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Three approaches to a bracket polynomial for singular links

Carmen Caprau, Alex Chichester and Patrick Chu

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In this paper we extend the Kauffman bracket to singular links. Specifically, we define a polynomial invariant for singular links, and in doing this, we consider three approaches to our extended Kauffman bracket polynomial: (1) using skein relations involving singular link diagrams, (2) using representations of the singular braid monoid, (3) via a Yang–Baxter state model. We also study some properties of the extended Kauffman bracket.

1. Introduction and background

Knot theory is one of the most active research areas in mathematics. In the recent years, there has been a great interest in the study of knot-like objects, including singular links, knotted graphs, virtual knots and pseudoknots, not only because of their connections to other areas in mathematics, but also because of their applications to physics, chemistry, and molecular biology.

In this paper we focus on singular links and construct an invariant for such objects, based on the skein relation defining the Kauffman bracket for classical knots and links. We hope our work will prove useful for young researchers interested in knot theory for its intrinsic beauty or for its possible applications.

Knot theory studies embeddings of circles in three-dimensional space. When more than one circle is embedded in $\mathbb{R}^3$, the resulting embedding is called a *link*; otherwise, it is called a *knot*. In particular, a link is a disjoint union of knots, and these knots are called the components of the link. For simplicity, whenever possible, we will refer to both knots and links as knots. A diagram of a knot is a projection of the knot into a plane, and the *crossings* of a knot diagram are artifacts of the projection. We consider only *regular diagrams*, in which all crossings are double points.

A *singular link* is an immersion of a disjoint union of circles in three-dimensional space, which has finitely many singularities, called *singular crossings*, that are all

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Keywords: Kauffman bracket, invariants for knots and links, singular braids and links, Yang–Baxter equation.

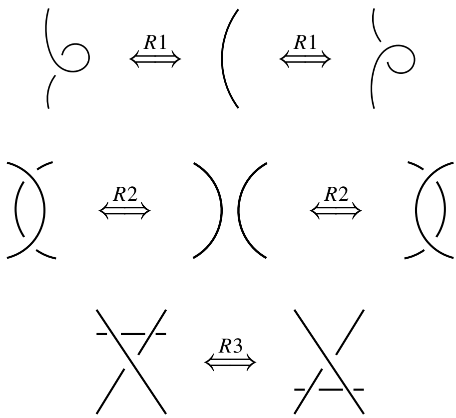


Figure 1. The Reidemeister moves.

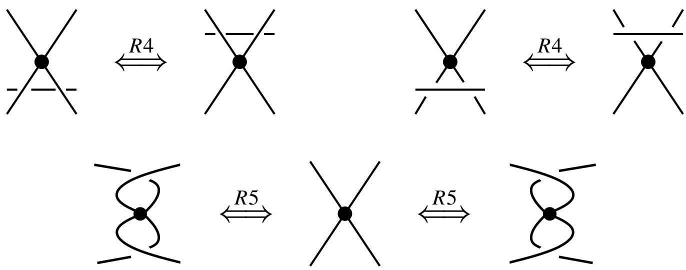


Figure 2. Additional moves for singular links.

transverse double points. A singular link can be regarded as an embedding in $\mathbb{R}^3$ of a four-valent graph with *rigid vertices*. We can think of such vertices as being rigid disks with four strands connected to it which turn as a whole when we flip the vertex by 180 degrees.

The goal of knot theory is to know whether or not two knots are isotopic. Two knots are called *ambient isotopic* if there is a continuously varying family of embeddings connecting one to the other. It is well known that two knot diagrams D_1 and D_2 represent ambient isotopic knots if and only if D_1 and D_2 are connected by a finite sequence of the Reidemeister moves, depicted in Figure 1. For more information on these and basic knot theory we refer the reader to the books [Adams 2004; Kauffman 2001; Murasugi 1996; Rolfsen 1976].

On the other hand, two singular link diagrams represent ambient isotopic singular links if their diagrams differ by a finite sequence of the Reidemeister moves together with the extended Reidemeister moves $R4$ and $R5$ shown in Figure 2; see [Kauffman 1989].

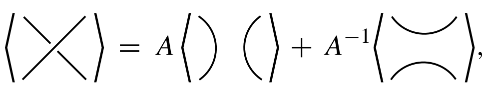
Any knot or link can be assigned an orientation, and there are two possible orientations for a knot and link component. The crossings of an oriented knot will have designated arrows due to the assigned orientation of the knot, and there are two types of crossings, namely positive and negative.

negative crossing positive crossing

Singular links may also be oriented or unoriented. If a singular link is oriented, then the singular crossings (or four-valent vertices) are crossing-type oriented, which is imposed by the fact that a singular link is an immersion in $\mathbb{R}^3$ of oriented circles with transversal double points.

In practice, it is tedious to work with Reidemeister moves to determine whether two diagrams represent equivalent knots (or singular links). Instead, one can work with an *invariant* for knots (or singular links), which is a quantity associated to the knot (or singular link) and is independent of the diagram of the knot (or singular link). Equivalently, if K_1 and K_2 are equivalent knots (or singular links), then $\mathrm{Inv}(K_1) = \mathrm{Inv}(K_2)$ for any invariant Inv. These invariants can be numbers, polynomials, groups, or more complex objects, such as homology theories. In this paper we are concerned with polynomial invariants.

The *Kauffman bracket* [1987] is a polynomial invariant for unoriented knots and links and is defined via a skein relation. A *skein relation* (as in (1-1)) is an identity involving knot diagrams (or singular link diagrams) that are the same except in a small neighborhood where they differ in the way indicated. The Kauffman bracket of a knot diagram K is denoted by $\langle K \rangle$, and is determined by

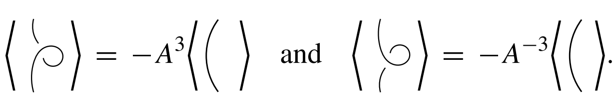


(1-1)

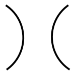
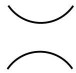
$$\langle \bigcirc \rangle = 1, \qquad \langle K \cup \bigcirc \rangle = (-A^2 - A^{-2})\,\langle K \rangle. \tag{1-2}$$

It is an enjoyable exercise to show that if two knot diagrams D_1 and D_2 differ by a Reidemeister move $R2$ or $R3$, then $\langle D_1 \rangle = \langle D_2 \rangle$. In other words, the Kauffman bracket is a *regular isotopy invariant* for knots. Note that if an invariant upholds the three Reidemeister moves it is called an *ambient isotopy invariant* for knots.

