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Symplectic embeddings of four-dimensional ellipsoids into polydiscs

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McDuff and Schlenk recently determined exactly when a four-dimensional symplectic ellipsoid symplectically embeds into a symplectic ball. Similarly, Frenkel and Müller recently determined exactly when a symplectic ellipsoid symplectically embeds into a symplectic cube. Symplectic embeddings of more complicated sets, however, remain mostly unexplored. We study when a symplectic ellipsoid $E(a,b)$ symplectically embeds into a polydisc $P(c,d)$. We prove that there exists a constant C depending only on d/c (here, d is assumed greater than c) such that if b/a is greater than C, then the only obstruction to symplectically embedding $E(a,b)$ into $P(c,d)$ is the volume obstruction. We also conjecture exactly when an ellipsoid embeds into a scaling of $P(1,b)$ for $b \geq 6$, and conjecture about the set of (a,b) such that the only obstruction to embedding $E(1,a)$ into a scaling of $P(1,b)$ is the volume. Finally, we verify our conjecture for $b = \frac{13}{2}$.

1. Introduction

Statement of results. Let (X_0, ω_0) and (X_1, ω_1) be symplectic manifolds. A *symplectic embedding* of (X_0, ω_0) into (X_1, ω_1) is a smooth embedding φ such that

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$\varphi^*(\omega_1) = \omega_0$. It is interesting to ask when one symplectic manifold embeds into another. For example, define the (open) four-dimensional symplectic *ellipsoid*

$$E(a, b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} < 1 \right\}, \tag{1-1}$$

and define the (open) *symplectic ball* $B(a) := E(a, a)$. These inherit symplectic forms by restricting the standard form $\omega = \sum_{k=1}^{2} dx_k dy_k$ on $\mathbb{R}^4 = \mathbb{C}^2$. McDuff and Schlenk [2012] determined exactly when a four-dimensional symplectic ellipsoid $E(a, b)$ embeds symplectically into a symplectic ball, and found that if b/a is small, then the answer involves an "infinite staircase" determined by the Fibonacci numbers with odd index, while if b/a is large then all obstructions vanish except for the volume obstruction.

To give another example, define the (open) four-dimensional *polydisc*

$$P(a, b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \pi |z_1|^2 < a,\ \pi |z_2|^2 < b \right\}, \tag{1-2}$$

where $a, b \geq 1$ are real numbers and the symplectic form is again given by restricting the standard symplectic form on $\mathbb{R}^4$. Frenkel and Müller [2012] determined exactly when a four-dimensional symplectic ellipsoid symplectically embeds into a *cube* $C(a) := P(a, a)$ and found that part of the expression involves the Pell numbers. Cristofaro-Gardiner and Kleinman [2013] studied embeddings of four-dimensional ellipsoids into scalings of $E\left(1, \frac{3}{2}\right)$ and also found that part of the answer involves an infinite staircase determined by a recursive sequence.

Here we study symplectic embeddings of an open four-dimensional symplectic ellipsoid $E(a, b)$ into an open four-dimensional symplectic polydisc $P(c, d)$. By scaling, we can encode this embedding question as the function

$$d(a, b) := \inf\left\{ \lambda : E(1, a) \overset{s}{\hookrightarrow} P(\lambda, b\lambda) \right\}, \tag{1-3}$$

where a and b are real numbers that are both greater than or equal to 1.

The function $d(a, b)$ always has a lower bound, $\sqrt{a/(2b)}$, the volume obstruction. Our first theorem states that for fixed b, if a is sufficiently large then this lower bound is sharp, i.e., all embedding obstructions vanish aside from the volume obstruction:

Theorem 1.1. *If $a \geq 9(b+1)^2/(2b)$, then $d(a, b) = \sqrt{a/(2b)}$.*

This is an analogue of a result of Buse and Hind [2013] concerning symplectic embeddings of one symplectic ellipsoid into another.

From the previously mentioned work of McDuff and Schlenk, Frenkel and Müller, and Cristofaro-Gardiner and Kleinman, one expects that if a is small then the function $d(a, b)$ should be more rich. Our results suggest that this is indeed the case. For example, we completely determine the graph of $d\left(a, \frac{13}{2}\right)$ (see Figure 1).

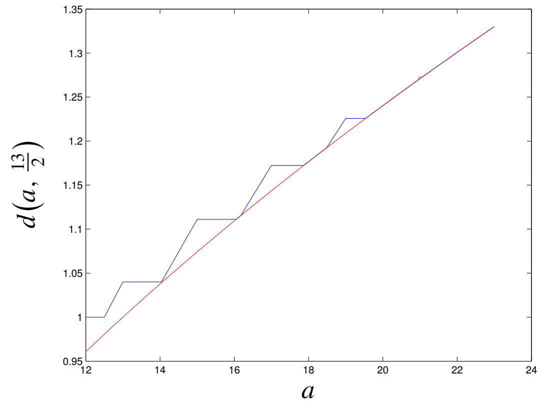

Figure 1. The graph of $d\big(a, \frac{13}{2}\big)$. The red line represents the volume obstruction.

Theorem 1.2. *For $b = \frac{13}{2}$, we have $d(a, b) \geq \sqrt{a/13}$ and is equal to this lower bound for all a except in the following cases*:

(i) $d\big(a, \frac{13}{2}\big) = 1$ *for all* $a \in \big[1, \frac{25}{2}\big]$.

