by using the time step $\Delta t = 1.41 \cdot 10^{-5} s$ and unbounded by using the time step $\Delta t = 1.42 \cdot 10^{-5} s$. Again the norms of the vectors $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$ divided by $\sqrt{N+1}$ are calculated and plotted (see Figure 7 for time step $\Delta t = 1.41 \cdot 10^{-5} s$ and Figure 8 for time step $\Delta t = 1.42 \cdot 10^{-5} s$).

From Figure 7, it can be seen that the perturbation remains bounded. For $\Delta t = 1.41 \cdot 10^{-5} s$ the numerical scheme is indeed stable.

From Figure 8, it can be concluded that the perturbation is unbounded. For $\Delta t = 1.42 \cdot 10^{-5} s$ it holds that the numerical scheme is unstable.

5. Conclusions

After examining the stability of the fourth-order Runge–Kutta and modified Sielecki numerical time-integration methods, it can be concluded that a bigger time step

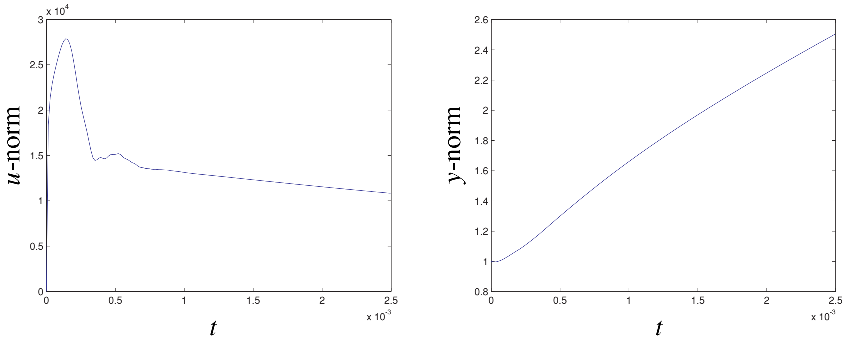

Figure 7. The norms of $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$, divided by $\sqrt{N+1}$, with $\Delta t = 1.41 \cdot 10^{-5} s$.

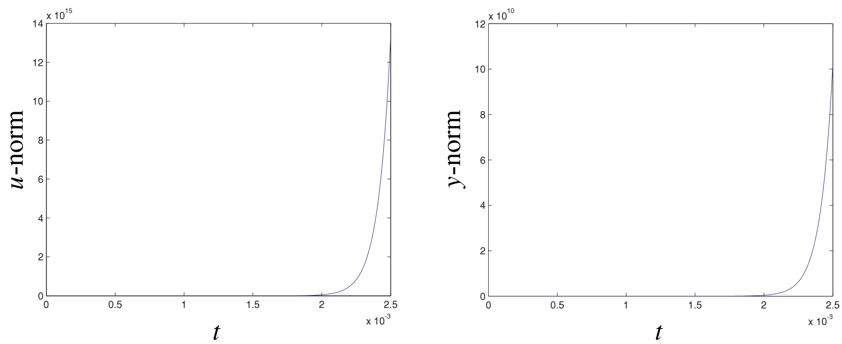

Figure 8. The norms of $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$, divided by $\sqrt{N+1}$, with $\Delta t = 1.42 \cdot 10^{-5} s$.

($\Delta t = 2.08 \cdot 10^{-5} s$) can be used for the Runge–Kutta four method than for the method modified Sielecki ($\Delta t = 1.41 \cdot 10^{-5} s$). This corresponds with the article by Diependaal, Duifhuis, Hoogstraten and Viergever [Diependaal et al. 1987].

When the time step $\Delta t = 2.08 \cdot 10^{-5} s$ is used, the fourth-order Runge–Kutta method is still stable, but the modified Sielecki method is then unstable. The modified Sielecki method already shows unstable behavior when a time step of $\Delta t = 1.42 \cdot 10^{-5} s$ is used.