It is not hard to check that the Kauffman bracket has the following behavior with respect to the Reidemeister move $R1$:



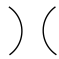
By the skein relation defining the Kauffman bracket, every crossing in a knot diagram L is locally replaced with one of the two possible *smoothings*,

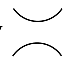
 or  ,

which will result in a finite number of disjoint circles, called a *state* of K. Note that if K contains n crossings, then there are 2^n states associated with K. The Kauffman bracket polynomial is thus a *state model* polynomial. In this state model, the polynomial $\langle K\rangle \in \mathbb{Z}[A, A^{-1}]$ is given by

$$\langle K\rangle = \sum_{\sigma} A^{\alpha(\sigma)-\beta(\sigma)}(-A^2 - A^{-2})^{\gamma(\sigma)-1},$$

where the sum is taken over all states σ of the knot diagram K, and where

$\alpha(\sigma)$ is the number of crossings  that have been replaced by  ,

$\beta(\sigma)$ is the number of crossings  that have been replaced by  ,

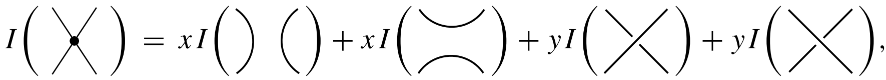
and $\gamma(\sigma)$ is the number of disjoint loops in the state σ. Sometimes, $\alpha(\sigma)$ and $\beta(\sigma)$ are referred to as the numbers of the A-smoothings and A^{-1}-smoothings, respectively.

One can use the Kauffman bracket polynomial to obtain an ambient isotopy invariant for oriented knots by counteracting the behavior of $\langle\,\cdot\,\rangle$ with respect to the move $R1$. This is done by defining the Kauffman X polynomial of an oriented knot K given by

$$X(K) := (-A^3)^{-w(K)}\langle K\rangle,$$

where $w(K)$ denotes the *writhe* of the oriented knot diagram K, given by the number of positive crossings minus the number of negative crossings, and where $\langle K\rangle$ is the Kauffman bracket of the unoriented knot diagram obtained from K. Since $w(K)$ and $\langle K\rangle$ are invariant under the moves $R2$ and $R3$, it follows that $X(K)$ is invariant under all three Reidemeister moves. Therefore, the polynomial X is an ambient isotopy invariant for oriented knots.

It is well-known that any polynomial invariant for classical links extends (in various ways) to an invariant of rigid-vertex isotopy for knotted four-valent graphs; see, for example, [Jonish and Millett 1991; Kauffman 1989; 2005; Kauffman and Magarshak 1995; Kauffman and Mishra 2013; Kauffman and Vogel 1992]. (Recall that a singular link can be regarded as a knotted four-valent graph with rigid vertices.) In particular, Kauffman and Vogel [1992] showed that if $I(K)$ is a regular isotopy polynomial invariant for unoriented knots and links, then imposing the skein relation



where x and y are commuting algebraic variables, yields a polynomial invariant, $I(G)$, of rigid-vertex regular isotopy for unoriented knotted graphs G (equivalently, it yields a regular isotopy invariant for unoriented singular links). This method certainly applies to the Kauffman bracket, and we start this paper by borrowing this approach with $x = 1$ and $y = 0$.

We remind the reader that one can also consider a regular isotopy invariant for oriented knots and links and extend it to oriented singular links by applying three local replacements at each singular crossings (that is, at each oriented vertex) and then taking a linear combination of the corresponding replacements. The three replacements are the positive crossing, the negative crossing, and the oriented smoothing at the vertex. For more details on this we refer the reader to [Kauffman 1989; Kauffman and Vogel 1992]. The work in [Kauffman and Magarshak 1995] contains possible applications to molecular biology of invariants of knotted rigid-vertex graphs. More recently, Kauffman and Mishra [2013] introduced a new method for constructing invariants of rigid vertex graph embeddings by using nonlocal combinatorial information that is available at each vertex. In particular, this paper uses the notions of Gauss code and parity for rigid-vertex graphs, and thus it is fundamentally different from the method mentioned earlier.

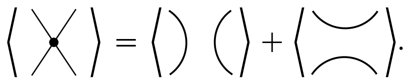
In this paper we work with a variant of the skein relation above to arrive at a version of the Kauffman bracket for singular links. The main scope of this paper is to show that the resulting polynomial for singular links can be defined in at least two more ways. By providing three approaches to the same polynomial invariant for singular links, we hope that a young researcher reading our paper will find a great deal of information which is educational and interesting, as it reveals beautiful connections between knot theory, combinatorics, abstract algebra, and statistical mechanics.

In Section 2 we give a detailed proof that using $x = 1$ and $y = 0$ in the above skein relation with $I(K) = \langle K \rangle$ yields an invariant for unoriented singular links. We refer to the resulting polynomial as the extended Kauffman bracket. In Section 3 we provide some properties of the extended Kauffman bracket and its associated ambient isotopy invariant for oriented singular links. In Section 4 we define a representation of the singular braid monoid into the Temperley–Lieb algebra, and use it to define a bracket polynomial for singular braids and ultimately recover the extended Kauffman bracket for singular links. Finally, in Section 5 we provide another method for constructing our extended Kauffman bracket; this method relies on a solution to the Yang–Baxter equation. By interpreting singular link diagrams as abstract tensor diagrams, we arrive at a Yang–Baxter state model for the extended Kauffman bracket.

2. An invariant for singular links

In this section, we extend the Kauffman bracket to singular links. For our purpose, we need to associate a skein relation to a singular crossing, and then check that the resulting polynomial is invariant under the extended Reidemeister moves $R4$ and $R5$.

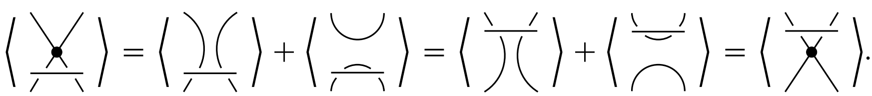
Given a singular link diagram L, we resolve each singular crossing in L using the skein relation



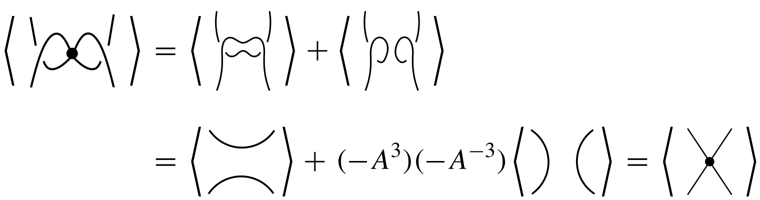
(2-1)

This process results in writing $\langle L\rangle$ as a $\mathbb{Z}[A, A^{-1}]$-linear combination of bracket evaluations of knots and links, which are then evaluated using the rules in (1-1) and (1-2). This yields a Laurent polynomial $\langle L\rangle \in \mathbb{Z}[A, A^{-1}]$.