(ii) *For $k \in \mathbb{Z}$, with* $0 \leq k \leq 4$,

$$d(a,b) = \begin{cases} \dfrac{2a}{25+2k} & \text{if } a \in [\alpha_k, 13+2k], \\ \dfrac{26+4k}{25+2k} & \text{if } a \in [13+2k, \beta_k], \end{cases}$$

where

$$\alpha_0 = \tfrac{25}{2}, \quad \alpha_1 = \tfrac{351}{25}, \quad \alpha_2 = \tfrac{841}{52}, \quad \alpha_3 = \tfrac{961}{52}, \quad \alpha_4 = \tfrac{1089}{52},$$
$$\beta_0 = \tfrac{351}{25}, \quad \beta_1 = \tfrac{1300}{81}, \quad \beta_2 = \tfrac{15028}{841}, \quad \beta_3 = \tfrac{18772}{961}, \quad \beta_4 = \tfrac{2548}{121}.$$

Interestingly, the graph of $d\big(a, \frac{13}{2}\big)$ has only finitely many nonsmooth points, in contrast to the infinite staircases in [McDuff and Schlenk 2012; Frenkel and Müller 2012; Cristofaro-Gardiner and Kleinman 2013]. This appears to be the case for many values of b. For example, we conjecture what the function $d(a, b)$ is for all $b \geq 6$; see Conjecture 6.3.

Our proofs rely on the following remarkable theorem of Frenkel and Müller [2012]. Let $N(a, b)$ be the sequence (indexed starting at 0) of all nonnegative integer linear combinations of a and b, arranged with repetitions in nondecreasing order, and let $M(a, b)$ be the sequence whose k-th term is

$$\min\big\{ma + nb \,:\, (m+1)(n+1) \geq k+1\big\},$$

where $k, m, n \in \mathbb{Z}_{\geq 0}$. Write $N(a, b) \leq M(c, d)$ if each term in the sequence $N(a, b)$ is less than or equal to the corresponding term in $M(c, d)$. Frenkel and Müller showed that embeddings of an ellipsoid into a polydisc are completely determined by the sequences M and N:

Theorem 1.3 [Frenkel and Müller 2012]. *There is a symplectic embedding*

$$E(a,b) \overset{s}{\hookrightarrow} P(c,d)$$

if and only if $N(a,b) \le M(c,d)$.

To motivate the sequences M and N, note that N is the sequence of *ECH capacities* of the symplectic ellipsoid $E(a,b)$, while M is the sequence of ECH capacities of the symplectic polydisc $P(c,d)$. The ECH capacities are a sequence of nonnegative (possibly infinite) real numbers, defined for any symplectic four-manifold, that obstruct symplectic embeddings. We will not discuss ECH capacities here; see [Hutchings 2014] for a survey. Theorem 1.3 is equivalent to the statement that the ECH capacities give sharp obstructions to embeddings of an ellipsoid into a polydisc.

2. Proof of Theorem 1.1

Weight sequences and the #-operation. We begin by describing the machinery that will be used to prove Theorem 1.1.

Let a^2 be a rational number. McDuff [2011] showed that there is a finite sequence

$$W(1,a^2) = (a_1, \dots, a_n),$$

called the *(normalized) weight sequence for* a^2, such that $E(1,a^2)$ embeds into a symplectic ellipsoid if and only if the disjoint union $\bigsqcup B(W) := \bigsqcup B(a_i)$ embeds into that ellipsoid.

To describe the weight sequence, let

$$W(a^2,1) = (X_0^{\times \ell_0}, X_1^{\times \ell_1}, \dots, X_k^{\times \ell_k}), \tag{2-1}$$

where $X_0 > X_1 > \dots > X_k$ and $\ell_k \ge 2$. The ℓ_i are the multiplicities of the entries X_i and come from the continued fraction expansion

$$a^2 = \ell_0 + \cfrac{1}{\ell_1 + \frac{1}{\ell_2 + \dots + \frac{1}{\ell_k}}} := [\ell_0; \ell_1, \dots, \ell_k].$$

The entries of (2-1) are defined as

$$X_{-1} := a^2, \qquad X_0 = 1, \qquad X_{i+1} = X_{i-1} - \ell_i X_i \quad \text{for } i \ge 0.$$

Important properties of the weight sequence include

$$\sum_i a_i^2 = a^2, \tag{2-2}$$

$$\sum_i a_i = a^2 + 1 - \frac{1}{q}, \tag{2-3}$$

where for all i, we have $a_i \le 1$ and $a = p/q$.

We will also make use of a helpful operation, #, as in [McDuff 2011]. Suppose s_1 and s_2 are sequences indexed with $k \in \mathbb{Z}$, starting at 0. Then,

$$(s_1 \# s_2)_k = \sup_{i+j=k} (s_1)_i + (s_2)_j.$$

A useful application of # is the following lemma:

Lemma 2.1 [McDuff 2011]. *For all* $a, b > 0$, *we have*

$$N(a, a) \# N(a, b) = N(a, a+b).$$

More generally, for all $\ell \geq 1$, *we have*

$$(\#^{\ell} N(a, a)) \# N(a, b) = N(a, b + \ell a).$$

This lemma together with the weight sequence and scaling implies that

$$N(1, a^2) = N(a_1, a_1) \# \cdots \# N(a_n, a_n). \tag{2-4}$$

Similar to [McDuff 2011], this machinery allows us to reduce Theorem 1.1 to a ball-packing problem.

Proof of Theorem 1.1. We begin by noting that the ECH capacities for $B(a)$ are

$$N(a, a) = (0, a, a, 2a, 2a, 2a, 3a, 3a, 3a, 3a, \dots),$$

where the terms $N_k(a, a)$ of this sequence are of the form da and for each d there are $d + l$ entries occurring at

$$\tfrac{1}{2}(d^2 + d) \leq k \leq \tfrac{1}{2}(d^2 + 3d). \tag{2-5}$$

Similarly, for the sequence $(a/\sqrt{2b})M(1, b)$, each term $(a/\sqrt{2b})M_k(1, b)$ is of the form $d(a/\sqrt{2b})$, where

$$k \leq \frac{d^2}{4b} + \frac{(1+b)d}{2b} + \frac{b^2 - 2b + 1}{4b}. \tag{2-6}$$

By continuity, it suffices to study $d(a^2, b)$ with a^2 rational. So, we can prove that the volume obstruction is the only obstruction when $a \geq 3(b+1)/\sqrt{2b}$ by showing that

$$N(1, a^2) \leq \frac{a}{\sqrt{2b}} M(1, b) \tag{2-7}$$

for said a-values.