Our numerical stability analysis for both time integration methods showed the following results:

- *Fourth-order Runge–Kutta:* when we use a time step of $2.08 \cdot 10^{-5} s$, the system is numerically stable, but numerically unstable for a time step of $2.09 \cdot 10^{-5} s$.
- *Modified Sielecki:* a time step of $1.41 \cdot 10^{-5} s$ causes the system to be numerically stable, and a time step of $1.42 \cdot 10^{-5} s$ causes it to be numerically unstable.

Following this numerical stability analysis, we tried to verify these results with a numerical test (Figure 4, 5, 7 and 8), and saw that the numerical tests supported the results from our analysis. Thus, the theoretical analyses and experimental analyses coincide.

List of symbols

- $p(x,t)$: transmembrane fluid pressure
- $y(x,t)$: excitation of the basilar membrane
- m_s: specific acoustic mass of the basilar membrane ($m_s = m \cdot b \cdot \Delta X$)
- $d_s(x)$: specific acoustic damping of the basilar membrane at place x ($d_s(x) = d(x) \cdot b \cdot \Delta X$ and $d_s(x_n) = d_{s_{\text{AMP}}}(x_n) \cdot d_{s_{\text{PROF}}}(x_n)$)
- $d_{s_{\text{AMP}}}(x)$: ensures that the damping in the cochlea is uniform everywhere ($d_{s_{\text{AMP}}}(x) = \epsilon\sqrt{m_s s_s(x)}$)
- $d_{s_{\text{PROF}}}(x)$: makes it possible to locally vary the (negative) damping ($d_{s_{\text{PROF}}}(x) = 1$ in the linear case)
- ϵ: models strength impulse response (Matlab: $\epsilon = 5 \cdot 10^{-2}$)
- $s_s(x)$: specific acoustic stiffness of the basilar membrane at place x ($s_s(x) = s(x) \cdot b \cdot \Delta X$ and $s_s(x_n) = s_0 \cdot e^{-\lambda x_n}$)
- s_0: specific acoustic stiffness constant (Matlab: $s_0 = 1 \cdot 10^{10}\ ^{\text{Pa}}/_{\text{m}}$)
- λ: value which determines place-frequency relation in the cochlea (Matlab: $\lambda = 300\,\text{m}^{-1}$)
- ρ: density of the cochlear fluid
- h: height of a scala ($h = S_{sc}/b$)
- m: acoustic mass of the basilar membrane
- $d(x)$: acoustic damping of the basilar membrane at place x
- $s(x)$: acoustic stiffness of the basilar membrane at place x
- ΔX: length of a subinterval ($\Delta X = L/(N+1)$)
- S_{sc}: surface of a scala
- b: width of a scala
- m_{c01}: acoustic mass of the cochlear fluid between the oval window and the first oscillator ($m_{c01} = \rho \Delta X_{01}/S_{sc}$)
- ΔX_{01}: distance between the oval window and the first oscillator
- m_m: acoustic mass of the middle ear ($m_m = n_t^2 Z_s/(S_t \omega_{rm} \delta)$)
- m_{sm}: specific acoustic mass of the middle ear ($m_{sm} = m_m S_{ST}$)
- Z_s: specific acoustic impedance of air
- S_t: surface of the eardrum

- ω_{rm}: resonance frequency of the middle ear
- δ: reciprocal value of the quality factor Q of the middle ear ($\delta = d_m/\sqrt{s_m m_m}$)
- d_m: acoustic damping of the middle ear
- s_m: acoustic stiffness of the middle ear
- n_t: transformation factor of the middle ear
- $p_e(t)$: form of the stimulus
- m_c: acoustic mass of the cochlear fluid between two oscillators ($m_c = \rho \Delta X / S_{sc}$)
- $u(x, t)$: velocity of the basilar membrane ($u(x, t) = \dot{y}(x, t)$)
- S_{ST}: surface of the stapes (Matlab: $S_{ST} = 3 \cdot 10^{-6}\,\mathrm{m}^2$)
- Z_a: acoustic impedance of air ($Z_a = Z_s/S_t$)
- s_{sm}: specific acoustic stiffness of the middle ear
- Δt: time step used in the fourth-order Runge–Kutta and modified Sielecki numerical methods

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