Note that $\langle L\rangle$ is already invariant under the Reidemeister moves $R2$ and $R3$, since $\langle\cdot\rangle$ is a regular isotopy invariant for knots. Thus, we only need to check that $\langle L\rangle$ is invariant under the moves $R4$ and $R5$. We show this below, where along the way, we use the fact that $\langle\cdot\rangle$ is invariant under the move $R2$ and the behavior of $\langle\cdot\rangle$ with respect to the move $R1$:

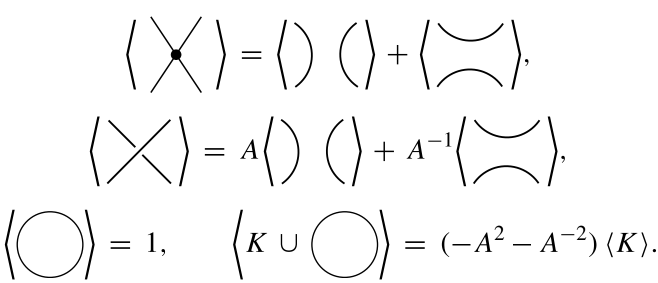


In addition,

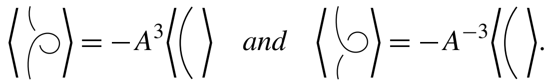


It follows that $\langle L\rangle$ is a regular isotopy polynomial invariant for singular links, which we call the *extended Kauffman bracket*. We have proved the statement below.

Theorem 1. *Let L be a singular link diagram and $\langle L\rangle \in \mathbb{Z}[A, A^{-1}]$ be the polynomial given by the following rules:*



Then $\langle L\rangle$ is a regular isotopy invariant for L and satisfies



We can define the writhe of an oriented singular link diagram in a similar manner as for the case of oriented knot diagrams. That is, the writhe $w(L)$ of an oriented singular link diagram L is given by the number of positive crossings minus the number of negative crossings. Note that $w(L)$ is independent of the number of singular crossings in L.

Theorem 2. *Let L be an oriented singular link diagram, and let $X(L)$ be the Laurent polynomial defined by*

$$X(L) := (-A^3)^{-w(L)}\langle L\rangle$$

where $\langle L\rangle$ is the extended Kauffman bracket of the unoriented singular link diagram represented by L. Then $X(L)$ is an ambient isotopy invariant for L.

3. Some properties of the extended Kauffman bracket

The goal of this section is to study the behavior of the extended Kauffman bracket polynomial and the polynomial X for singular links with respect to disjoint unions, connected sums, and mirror images of singular links.

For this purpose, we observe first that the extended Kauffman bracket of a singular link can also be defined using a state-sum formula. Let L be a singular link diagram with n classical crossings and m singular crossings. By resolving the classical and singular crossings in L using the first two skein relations in Theorem 1, we write $\langle L\rangle$ as a $\mathbb{Z}[A, A^{-1}]$-linear combination of bracket evaluations of the states associated with L. Note that L has 2^{n+m} states and that each state is a disjoint union of closed loops. Then

$$\langle L\rangle = \sum_{\sigma} A^{\alpha(\sigma)-\beta(\sigma)}(-A^2 - A^{-2})^{\gamma(\sigma)-1},$$

where the sum is taken over all states σ associated with the singular link diagram L, where $\gamma(\sigma)$ is the number of disjoint loops in a state σ, and where $\alpha(\sigma)$ and $\beta(\sigma)$ are, respectively, the numbers of A-smoothings and A^{-1}-smoothings in the state σ. (Observe that these smoothings correspond to classical crossings in L.)

Proposition 3. *Let $L_1 \cup L_2$ be the disjoint union of singular link diagrams L_1 and L_2. Then*

$$\langle L_1 \cup L_2\rangle = (-A^2 - A^{-2})\langle L_1\rangle\langle L_2\rangle.$$

Figure 3. A pair of disjoint links (left) and their connected sum (right).

Proof. Let $L = L_1 \cup L_2$ and let S be the set of all of the states corresponding to L. We have

$$\langle L\rangle = \langle L_1 \cup L_2\rangle = \sum_{\sigma\in S} A^{\alpha(\sigma)-\beta(\sigma)}(-A^2 - A^{-2})^{\gamma(\sigma)-1}.$$

Let S_1 and S_2 represent the set of all of the states associated with L_1 and L_2, respectively. Observe that the disjoint union of two singular links does not introduce any new crossings and that there is a canonical one-to-one correspondence between $S_1 \times S_2$ and S. For $\sigma_1 \in S_1$, $\sigma_2 \in S_2$, denote by $\sigma \in S$ the state of L which corresponds to (σ_1, σ_2). Then

$$\alpha(\sigma) = \alpha(\sigma_1) + \alpha(\sigma_2), \quad \beta(\sigma) = \beta(\sigma_1) + \beta(\sigma_2), \quad \gamma(\sigma) = \gamma(\sigma_1) + \gamma(\sigma_2).$$

Therefore,

$$\begin{aligned}
\langle L\rangle &= \sum_{\sigma\in S} A^{\alpha(\sigma)-\beta(\sigma)}(-A^2 - A^{-2})^{\gamma(\sigma)-1} \\
&= \sum_{(\sigma_1,\sigma_2)\in S_1\times S_2} A^{\alpha(\sigma_1)+\alpha(\sigma_2)-\beta(\sigma_1)-\beta(\sigma_2)}(-A^2 - A^{-2})^{\gamma(\sigma_1)+\gamma(\sigma_2)-1} \\
&= \sum_{(\sigma_1,\sigma_2)\in S_1\times S_2} A^{\alpha(\sigma_1)-\beta(\sigma_1)}(-A^2 - A^{-2})^{\gamma(\sigma_1)-1} A^{\alpha(\sigma_2)-\beta(\sigma_2)}(-A^2 - A^{-2})^{\gamma(\sigma_2)-1+1} \\
&= \langle L_1\rangle\langle L_2\rangle(-A^2 - A^{-2}).
\end{aligned}$$

□

Corollary 4. *Let $L_1 \cup L_2$ be the disjoint union of oriented singular link diagrams L_1 and L_2. Then,*

$$X(L_1 \cup L_2) = (-A^2 - A^{-2})X(L_1)X(L_2).$$

Proof. Note that $w(L_1 \cup L_2) = w(L_1) + w(L_2)$. Combining this and making use of Proposition 3,

$$\begin{aligned}
X(L_1 \cup L_2) &= (-A^3)^{-w(L_1\cup L_2)}\langle L_1 \cup L_2\rangle \\
&= (-A^3)^{-w(L_1)}\langle L_1\rangle \cdot (-A^3)^{w(L_2)}\langle L_2\rangle \cdot (-A^2 - A^{-2}) \\
&= (-A^2 - A^{-2})X(L_1)X(L_2).
\end{aligned}$$

□

A singular link diagram L is a *connected sum*, denoted by $L = L_1 \# L_2$, if it is displayed as two disjoint singular link diagrams L_1 and L_2 connected by parallel embedded arcs, up to planar isotopy. Figure 3 shows a connected sum of oriented diagrams.