By (2-5) and (2-6), it is therefore sufficient to show that

$$\sum_i d_i a_i \leq \frac{a}{\sqrt{2b}} d \tag{2-8}$$

whenever $d_1, \dots, d_m, d$ are nonnegative integers such that

$$\sum_i (d_i^2 + d_i) \le 2\left(\frac{d^2}{4b} + \frac{(1+b)d}{2b} + \frac{b^2-2b+1}{4b}\right). \tag{2-9}$$

We do so by considering the following cases:

Case 1: $\sum_i d_i^2 \le d^2/(2b)$. In this case, the Cauchy–Schwarz inequality along with (2-2) implies (2-8).

Case 2: $\sum_i d_i^2 > d^2/(2b)$. This case, along with (2-9), implies

$$\sum_i d_i a_i \le \sum_i d_i \le \frac{(1+b)d}{b} + \frac{b^2-2b+1}{2b}.$$

So, we need

$$\frac{(1+b)d}{b} + \frac{b^2-2b+1}{2b} \le \frac{a}{\sqrt{2b}} d.$$

It follows that

$$a \ge \frac{b+1}{\sqrt{2b}}\left(2 + \frac{b+1}{d}\right). \tag{2-10}$$

Now let $d = b+1$. We see that (2-6) is equivalent to

$$k \le b + 1 + \frac{1}{4b}.$$

It is easy to see that $N_k(1, a^2) \le (a/\sqrt{2b}) M_k(1, b)$ for all such k values. As such, we can apply $d = b+1$ to (2-10) to get

$$a \ge \frac{3(b+1)}{\sqrt{2b}}, \tag{2-11}$$

and hence the desired result. □

Remark 2.2. We allow $d = b+1$ in the statement of Theorem 1.1. However, if we show $N_k(1, a^2) \le (a/\sqrt{2b}) M_k(1, b)$ for all

$$k \le \frac{d^2}{4b} + \frac{(1+b)d}{2b} + \frac{b^2-2b+1}{4b},$$

then we can use this d in (2-10) to achieve a sharper bound for a.

3. Proof of Theorem 1.2, Part I

We begin by computing $d\left(a, \frac{13}{2}\right)$ on the regions where it is linear.

Nondifferentiable points and Ehrhart polynomials. We first compute d at certain values. These will eventually be the points a where $d\left(a, \frac{13}{2}\right)$ is not differentiable.

Proposition 3.1. *We have*

$$\begin{aligned}
&d\big(1,\tfrac{13}{2}\big)=1, && d\big(\tfrac{25}{2},\tfrac{13}{2}\big)=1, && d\big(13,\tfrac{13}{2}\big)=\tfrac{26}{25},\\
&d\big(\tfrac{351}{25},\tfrac{13}{2}\big)=\tfrac{26}{25}, && d\big(15,\tfrac{13}{2}\big)=\tfrac{10}{9}, && d\big(\tfrac{1300}{81},\tfrac{13}{2}\big)=\tfrac{10}{9},\\
&d\big(\tfrac{841}{52},\tfrac{13}{2}\big)=\tfrac{29}{26}, && d\big(17,\tfrac{13}{2}\big)=\tfrac{34}{29}, && d\big(\tfrac{15028}{841},\tfrac{13}{2}\big)=\tfrac{34}{29},\\
&d\big(\tfrac{961}{52},\tfrac{13}{2}\big)=\tfrac{31}{26}, && d\big(19,\tfrac{13}{2}\big)=\tfrac{38}{31}, && d\big(\tfrac{18772}{961},\tfrac{13}{2}\big)=\tfrac{38}{31},\\
&d\big(\tfrac{1089}{52},\tfrac{13}{2}\big)=\tfrac{33}{26}, && d\big(21,\tfrac{13}{2}\big)=\tfrac{42}{33}, && d\big(\tfrac{2548}{121},\tfrac{13}{2}\big)=\tfrac{42}{33}.
\end{aligned}$$

To prove the proposition, the main difficulty comes from the fact that applying Theorem 1.3 in principle requires checking infinitely many ECH capacities. Our strategy for overcoming this difficulty is to study the growth rate of the terms in the sequences M and N. We will find that in every case needed to prove Proposition 3.1, one can bound these growth rates to conclude that only finitely many terms in the sequences need to be checked. This is then easily done by computer. The details are as follows:

Proof. **Step 1:** For the sequence $N(a,b)$, let $k(a,b,t)$ be the largest k such that $N_k(a,b)\le t$. Similarly, for the sequence $M(c,d)$, let $l(c,d,t)$ be the largest l such that $M_l(c,d)\le t$. To show that $E(a,b)\overset{s}{\hookrightarrow}P(c,d)$, by Theorem 1.3, we just have to show that for all t, we have $k(a,b,t)\ge l(c,d,t)$.

Step 2: We can estimate $k(a,b,t)$ by applying the following proposition:

Proposition 3.2. *If a, b, r, and t are all positive integers, then*

$$k\left(\frac{a}{r},\frac{b}{r},t\right)=\frac{1}{2ab}(tr)^2+\frac{1}{2}(tr)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{ab}\right)+\frac{1}{4}\left(1+\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{12}\left(\frac{a}{b}+\frac{b}{a}+\frac{1}{ab}\right)$$
$$+\frac{1}{a}\sum_{j=1}^{a-1}\frac{\xi_a^{j(-tr)}}{(1-\xi_a^{jb})(1-\xi_a^{j})}+\frac{1}{b}\sum_{l=1}^{b-1}\frac{\xi_b^{l(-tr)}}{(1-\xi_b^{la})(1-\xi_b^{l})},\tag{3-1}$$

where $\xi_d=e^{2\pi i/d}$.