Proposition 5. *Let L be a singular link diagram with the property that $L = L_1 \# L_2$ for some singular link diagrams L_1 and L_2. Then the polynomial $\langle L \rangle$ can be computed as*

$$\langle L \rangle = \langle L_1 \rangle \langle L_2 \rangle.$$

Proof. For every state σ of L, there is a pair of states σ_1 and σ_2 of L_1 and L_2, respectively, such that $\sigma = \sigma_1 \# \sigma_2$. Therefore, $\gamma(\sigma) = \gamma(\sigma_1) + \gamma(\sigma_2) - 1$, while

$$\alpha(\sigma) = \alpha(\sigma_1) + \alpha(\sigma_2) \quad \text{and} \quad \beta(\sigma) = \beta(\sigma_1) + \beta(\sigma_2).$$

Using a similar approach to that in the proof of Proposition 3, we have

$$\begin{aligned}\langle L_1 \# L_2 \rangle &= \sum_{\sigma} A^{\alpha(\sigma)-\beta(\sigma)}(-A^2 - A^{-2})^{\gamma(\sigma)-1} \\ &= \sum_{\sigma_1} A^{\alpha(\sigma_1)-\beta(\sigma_1)}(-A^2 - A^{-2})^{\gamma(\sigma_1)-1} \sum_{\sigma_2} A^{\alpha(\sigma_2)-\beta(\sigma_2)}(-A^2 - A^{-2})^{\gamma(\sigma_2)-1} \\ &= \langle L_1 \rangle \langle L_2 \rangle. \end{aligned}$$ □

Corollary 6. *Let L be an oriented singular link diagram such that $L = L_1 \# L_2$ for some oriented singular link diagrams L_1 and L_2. Then,*

$$X(L) = X(L_1) X(L_2).$$

Proof. The proof is similar to that of Corollary 4, and thus it is omitted. □

The *mirror image* of a singular link with diagram L is the singular link whose diagram L^* is obtained from L by changing the crossing type for all classical crossings in L. A singular link is *achiral* if it is ambient isotopic to its mirror image and *chiral* otherwise.

Proposition 7. *Let L^* denote the mirror image of a singular link diagram L. Then the extended Kauffman bracket of L^* is obtained from the extended Kauffman bracket of L by interchanging A and A^{-1}. That is,*

$$\langle L^* \rangle (A) = \langle L \rangle (A^{-1}).$$

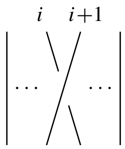
Proof. According to the state-sum formula defining the extended Kauffman bracket polynomial, it is easy to see that reversing the classical crossings in L replaces an A-smoothing with an A^{-1}-smoothing and vice versa. Hence, the statement follows at once. □

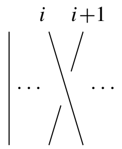
Corollary 8. *If $\langle L \rangle (A) \neq \langle L \rangle (A^{-1})$, then L is a chiral singular link.*

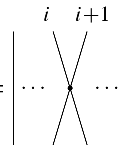
4. A representation of the singular braid monoid

In this section we provide a different approach to the extended Kauffman bracket for singular links, via a representation of the singular braid monoid.

4.1. ***The singular braid monoid.*** Let n be a positive integer, $n \geq 2$. Recall that the *singular braid monoid* on n strands, denoted SB_n, is the monoid generated by elements σ_i, σ_i^{-1}, and τ_i, for $1 \leq i \leq n-1$, where

$\sigma_i =$ 

, $\sigma_i^{-1} =$ 

, $\tau_i =$ 

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and satisfying the following relations, under the operation given by vertical concatenation of diagrams:

(1) $g_i h_j = h_j g_i$ for all $g_i, h_i \in \{\sigma_i, \sigma_i^{-1}, \tau_i\}$ and $1 \leq i, j \leq n-1$ with $|i-j| > 1$,

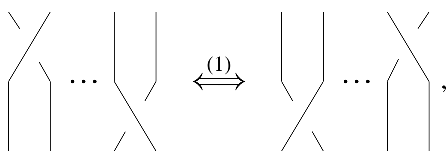
(2) $\sigma_i \sigma_i^{-1} = 1_n = \sigma_i^{-1} \sigma_i$ for all $1 \leq i \leq n-1$,

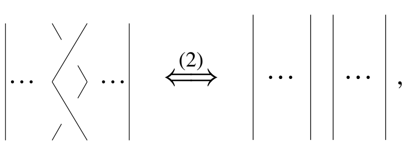
(3) $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ for all $1 \leq i, j \leq n-1$ with $|i-j| = 1$

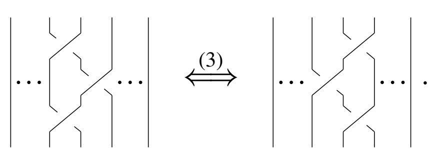
(4) $\tau_i \sigma_j \sigma_i = \sigma_j \sigma_i \tau_j$ for all $1 \leq i, j \leq n-1$ with $|i-j| = 1$,

(5) $\sigma_i \tau_i = \tau_i \sigma_i$ for all $1 \leq i \leq n-1$.

Note that the identity element, denoted 1_n, is represented by n vertical strands with no crossings. The geometric representations of the first three relations is given below (observe that relations (2) and (3) mimic the Reidemeister moves $R2$ and $R3$, respectively):

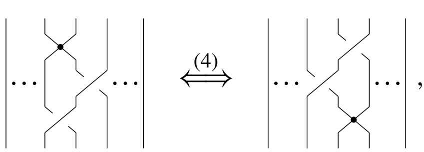


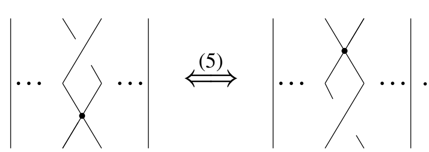




These three relations (where in (1) we exclude the relations involving the generators τ_i) are exactly the relations in the well-known Artin braid group.

In addition, note that relations (4) and (5) defining the singular braid monoid SB_n mimic, respectively, the moves $R4$ and $R5$ for singular link diagrams:





Due to Joan Birman [1993], we know that every singular link can be expressed as the *closure* of a singular braid, via ambient isotopy. Figure 4 displays the closure $\bar{\beta}$ of a braid β.

There are many different ways to represent a singular link as a closed singular braid. Bernd Gemein [1997] showed that two singular braids have isotopic closures if and only if there exists a finite sequence of singular braid relations and/or extended Markov moves (detailed below) transforming one singular braid into the other.

Let $w \in \mathrm{SB}_n$ be a braid on n strands and let w^* be the natural inclusion of w into SB_{n+1} obtained by adding an $(n+1)$-st strand to w. Then the following are called the *extended Markov moves*:

(M1) (a) $\tau_i w \sim w \tau_i$ for all $1 \le i \le n-1$,
(b) $\sigma_i w \sim w \sigma_i$ for all $1 \le i \le n-1$,

(M2) $w^* \sigma_n \sim w \sim w^* \sigma_n^{-1}$.

Figure 5 shows isotopic closed braids that differ by an extended Markov move.

Therefore, the works [Birman 1993; Gemein 1997] allow us to relate the theory of singular links with the theory of the singular braid monoid. In particular, we can study the extended Kauffman bracket via the singular braid monoid.

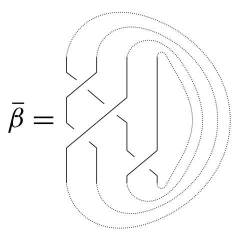


Figure 4. The closure of a braid.

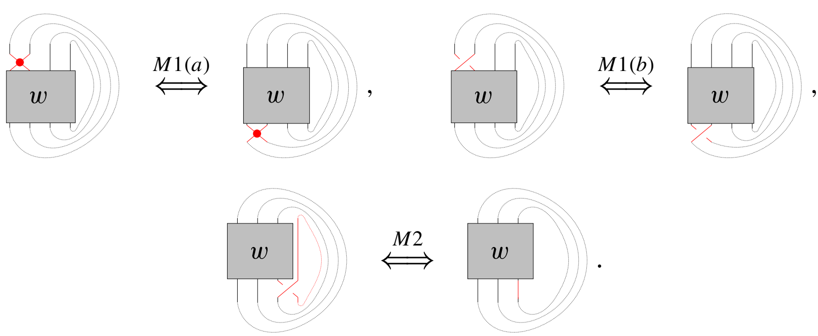
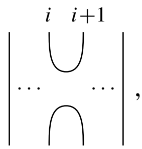


Figure 5. Equivalent singular links under extended Markov moves.

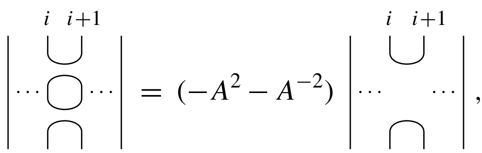
4.2. ***The Temperley–Lieb algebra.*** The Temperley–Lieb algebra played a central role in the discovery of the Jones polynomial [1985], and in the subsequent developments relating knot theory, topological quantum field theory, and statistical mechanics [Kauffman 2001]. Originally presented in terms of abstract generators and relations, it was combinatorially described by Kauffman as a planar diagram algebra in terms of his bracket polynomial for unoriented knots.

For each integer $n \geq 2$, the *n-strand Temperley–Lieb algebra*, denoted TL_n, is the unital, associative algebra over the ring $\mathbb{Z}[A, A^{-1}]$ generated by u_i, for $1 \leq i \leq n-1$, where

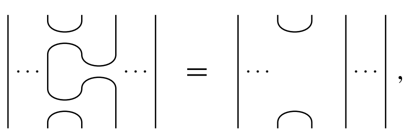
$u_i =$ 

along with the identity diagram, denoted 1_n, and subject to the following relations (where multiplication is given by vertical concatenation of diagrams):

- $u_i^2 = (-A^2 - A^{-2})u_i$ for all $1 \leq i \leq n-1$:



- $u_i u_j u_i = u_i$ for all $1 \leq i, j \leq n-1$ with $|i-j| = 1$:



- $u_i u_j = u_j u_i$ for all $1 \leq i, j \leq n-1$ with $|i-j| > 1$.

Observe that a generic element in TL_n is a formal $\mathbb{Z}[A, A^{-1}]$-linear combination of n-strand diagrams formed by multiplications of the generators u_i and the identity 1_n.

Define a *trace* function $\mathrm{tr} : \mathrm{TL}_n \to \mathbb{Z}[A, A^{-1}]$ given by $\mathrm{tr}(D) = (-A^2 - A^{-2})^{c-1}$, where c is the number of disjoint loops in the diagram $\overline{D}$ obtained by closing the diagram $D \in \mathrm{TL}_n$ in the same way that we close a braid or a singular braid. Then extend tr by linearity to all elements of TL_n.

It is easy to see that the function tr satisfies

$$\mathrm{tr}(xy) = \mathrm{tr}(x)\,\mathrm{tr}(y) \quad \text{for all } x, y \in \mathrm{TL}_n . \tag{4-1}$$

4.3. ***A representation of*** $\mathbf{SB}_n$**.** We observe that for any given homomorphism $\rho : \mathrm{SB}_n \to \mathrm{TL}_n$, we can compose it with the trace function tr to obtain, for any singular braid element $w \in \mathrm{SB}_n$, a polynomial $(\mathrm{tr} \circ \rho)(w) \in \mathbb{Z}[A, A^{-1}]$.

Inspired by the skein relations defining the extended Kauffman bracket for singular links, we define a homomorphism $\rho : \mathrm{SB}_n \to \mathrm{TL}_n$ as follows:

$$\begin{aligned} \tau_i &\overset{\rho}{\longmapsto} u_i + 1_n, \\ \sigma_i &\overset{\rho}{\longmapsto} A^{-1}u_i + A1_n, \\ \sigma_i^{-1} &\overset{\rho}{\longmapsto} Au_i + A^{-1}1_n. \end{aligned}$$

We can think of ρ as a function that resolves the crossings of the singular braid, since each σ_i, σ_i^{-1}, and τ_i represents a crossing of the strands in the singular braid.

Theorem 9. *The map ρ is a representation of the singular braid monoid* SB_n *into the Temperley–Lieb algebra* TL_n*. That is, ρ preserves the singular braid monoid relations.*

Proof. First, observe that ρ preserves the commuting relations in SB_n, since the generators for the algebra TL_n satisfy similar commuting relations. Note also that it must be the case that ρ preserves the relations (2)–(5) in SB_n, since the extended Kauffman bracket is invariant under the Reidemeister moves $R2$ and $R3$, as well as under the moves $R4$ and $R5$. However, we will check two of the singular braid monoid relations and leave the other relations as an exercise.

We start off by verifying that $\rho(\tau_i\sigma_j\sigma_i) = \rho(\sigma_j\sigma_i\tau_j)$. First, observe that

$$\begin{aligned} \rho(\tau_i\sigma_j\sigma_i) &= \rho(\tau_i)\rho(\sigma_j)\rho(\sigma_i) \\ &= (u_i + 1_n)(A^{-1}u_j + A1_n)(A^{-1}u_i + A1_n), \end{aligned}$$

$$\begin{aligned} \rho(\sigma_j\sigma_i\tau_j) &= \rho(\sigma_j)\rho(\sigma_i)\rho(\tau_j) \\ &= (A^{-1}u_j + A1_n)(A^{-1}u_i + A1_n)(u_j + 1_n). \end{aligned}$$

Employing the relations in TL_n, we have

$$\begin{aligned} &(u_i + 1_n)(A^{-1}u_j + A1_n)(A^{-1}u_i + A1_n) \\ &\quad = A^2u_i + u_i^2 + A^{-2}u_iu_ju_i + u_iu_j + A^{-2}u_ju_i + A^2 1_n + u_i + u_j \\ &\quad = A^2u_i + (-A^2 - A^{-2})u_i + A^{-2}u_i + u_iu_j + A^{-2}u_ju_i + A^2 1_n + u_i + u_j \\ &\quad = u_iu_j + A^{-2}u_ju_i + A^2 1_n + u_i + u_j \\ &\quad = A^2u_j + (-A^2 - A^{-2})u_j + A^{-2}u_j + u_iu_j + A^{-2}u_ju_i + A^2 1_n + u_i + u_j \\ &\quad = A^2u_j + u_j^2 + A^{-2}u_ju_iu_j + u_iu_j + A^{-2}u_ju_i + A^2 1_n + u_i + u_j \\ &\quad = (A^{-1}u_j + A1_n)(A^{-1}u_i + A1_n)(u_j + 1_n). \end{aligned}$$