Proof. The number of terms in $N(a/r,b/r)$ that are less than t is the same as the number of lattice points (m,n) in the triangle bounded by the positive x- and y-axes and the line $x(a/r)+y(b/r)\le t$. For integral t, this number can be computed by applying the theory of "Ehrhart polynomials". Proposition 3.2 follows by applying [Beck and Robins 2007, Theorem 2.10]. □

We will be most interested in this proposition in the case where $a=r$. Note that by the last two terms of the formula in Proposition 3.2, we have $k(a/r,b/r,t)$ is a periodic polynomial with period ab.

We also need an argument to account for the fact that Proposition 3.2 is only for integral t, whereas the argument in Step 1 involves real t. To account for this, we use an asymptotic argument. Specifically, for $E(1, a/r)$, with $a, r \in \mathbb{Z}_{\geq 1}$, we bound the right-hand side of (3-1) from below by taking the floor function of t. It is convenient for our argument to further bound this expression from below by

$$\frac{c_1}{r^2}(rt-1)^2 + \frac{c_2}{r}(rt-1) + c_3, \tag{3-2}$$

where the c_i are the coefficients of the right-hand side of (3-1) that do not involve t or r.

This is the lower bound that we will use for $k(1, a/r, t)$.

Step 3: To get an upper bound $l(c, d, t)$ for $M(c, d)$, recall that

$$M_l(c, d) = \min\{cm + dn \,:\, (m+1)(n+1) \geq l+1\}.$$

For $cm + dn = t$, we solve for m in terms of n and find

$$\left(\frac{t-dn}{c} + 1\right)(n+1) - 1 \geq l.$$

Considering $m, n \in \mathbb{R}$, we can take the derivative of the left side of the inequality with respect to n and then set the expression equal to 0 to maximize it. We do the same with m to obtain

$$\left(\frac{t}{2d} + \frac{c}{2d} + \frac{1}{2}\right)\left(\frac{t}{2c} + \frac{d}{2c} + \frac{1}{2}\right) - 1 \geq l.$$

By simplifying, we get that an upper bound for l is

$$l(c, d, t) = \frac{t^2}{4cd} + \frac{(c+d)t}{2cd} + \frac{(c-d)^2}{4cd}. \tag{3-3}$$

Our strategy now is to check that for each point in Proposition 3.1, we have $k(a, b, t) \geq l(c, d, t)$ asymptotically in t for the corresponding (a, b, c, d). From there, we can check that for a sufficient number of terms, $N(1, a) \leq M(\lambda, \lambda b)$.

Step 4: Since the rest of the proof amounts to computation, it is best summarized by Table 1. In the table, k_{t^2} and l_{t^2} denote the coefficients of the quadratic terms in the upper and lower bounds from Steps 2 and 3, while k_t and l_t denote the corresponding coefficients of the linear terms.

The t-column gives a sufficient number to check up to before the asymptotic bounds from the previous three steps are enough. Note that if k_{t^2} and l_{t^2} in any row are equal, then linear coefficients, k_t and l_t, are used to make an asymptotic argument; this explains the appearance of the "N/A"s in the table. It is simple to check by computer that the relevant N and M sequences in each row satisfy $N \leq M$ once one knows that the problem only has to be checked up to the t in the t-column.

The rightmost column of Table 1 gives an ECH capacity that shows that one cannot shrink λ further, i.e., the claimed embeddings are actually sharp. □

$E(1,a)\overset{s}{\hookrightarrow} P(\lambda,\lambda b)$	k_{t^2}	l_{t^2}	k_t	l_t	t	ECH obstruction
$E(1,\frac{25}{2})\overset{s}{\hookrightarrow} P(1,\frac{13}{2})$	$\frac{1}{25}$	$\frac{1}{26}$	N/A	N/A	51	1
$E(1,13)\overset{s}{\hookrightarrow} P(\frac{26}{25},\frac{169}{25})$	$\frac{1}{26}$	$\frac{625}{17576}$	N/A	N/A	33	13
$E(1,\frac{351}{25})\overset{s}{\hookrightarrow} P(\frac{26}{25},\frac{169}{25})$	$\frac{25}{702}$	$\frac{625}{17576}$	N/A	N/A	522	13
$E(1,15)\overset{s}{\hookrightarrow} P(\frac{10}{9},\frac{65}{9})$	$\frac{1}{30}$	$\frac{81}{2600}$	N/A	N/A	29	15
$E(1,\frac{1300}{81})\overset{s}{\hookrightarrow} P(\frac{10}{9},\frac{65}{9})$	$\frac{81}{2600}$	$\frac{81}{2600}$	$\frac{691}{1300}$	$\frac{27}{52}$	272	15
$E(1,\frac{841}{52})\overset{s}{\hookrightarrow} P(\frac{29}{26},\frac{29}{4})$	$\frac{26}{841}$	$\frac{26}{841}$	$\frac{447}{841}$	$\frac{15}{29}$	122	17
$E(1,17)\overset{s}{\hookrightarrow} P(\frac{34}{29},\frac{221}{29})$	$\frac{1}{34}$	$\frac{841}{30056}$	N/A	N/A	27	17
$E(1,\frac{15028}{841})\overset{s}{\hookrightarrow} P(\frac{34}{29},\frac{221}{29})$	$\frac{841}{30056}$	$\frac{841}{30056}$	$\frac{7935}{15028}$	$\frac{435}{884}$	32	17
$E(1,\frac{961}{52})\overset{s}{\hookrightarrow} P(\frac{31}{26},\frac{31}{4})$	$\frac{26}{961}$	$\frac{26}{961}$	$\frac{507}{961}$	$\frac{15}{31}$	23	19
$E(1,19)\overset{s}{\hookrightarrow} P(\frac{38}{31},\frac{247}{31})$	$\frac{1}{38}$	$\frac{961}{37544}$	N/A	N/A	7	19
$E(1,\frac{18772}{961})\overset{s}{\hookrightarrow} P(\frac{38}{31},\frac{247}{31})$	$\frac{961}{37544}$	$\frac{961}{37544}$	$\frac{759}{1444}$	$\frac{465}{988}$	28	19
$E(1,\frac{1089}{52})\overset{s}{\hookrightarrow} P(\frac{33}{26},\frac{33}{4})$	$\frac{26}{1089}$	$\frac{26}{1089}$	$\frac{571}{1089}$	$\frac{15}{33}$	14	21
$E(1,21)\overset{s}{\hookrightarrow} P(\frac{42}{33},\frac{273}{33})$	$\frac{1}{42}$	$\frac{121}{5096}$	N/A	N/A	26	21
$E(1,\frac{2548}{121})\overset{s}{\hookrightarrow} P(\frac{42}{33},\frac{273}{33})$	$\frac{121}{5096}$	$\frac{121}{5096}$	$\frac{1335}{2548}$	$\frac{165}{364}$	41	21