It follows that the fourth relation defining SB_n is preserved by the map ρ. Next we show that $\rho(\tau_i\sigma_i) = \rho(\sigma_i\tau_i)$. Using basic computations, we obtain

$$\begin{aligned}\rho(\tau_i\sigma_i) = \rho(\tau_i)\rho(\sigma_i) &= (u_i + 1_n)(A^{-1}u_i + A1_n)\\ &= A^{-1}u_i^2 + A^{-1}u_i + Au_i + A1_n\\ &= (A^{-1}u_i + A1_n)(-A^{-3})(u_i + 1_n)\\ &= \rho(\sigma_i)\rho(\tau_i) = \rho(\sigma_i\tau_i).\end{aligned}$$

This shows that ρ also preserves the fifth relation defining SB_n. $\square$

Remark 10. For any $a, b \in \mathbb{Z}[A, A^{-1}]$, the homomorphism $f : \mathrm{SB}_n \to \mathrm{TL}_n$ given by

$$\tau_i \overset{f}{\mapsto} au_i + b1_n \quad \text{and} \quad \sigma_i^{\pm 1} \overset{f}{\mapsto} A^{\mp 1}u_i + A^{\pm 1}1_n$$

also defines a representation of the singular braid monoid SB_n into the Temperley–Lieb algebra TL_n. The proof that f preserves the singular braid monoid relations follows verbatim as that for the map ρ.

4.4. ***The bracket polynomial of a singular braid.*** In this section, we show how to recover the extended Kauffman bracket of singular links by making use of the map ρ and the trace function tr.

Let $\beta \in \mathrm{SB}_n$ be a singular braid on n strands and denote by $\mathrm{wr}(\beta)$ the writhe of β, defined as the sum of the number of generators of type σ_i minus the sum of the generators of type σ_j^{-1} in the expression of β.

Define the function $\langle \cdot \rangle : \mathrm{SB}_n \to \mathbb{Z}[A, A^{-1}]$, given by the formula

$$\langle \beta \rangle = (-A^3)^{-\mathrm{wr}(\beta)}(\mathrm{tr} \circ \rho)(\beta).$$

We call $\langle \beta \rangle$ the bracket polynomial of the singular braid β.

Proposition 11. *The bracket polynomial of a singular braid is well-defined on singular braids, and is invariant under the extended Markov moves. Moreover, if L is a singular link diagram in braid form such that $L = \bar{\beta}$ for some $\beta \in \mathrm{SB}_n$, then*

$$\langle \beta \rangle = (-A^3)^{-\mathrm{wr}(\beta)}\langle \bar{\beta} \rangle = (-A^3)^{-\mathrm{wr}(\beta)}\langle L \rangle.$$

Proof. Since ρ is a representation of SB_n and the writhe of the singular braid is invariant under the relations in SB_n, it follows that the bracket polynomial of a singular braid is well-defined on singular braids. The trace function tr satisfies (4-1), and thus

$$(\mathrm{tr} \circ \rho)(\tau_i w) = \mathrm{tr}(\rho(\tau_i)\rho(w)) = \mathrm{tr}(\rho(w)\rho(\tau_i)) = (\mathrm{tr} \circ \rho)(w\tau_i),$$

and similarly,

$$(\mathrm{tr} \circ \rho)(\sigma_i w) = (\mathrm{tr} \circ \rho)(w\sigma_i)$$

for all $\tau_i, \sigma_i, w \in \mathrm{SB}_n$. Thus $\langle \cdot \rangle$ is invariant under the extended Markov moves of type (M1). Moreover, the coefficient $(-A^3)^{-\operatorname{wr}(\beta)}$ in the expression of $\langle \cdot \rangle$ cancels the effect of a Markov move of type (M2):

$$\langle w^* \sigma_n \rangle = (-A^3)^{-\operatorname{wr}(\beta)-1} (\operatorname{tr} \circ \rho)(w^* \sigma_n) = (-A^3)^{-\operatorname{wr}(\beta)} (\operatorname{tr} \circ \rho)(w) = \langle w \rangle.$$

Finally, due to [Birman 1993; Gemein 1997] and the definitions for the maps ρ and tr, the second part of the statement follows immediately. □

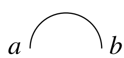
5. The Yang–Baxter equation and the extended Kauffman bracket

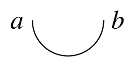
We will show now how to arrive at the extended Kauffman bracket by interpreting singular link diagrams as *abstract tensor diagrams* and employing a solution to the Yang–Baxter equation.

5.1. ***A Yang–Baxter model for the extended Kauffman bracket.*** Our approach here is an extension from classical knots to singular links of the Yang–Baxter state model for the Kauffman bracket, as introduced in [Kauffman 2001].

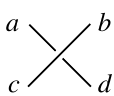
A singular link diagram D can be decomposed with respect to a height function into minima (creations), maxima (annihilations) and crossings (interactions), as illustrated in Figure 6. That is, the diagram D is constructed from interconnected maxima, minima, and crossings (there might be some curves with no critical points vis-a-vis the height function), and we want to associate to them square matrices with entries in the ring $\mathbb{Z}[A, A^{-1}]$. We start by labeling the edges of the diagram D with *spins* from the index set $I = \{1, 2\}$.

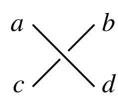
We will denote the following portions of the link diagram as follows:

$M_{a,b} \longleftrightarrow$ 

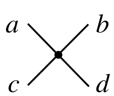
, $M^{a,b} \longleftrightarrow$ 

,

$R^{a,b}_{c,d} \longleftrightarrow$ 

, $\bar{R}^{a,b}_{c,d} \longleftrightarrow$ 

,

$Q^{a,b}_{c,d} \longleftrightarrow$ 

, $\delta^a_b \longleftrightarrow$ 

,

where $a, b, c, d \in I$ and

$$\delta^a_b = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{if } a \neq b. \end{cases}$$

Using these conventions, we wish to associate to any singular link diagram D a polynomial $\tau(D) \in \mathbb{Z}[A, A^{-1}]$ so that $\tau(D)$ recovers the extended Kauffman bracket $\langle D \rangle$. The expression $\tau(D)$ is obtained by taking the sum over all internal labels (spins on the arcs of the diagram D) of the products of symbols representing maxima, minima, and crossings (classical and singular).

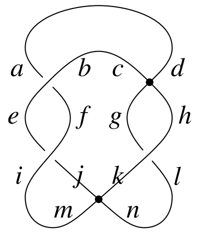


Figure 6. An abstract tensor singular link diagram.

For example, for the diagram D in Figure 6, $\tau(D)$ is given by the following sum of products of abstract tensor symbols:

$$\tau(D) = \sum_{a,b,\dots,n\in I} M_{a,d}M_{b,c}R^{a,b}_{e,f}R^{e,f}_{i,j}M^{i,m}Q^{j,k}_{m,n}M^{n,l}R^{g,h}_{k,l}Q^{c,d}_{g,h},$$

where the sum is over all possible choices of indices (spins from I) in the expression. Note that the order of the factors in a product of abstract tensors does not matter, since the abstract tensors are elements of the commutative ring $\mathbb{Z}[A, A^{-1}]$.

We will use the following notational conventions:

$$X = (X)^{a,b}_{c,d} = \begin{bmatrix} X^{1,1}_{1,1} & X^{1,1}_{1,2} & X^{1,1}_{2,1} & X^{1,1}_{2,2} \\ X^{1,2}_{1,1} & X^{1,2}_{1,2} & X^{1,2}_{2,1} & X^{1,2}_{2,2} \\ X^{2,1}_{1,1} & X^{2,1}_{1,2} & X^{2,1}_{2,1} & X^{2,1}_{2,2} \\ X^{2,2}_{1,1} & X^{2,2}_{1,2} & X^{2,2}_{2,1} & X^{2,2}_{2,2} \end{bmatrix}$$

and

$$(B)_{a,b} = (B)^{a,b} = (B)^a_b = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}.$$

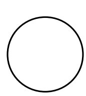
Observe that

$$\sum_{c,d\in I} X^{a,b}_{c,d}Y^{c,d}_{e,f} = (XY)^{a,b}_{e,f} \quad \text{for all } a, b, e, f \in I.$$

This can be easily seen by rewriting $X^{a,b}_{c,d}$ as X^i_j, where $i = b + 2(a-1)$ and $j = d + 2(c-1)$, since

$$\sum_{j=1}^{4} X^i_j Y^j_k = (XY)^i_k.$$

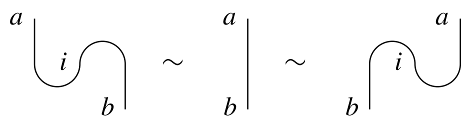
To arrive at the bracket polynomial, the matrices corresponding to maxima and minima need to satisfy

$$\sum_{a,b\in I} M_{a,b}M^{a,b} \longleftrightarrow$$  $$\longleftrightarrow -A^2 - A^{-2}.$$

By imposing $M_{a,b} = M^{a,b}$ for $a, b \in I$, the above equality becomes

$$\sum_{a,b\in I} (M_{a,b})^2 = -A^2 - A^{-2} = \sum_{a,b\in I} (M^{a,b})^2. \tag{5-1}$$

Since we want $\tau(D)$ to be a topological invariant, pairs of maxima and minima should cancel as shown:



Therefore, we need that

$$\sum_{i\in I} M^{a,i} M_{i,b} = \delta^a_b = \sum_{i\in I} M_{b,i} M^{i,a}$$

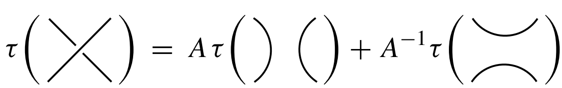
or, equivalently,

$$\sum_{i\in I} M_{a,i} M_{i,b} = \delta^a_b = \sum_{i\in I} M_{b,i} M_{i,a}. \tag{5-2}$$

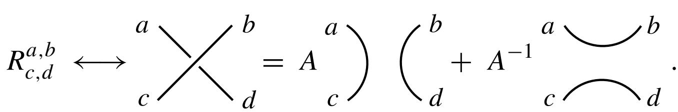
It follows that the matrix $M = (M_{a,b})$ should be its own inverse. The following matrix satisfies (5-1) and (5-2):

$$M = \begin{bmatrix} 0 & iA \\ -iA^{-1} & 0 \end{bmatrix}, \quad \text{where } i^2 = -1.$$

We wish $\tau(D)$ to satisfy the Kauffman bracket skein relation



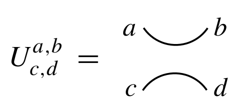
and thus the R-matrix should satisfy



Therefore,

$$R^{a,b}_{c,d} = A\delta^a_c \delta^b_d + A^{-1} M^{a,b} M_{c,d} \quad \text{for all } a, b, c, d \in I.$$

Note that the matrix $U = (U^{a,b}_{c,d}) := (M^{a,b} M_{c,d})$, where



has the following expression:

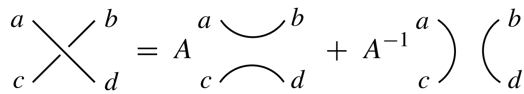
$$U=\begin{bmatrix} M^{1,1}M_{1,1} & M^{1,1}M_{1,2} & M^{1,1}M_{2,1} & M^{1,1}M_{2,2}\\ M^{1,2}M_{1,1} & M^{1,2}M_{1,2} & M^{1,2}M_{2,1} & M^{1,2}M_{2,2}\\ M^{2,1}M_{1,1} & M^{2,1}M_{1,2} & M^{2,1}M_{2,1} & M^{2,1}M_{2,2}\\ M^{2,2}M_{1,1} & M^{2,2}M_{1,2} & M^{2,2}M_{2,1} & M^{2,2}M_{2,2} \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & -A^2 & 1 & 0\\ 0 & 1 & -A^{-2} & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Moreover, observe that

$$(\delta^a_c\,\delta^b_d)=\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix}=(\delta^a_c)\otimes(\delta^b_d),$$

where $\otimes$ represents the Kronecker product of matrices.

Furthermore, the $\bar{R}$-matrix should satisfy

$\bar{R}^{a,b}_{c,d} \longleftrightarrow$ 

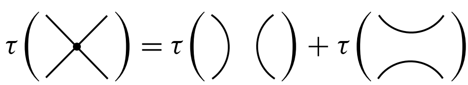
and thus

$$\bar{R}^{a,b}_{c,d}=AM^{a,b}M_{c,d}+A^{-1}\delta^a_c\,\delta^b_d \quad \text{for all } a,b,c,d\in I.$$

We arrive at the following matrices associated with classical crossings:

$$R=\begin{bmatrix} A & 0 & 0 & 0\\ 0 & 0 & A^{-1} & 0\\ 0 & A^{-1} & A-A^{-3} & 0\\ 0 & 0 & 0 & A \end{bmatrix} \quad \text{and} \quad \bar{R}=\begin{bmatrix} A^{-1} & 0 & 0 & 0\\ 0 & A^{-1}-A^3 & A & 0\\ 0 & A & 0 & 0\\ 0 & 0 & 0 & A^{-1} \end{bmatrix}.$$

Finally, we wish $\tau(D)$ to also satisfy



which forces the matrix Q associated with a singular crossing to be given by

$$Q^{a,b}_{c,d}=\delta^a_c\,\delta^b_d+M^{a,b}M_{c,d} \quad \text{for all } a,b,c,d\in I.$$

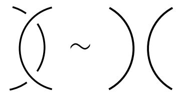
Equivalently,

$$Q=(Q^{a,b}_{c,d})=\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1-A^2 & 1 & 0\\ 0 & 1 & 1-A^{-2} & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now that we have defined $\tau(D)$ for a given singular link diagram D, we need to make sure that it is a regular isotopy invariant for D. That is, we need to verify that if D_1 and D_2 are singular link diagrams that differ by a Reidemeister move $R2$ or $R3$, or by an extended Reidemeister move $R4$ or $R5$, then $\tau(D_1) = \tau(D_2)$.