Table 1. The computations from Step 4 of the proof of Proposition 3.1.

The linear steps. Given the computations from the previous section, the computation of $d(a,\frac{13}{2})$ for all the "linear steps", i.e., those portions of the graph of d for which d is linear, is straightforward. Indeed, we have the following two lemmas:

Lemma 3.3. *For fixed b, the function $d(a,b)$ is monotonically nondecreasing.*

Proof. This follows from the fact that $E(1,a)\overset{s}{\hookrightarrow} E(1,a')$ if $a \le a'$. □

Lemma 3.4 (subscaling). $\quad d(\lambda a, b) \le \lambda d(a,b)$.

Proof. This follows from the fact that $E(1,\lambda a)\overset{s}{\hookrightarrow} E(\lambda,\lambda a)$ for $\lambda \ge 1$. □

By monotonicity, we know that $d(a,\frac{13}{2})$ is constant on the intervals

$$[1,\tfrac{25}{2}],\quad [13,\tfrac{351}{25}],\quad [15,\tfrac{1300}{81}],\quad [17,\tfrac{15028}{841}],\quad [19,\tfrac{18772}{961}],\quad [21,\tfrac{2548}{121}].$$

We now explain why for $k \in \mathbb{Z}$, with $0 \le k \le 4$, we have

$$d\big(a,\tfrac{13}{2}\big) = \frac{2a}{25+2k} \quad \text{for } a \in [\alpha_k, 13+2k],$$

where $\alpha_0 = \frac{25}{2}$, $\alpha_1 = \frac{351}{25}$, $\alpha_2 = \frac{841}{52}$, $\alpha_3 = \frac{961}{52}$, and $\alpha_4 = \frac{1089}{52}$.

Given the critical points we have determined, along with the subscaling lemma, we have $2a/(25+2k)$ as an upper bound for $d\big(a, \frac{13}{2}\big)$ on the above intervals.

Intervals on which $d\big(a, \frac{13}{2}\big)$ is linear. We also know that

$$d\big(a, \tfrac{13}{2}\big) = \sup\left\{ \frac{N_x(1,a)}{M_x\big(1, \frac{13}{2}\big)} : x \in \mathbb{N} \right\} \geq \frac{N_l(1,a)}{M_l\big(1, \frac{13}{2}\big)} \quad \text{for any } l.$$

Here is a representative example of our method:

Example 3.5. To illustrate how this can give us a suitable lower bound, consider the case where $x = 13$:

$$\sup\left\{ \frac{N_x(1,a)}{M_x\big(1, \frac{13}{2}\big)} : x \in \mathbb{N} \right\} \geq \frac{N_{13}(1,a)}{M_{13}\big(1, \frac{13}{2}\big)} = \frac{2a}{25} \quad \text{for } a \in \big[\tfrac{25}{2}, 13\big].$$

This lower bound equals the upper bound given by Lemma 3.4, so we have proven Theorem 1.2 for $a \in \big[\frac{25}{2}, 13\big]$.

The general method is similar: given $a \in [\alpha_k, 13+2k]$, we can find an l such that

$$\frac{N_l(1,a)}{M_l\big(1, \frac{13}{2}\big)} = \frac{2a}{25+2k}.$$

Such obstructing values of l are given in the following table:

k	$\frac{2}{25+2k}$	l
0	$\frac{2}{25}$	13
1	$\frac{2}{27}$	15
2	$\frac{2}{29}$	17
3	$\frac{2}{31}$	19
4	$\frac{2}{33}$	21

Given $a \in [\alpha_k, 13+2k]$ for each integer $k \in [0, 4]$, we have found that the upper and lower bounds of $d\big(a, \frac{13}{2}\big)$ equal $2a/(25+2k)$. Thus, we have proven our claim for these intervals.

4. Proof of Theorem 1.2, Part II

To complete the proof of Theorem 1.2, we need to show that aside from the linear steps described in the previous section, the graph of $d\big(a, \frac{13}{2}\big)$ is equal to the graph of the volume obstruction. To do this, we adapt some of the ideas from [McDuff and Schlenk 2012] in a purely combinatorial way. This will be needed to complete the proof of Theorem 1.2. Our combinatorial perspective on the techniques from [McDuff and Schlenk 2012] borrows many ideas from [McDuff 2011].

Preliminaries. This section collects the main combinatorial machinery that will be used to complete the proof. The basic idea behind our proof will be to reduce to a ball-packing problem, as in the proof of Theorem 1.1. The machinery we develop here will be useful for approaching this ball-packing problem.

We begin with two definitions:

Definition 4.1. Let $\mathrm{Cr}(d, d_i) = (d', d_i')$, where $d' = 2d - d_1 - d_2 - d_3$, $d_i' = d - d_j - d_k$ for $i, j, k = 1, 2, 3$ and $d_i' = d_i$ for all $i \geq 4$. We call Cr the *Cremona transform*.

Definition 4.2. We say $(d, d_i) \in \mathbb{R}^{1+n}$ is

(i) *positive* if $d, d_i \geq 0$ for all i,

(ii) *ordered* if $d_i, d_{i+1} \neq 0$ implies $d_i \geq d_{i+1}$ and $d_i \neq 0$, $d_j = 0$ implies $i < j$,

(iii) *reduced* if positive, ordered, and $d \geq d_1 + d_2 + d_3$.

Remark 4.3. It will be important to note that $\mathrm{Cr}(\mathrm{Cr}(d, d_i)) = (d, d_i)$.

We now define a product analogous to the intersection product in [McDuff and Schlenk 2012]:

Definition 4.4.
$$(x, x_i) \cdot (y, y_i) = xy - \sum_i x_i y_i.$$

We also define a vector $-K \in \mathbb{R}^{1+n}$ that is motivated by the standard anticanonical divisor in the M-fold blow up of $\mathbb{C}P^2$.

Definition 4.5.
$$-K = (3, 1, 1, \dots, 1).$$

The following is a combinatorial analogue of "positivity of intersections" that will be useful:

Lemma 4.6. *If* (x, x_i) *is reduced*, (d, d_i) *is positive*, $-K \cdot (d, d_i) \geq 0$, *and* $d \geq \max(d_i)$, *then* $(x, x_i) \cdot (d, d_i) \geq 0$.

Proof. Let (d', d_i') be the vector obtained from ordering d_i. As

$$(x, x_i) \cdot (d, d_i) \geq (x, x_i) \cdot (d', d_i'),$$

we can assume without loss of generality that (d, d_i) is ordered. If $x_3 = 0$ then $x_i = 0$ for $i \geq 3$ and

$$(x, x_i) \cdot (d, d_i) = xd - x_1 d_1 - x_2 d_2.$$

As $d \geq \max(d_i)$, we know that this expression is greater than or equal to

$$(x - x_1 - x_2) d.$$

As (x, x_i) is reduced, this is greater than or equal to 0.

We now assume without loss of generality that $x_3 = 1$. Hence, $x_i \leq 1$ for $i \geq 3$. Let $e_1 = x_1 - 1$ and $e_2 = x_2 - 1$. Then

$$xd \geq (3 + e_1 + e_2)d$$

as (x, x_i) is reduced. This expression is equal to

$$3d + de_1 + de_2.$$

As $d \geq d_1, d_2$, we now have the following chain of inequalities:

$$\begin{aligned} 3d + de_1 + de_2 &\geq 3d + d_1 e_1 + d_2 e_2 \geq \sum_i d_i + d_1 e_1 + d_2 e_2 \\ &= d_1 x + d_2 x + \sum_{i \geq 3} d_i \geq d_1 x_1 + d_2 x_2 + \sum_{i \geq 3} x_i d_i = \sum_i d_i x_i. \quad \square \end{aligned}$$

In [McDuff and Schlenk 2012], Cremona transformations preserve the intersection product. Here we prove an analogous result.

Lemma 4.7. $\mathrm{Cr}(x, x_i) \cdot \mathrm{Cr}(y, y_i) = (x, x_i) \cdot (y, y_i).$

Proof.

$$\begin{aligned} \mathrm{Cr}(x, x_i) \cdot \mathrm{Cr}(y, y_i) &= x' y' - \sum_i x'_i y'_i \\ &= (2x - x_1 - x_2 - x_3)(2y - y_1 - y_2 - y_3) - (x - x_2 - x_3)(y - y_2 - y_3) \\ &\quad - (x - x_1 - x_3)(y - y_1 - y_3) - (x - x_2 - x_3)(y - y_2 - y_3) - \sum_{i > 3} x_i y_i \\ &= xy - x_1 y_1 - x_2 y_2 - x_3 y_3 - \sum_{i > 3} x_i y_i \\ &= xy - \sum_i x_i y_i \\ &= (x, x_i) \cdot (y, y_i). \end{aligned}$$

$\square$

The following sets will also be useful:

Definition 4.8. $F = \{(d, d_i) : (d, d_i) \cdot (-K + (d, d_i)) \geq 0,\ d, d_i \in \mathbb{Z}\}.$

Definition 4.9. $F^+ = \{(d, d_i) : (d, d_i) \in F,\ d, d_i \geq 0\}.$

Definition 4.10. $E = \{(d, d_i) : (d, d_i) \cdot (d, d_i) \geq -1, -K \cdot (d, d_i) = 1,\ d, d_i \in \mathbb{Z}\}.$

Remark 4.11. Observe $\mathrm{Cr}(F) \subset F$ and $\mathrm{Cr}(E) \subset E$. Additionally, F, F^+, and E are invariant under permutations of d_i.

Remark 4.12. Note that $(0, -1, 0, \ldots, 0) \in E$.

Definition 4.13. Let C be the set of (x, x_i) such that $x, x_i \in \mathbb{Z}$ and

(a) $(x, x_i) \cdot (x, x_i) \geq 0$,

(b) $(x, x_i) \cdot (d, d_i) \geq 0$ for all $(d, d_i) \in E$.

Both Li and Li [2002] and McDuff and Schlenk [2012] have found that compositions of Cremona transformations and permutations can reduce certain classes. Here we prove a combinatorial version of those lemmas.

Lemma 4.14. *If* $(x, x_i) \in C$ *then by a sequence of Cremona transforms and permutations of* x_i*, we can transform* (x, x_i) *to* (x', x_i')*, where* (x', x_i') *is reduced.*

Proof. We begin with some helpful results:

Sublemma 4.15. $\mathrm{Cr}(C) \subset C.$

Proof. The fact that Cr preserves (a) follows from the fact that Cr preserves the intersection product. To complete the sublemma, note that if $(d, d_i) \in E$, then

$$\mathrm{Cr}(x, x_i) \cdot (d, d_i) = \mathrm{Cr}^2(x, x_i) \cdot (d', d_i') = (x, x_i) \cdot (d', d_i') \geq 0 \quad \text{as } (d', d_i') \in E. \quad \square$$

Sublemma 4.16. *If* P *is some permutation,* $P(C) \subset C$.