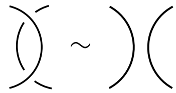
An easy check shows that matrices R and $\bar{R}$ are inverses of each other, since $R\bar{R} = I_{4\times 4} = \bar{R}R$. Equivalently,

$$\sum_{i,i\in I} R^{a,b}_{i,j}\bar{R}^{i,j}_{c,d} \longleftrightarrow$$



$$\longleftrightarrow \delta^a_c\delta^b_d,$$

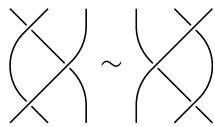
$$\sum_{i,i\in I} \bar{R}^{a,b}_{i,j}R^{i,j}_{c,d} \longleftrightarrow$$



$$\longleftrightarrow \delta^a_c\delta^b_d.$$

Hence, $\tau(D)$ is invariant under the Reidemeister move $R2$. Moreover, we have that

$$\sum_{i,j,k\in I} R^{a,b}_{i,j}R^{j,c}_{k,f}R^{i,k}_{d,e} \longleftrightarrow$$



$$\longleftrightarrow \sum_{i,j,k\in I} R^{b,c}_{i,j}R^{a,i}_{d,k}R^{k,j}_{e,f}.$$

The latter relation is the *Yang–Baxter equation* (YBE):

$$\sum_{i,j,k\in I} R^{a,b}_{i,j}R^{j,c}_{k,f}R^{i,k}_{d,e} = \sum_{i,j,k\in I} R^{b,c}_{i,j}R^{a,i}_{d,k}R^{k,j}_{e,f},$$

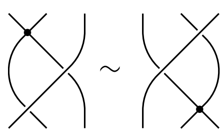
which can be rewritten as

$$(R\otimes I)(I\otimes R)(R\otimes I) = (I\otimes R)(R\otimes I)(I\otimes R).$$

That is, the R-matrix as defined above is a solution of the YBE. Similarly, one can easily verify that the matrix $\bar{R}$ is a solution of the YBE. It follows that $\tau(D)$ is invariant under the Reidemeister move $R3$.

Furthermore, it is not hard to check that the following holds:

$$\sum_{i,j,k\in I} Q^{a,b}_{i,j}R^{j,c}_{k,f}R^{i,k}_{d,e} \longleftrightarrow$$



$$\longleftrightarrow \sum_{i,j,k\in I} R^{b,c}_{i,j}R^{a,i}_{d,k}Q^{k,j}_{e,f},$$

or, equivalently,

$$(Q\otimes I)(I\otimes R)(R\otimes I) = (I\otimes R)(R\otimes I)(I\otimes Q).$$

A similar relation holds for R being replaced by $\bar{R}$. Hence, $\tau(D)$ is invariant under the extended Reidemeister move $R4$.

Finally, observe that $RQ = QR$ and $\bar{R}Q = Q\bar{R}$, or equivalently,

$$RQ\bar{R} = Q \quad \text{and} \quad \bar{R}QR = Q.$$

Therefore, $\tau(D)$ is invariant under the extended Reidemeister move $R5$.

According to the above discussion, we have proved the following statement.

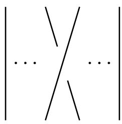
Theorem 12. *The polynomial $\tau(D) \in \mathbb{Z}[A, a^{-1}]$ is an invariant of regular isotopy for singular links.*

Remark 13. We note that $\tau(D)$ is the unnormalized extended Kauffman bracket. That is,

$$\tau(D) = (-A^2 - A^{-2})\langle D\rangle,$$

where $\langle D\rangle$ is the extended Kauffman bracket introduced in Section 2.

5.2. ***Yet another representation of* $\mathbf{SB}_n$.** We can use the matrices R, $\bar{R}$, and Q to define a representation of the singular braid monoid SB_n into a matrix algebra over the ring $\mathbb{Z}[A, A^{-1}]$. Observe first that we can regard a generator for SB_n as an abstract tensor diagram. For example,

$\sigma_i =$  $\longleftrightarrow \delta^{a_1}_{b_1} \cdots R^{a_i,a_{i+1}}_{b_i,b_{i+1}} \cdots \delta^{a_n}_{b_n} \in M_{2^n\times 2^n}(\mathbb{Z}[A, A^{-1}]).$

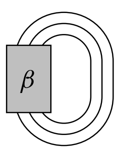
Inspired by this, we define a homomorphism $\Delta : \mathrm{SB}_n \to M_{2^n\times 2^n}(\mathbb{Z}[A, A^{-1}])$ given by

$$\begin{aligned}
\sigma_i &\mapsto I^{\otimes(i-1)} \otimes R \otimes I^{\otimes(n-i-1)},\\
\sigma_i^{-1} &\mapsto I^{\otimes(i-1)} \otimes \bar{R} \otimes I^{\otimes(n-i-1)},\\
\tau_i &\mapsto I^{\otimes(i-1)} \otimes Q \otimes I^{\otimes(n-i-1)}.
\end{aligned}$$

Since the polynomial $\tau(D)$ is a regular isotopy invariant for singular links, it follows that the map Δ preserves the singular braid monoid relations. Therefore, the following statement holds.

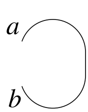
Proposition 14. *The mapping Δ is a representation of the singular braid monoid SB_n into the matrix algebra $M_{2^n\times 2^n}(\mathbb{Z}[A, A^{-1}])$.*

The mapping Δ provides yet another method for obtaining the extended bracket polynomial of a singular link. Let L be a singular link diagram in braid form and let $\beta \in \mathrm{SB}_n$ be the singular braid whose closure is L. That is, $L = \bar{\beta}$:

$L =$ 

.

When closing a braid, each braid strand contributes a diagram and an associated matrix of the form

$\eta^a_b =$ 

$\longleftrightarrow \sum_{c\in I} M_{a,c}M^{b,c}, \quad$ where $a, b \in I$.

The matrix $\eta = (\eta_b^a)$ is

$$\eta = \begin{bmatrix} M_{1,1}M^{1,1} + M_{1,2}M^{1,2} & M_{1,1}M^{2,1} + M_{1,2}M^{2,2} \\ M_{2,1}M^{1,1} + M_{2,2}M^{1,2} & M_{2,1}M^{2,1} + M_{2,2}M^{2,2} \end{bmatrix} = \begin{bmatrix} -A^2 & 0 \\ 0 & -A^{-2} \end{bmatrix}.$$

Observe that Trace(η), the trace of the matrix η, is $-A^2 - A^{-2}$. Moreover,

$$\tau(L) = \mathrm{Trace}(\eta^{\otimes n}\Delta(\beta)),$$

whenever $\beta \in \mathrm{SB}_n$ and $\bar{\beta} = L$.

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