Proof. If $(d, d_i) \in E$, then

$$P(x, x_i) \cdot (d, d_i) = (x, x_i) \cdot P^{-1}(d, d_i) \quad \text{as } P^{-1}(E) \subset E. \qquad \square$$

Sublemma 4.17. *If* $(x, x_i) \in C$*, then* $x, x_i \geq 0$.

Proof. If $d_i = (-\delta_{ij})$ and $(0, d_i) \in E$ then we have $j \leq \mathrm{length}(d_i)$ for all j. So, $(x, x_i) \cdot (0, d_i) = x_j \geq 0$. We also have $(x, x_i) \cdot (1, 1, 1, 0, 0, \dots, 0) = x - x_1 - x_2 \geq 0$. As $x_1, x_2 \geq 0$, this implies that $x \geq 0$. $\square$

Let oCr denote the transformation Cr followed by ordering the d_i. Fix $(x, x_i) \in C$. Let $(x^k, x_i^k) = \mathrm{oCr}^k(x, x_i)$. Let $\alpha(k) = x^k - x_1^k - x_2^k - x_3^k$. It suffices to show $\alpha(k) \geq 0$ for some k. Assume not. Then $\alpha(k) \leq -1$ for all k. By Sublemmas 4.15 and 4.16, $\mathrm{oCr}(C) \subset C$. For $k \geq 1$,

$$x^k = x^{k-1} + \alpha(k-1) \leq x^{k-1} - 1.$$

Thus, there exists k such that $x^k < 0$. This contradicts Sublemma 4.17, completing the proof that we may reduce (x, x_i). $\square$

We now prove Lemma 4.18, a result analogous to [McDuff and Schlenk 2012, Proposition 1.2.12(i)].

Lemma 4.18. *If* $(x, x_i) \in C$ *then* $(x, x_i) \cdot (d, d_i) \geq 0$ *for all* $(d, d_i) \in F$.

Proof. By Lemma 4.14 there exists A, a composition of Cr and permutations, such that $A(x, x_i) = (x', x_i')$ with (x', x_i') reduced. For $(d, d_i) \in F$, let $A(d, d_i) = (d', d_i') \in F$. So,

$$(x, x_i) \cdot (d, d_i) = A(x, x_i) \cdot A(d, d_i) = (x', x_i') \cdot (d', d_i').$$

Let $e = d$, $e_i = d_i$ if $d_i > 0$ and $e_i = 0$ if $d_i \leq 0$. We note $(e, e_i) \in F$ and

$$(x', x_i') \cdot (d', d_i') \geq (x', x_i') \cdot (e, e_i).$$

If $(e, e_i)\cdot(e, e_i) \geq 0$ then the Cauchy–Schwarz inequality shows $(x', x_i')\cdot(e, e_i) \geq 0$. Otherwise, $(e, e_i)\cdot(-K) \geq 0$. Then

$$\sum_i e_i^2 + e_i \leq e^2 + 3e$$

implies $e \geq e_i$, so Lemma 4.6 shows $(x', x_i')\cdot(e, e_i) \geq 0$. □

Remark 4.19. By scaling, Lemma 4.18 extends to (x, x_i) that satisfy (a) and (b) of Definition 4.13 with $x, x_i \in \mathbb{Q}$.

A key lemma. We now use the combinatorial machinery from the previous section, together with a reduction to the ball-packing problem, to prove the key lemma needed to complete the proof of Theorem 1.2; see part (iii) of Lemma 4.24 below.

To reduce to a ball-packing problem, note that [Frenkel and Müller 2012, Proposition 1.4] states that for rational a, we have that

$$E(1, a) \overset{s}{\hookrightarrow} P(\lambda, c\lambda)$$

if and only if

$$E(1, a) \sqcup B(\lambda) \sqcup B(c\lambda) \overset{s}{\hookrightarrow} B((1+c)\lambda), \tag{4-1}$$

where $\sqcup$ denotes disjoint union. Since, as explained in [Hutchings 2014], one can compute the ECH capacities of the disjoint union in terms of the #-operation, we know that the embedding in (4-1) exists if and only if

$$N(1, a) \# N(\lambda, \lambda) \# N(c\lambda, c\lambda) \leq N\big((1+c)\lambda, (1+c)\lambda\big). \tag{4-2}$$

For the rest of the proof of Theorem 1.2, we are looking at intervals for a on which the graph of d is equal to the volume obstruction; we therefore want to show that (4-2) holds with $\lambda = \sqrt{a/(2c)}$ (of course, for our proof one can specify $c = \frac{13}{2}$, but we state things here in slightly greater generality). By an argument analogous to the argument used in the proof of Theorem 1.1, it is sufficient to show

$$\left(\sum_i d_i^2 + d_i\right) + e_1^2 + e_1 + e_2^2 + e_2 \leq d^2 + 3d$$

implies

$$\left(\sum_i a_i d_i\right) + c\lambda e_1 + \lambda e_2 \leq (1+c)\lambda d$$

for all nonnegative integers d, d_i, e_1, e_2. Let $m_1 = e_1, m_2 = e_2$ and $m_i = d_{i-2}$ for $i \geq 3$ and let $w_1(a) = c\lambda$, $w_2(a) = \lambda$ and $w_i(a) = a_{i-2}$ for $i \geq 3$. Hence, it is enough to show

$$\sum_i m_i^2 + m_i \leq d^2 + 3d$$

implies

$$m \cdot w(a) \leq (1+c)\lambda d. \tag{4-3}$$

Let $\mu(d;m)(a) = (m \cdot w(a))/d$. Then (4-3) is equivalent to $\mu(d;m)(a) \le (1+c)\lambda$. By Lemma 4.18, it is sufficient to check the case

$$\sum_i m_i^2 = d^2 + 1, \tag{4-4}$$

$$\sum_i m_i = 3d - 1. \tag{4-5}$$

Let E be the set of $(d;m)$ satisfying (4-4) and (4-5) with d, m_i nonnegative integers. Define ε by

$$m = \frac{d}{(1+c)\lambda} w(a) + \varepsilon.$$

We now have a series of lemmas, culminating in the key lemma, Lemma 4.24.

Lemma 4.20. *For $(d;m) \in E$, we have*:

(i) $\mu(d;m)(a) \le (1+c)\lambda\sqrt{1+\frac{1}{d^2}}$.

(ii) $\mu(d;m)(a) > (1+c)\lambda$ *if and only if* $\varepsilon \cdot w > 0$.

(iii) $\mu(d;m)(a) > (1+c)\lambda$ *implies* $\sum_i \varepsilon_i^2 < 1$.

(iv) *Let* $y(a) = a + 1 - 2(1+c)\lambda$. *Then*

$$-\sum_i \varepsilon_i = 1 + \frac{d}{(1+c)\lambda}\left(y(a) - \frac{1}{q}\right),$$

where $a = p/q$.

Proof. Part (i) follows from $\sum_i w_i^2 = c^2\lambda^2 + \lambda^2 + \sum_i a_i^2 = (1+c)^2\lambda^2$ and the Cauchy–Schwarz inequality. To prove (ii), note

$$\begin{aligned}
\varepsilon \cdot w &= m \cdot w - \frac{d}{(1+c)\lambda} w \cdot w \\
&= d\left(\frac{m \cdot w}{d} - (1+c)\lambda\right) \\
&= d\big(\mu(d;m)(a) - (1+c)\lambda\big).
\end{aligned}$$

To prove (iii), note

$$\begin{aligned}
\sum_i \varepsilon_i^2 = \varepsilon \cdot \varepsilon &= m \cdot m + \frac{d^2}{(1+c)^2\lambda^2} w \cdot w - \frac{2d}{(1+c)\lambda} m \cdot w \\
&= 1 + d^2\left(2 - \frac{2}{(1+c)\lambda}\frac{m \cdot w}{d}\right) \\
&< 1 \quad \text{if } \mu(d;m)(a) > (1+c)\lambda.
\end{aligned}$$

To prove (iv), note

$$-\sum_i \varepsilon_i = \frac{d}{(1+c)\lambda}\sum_i w_i - \sum_i m_i$$
$$= \frac{d}{(1+c)\lambda}\left(a+1-\frac{1}{q}+c\lambda+\lambda\right)-3d-1$$
$$= 1+\frac{d}{(1+c)\lambda}\left(a+1-\frac{1}{q}-2(1+c)\lambda\right). \qquad \square$$

Lemma 4.21. *Let $(d;m)\in E$ and suppose that I is the maximal nonempty open interval such that $\mu(d;m)(a) > (1+c)\lambda$ for all $a\in I$. Then there exists a unique $a_0\in I$ such that $l(a_0)=l(m)$, where $l(a_0)$ is the length of $w_i(a)$ and $l(m)$ is the number of nonzero terms in m. Furthermore, $l(a)\geq l(m)$ for all $a\in I$.*

Proof. We adapt the proof of Lemma 2.1.3 in [McDuff and Schlenk 2012]. For $i\geq 3$, $w_i(a)$ is piecewise linear and is linear on open intervals that do not contain an element a' with length $l(a')\leq i$. Therefore, if $l(a)>l(m)$ for all $a\in I$,

$$\mu(d;m)(a)-\frac{c\lambda m_1+\lambda m_2}{d}$$

is linear on I. This is impossible as $c\lambda(1-m_1/d)+\lambda(1-m_2/d)$ is concave and I is bounded. Thus there exists $a_0\in I$ with $l(a_0)\leq l(m)$. If $l(a)<l(m)$ then $\sum_{i\leq l(a)} m_i^2 < d^2+1$, which implies

$$m\cdot w \leq \|w\|\sqrt{\sum_{i\leq l(a)} m_i^2}\leq d\|w\| = (1+c)\lambda d,$$

which is impossible for $a\in I$. The proof of uniqueness is the same as in [McDuff and Schlenk 2012, Lemma. 2.1.3]. $\square$

Lemma 4.22. *Let $(d;m)$ be in E with $\mu(d;m)(a) > (1+c)\lambda$ for some a. Let $J=k,\dots,k+s-1$ be a block of $s\geq 2$ consecutive integers such that $w_i(a)$ is constant for $i\in J$. Then*:

(i) *One of the following holds*:
- $m_k=\cdots=m_{k+s-1}$.
- $m_k=\cdots=m_{k+s-2}=m_{k+s-1}+1$.
- $m_k-1=m_{k+1}=\cdots=m_{k+s-1}$.

(ii) *There is at most one block of length $s\geq 2$ on which the m_i are not all equal.*

(iii) *If there is a block J of length $s\geq 2$ on which the m_i are not all equal then*

$$\sum_{i\in J}\varepsilon_i^2 \geq \frac{s-1}{s}.$$

Proof. See the proof of [McDuff and Schlenk 2012, Lemma. 2.1.7]. McDuff and Schlenk consider the case of embedding an ellipsoid into a ball, but their proof generalizes without change to our situation. □

Lemma 4.23. *Let $(d; m) \in E$ be such that $\mu(d; m) > (1+c)\lambda$ for some a with $l(a) = l(m) = M$. Let $w_{k+1}, \dots, w_{k+s}$ be a block, but not the first block, of $w(a)$ (the first two terms of $w(a)$ are not considered to be part of any block).*

(i) *If this block is not the last block, then*

$$\left|m_k - (m_{k+1} + \cdots + m_{k+s} + m_{k+s+1})\right| < \sqrt{s+2}.$$

If this block is the last block, then

$$\left|m_k - (m_{k+1} + \cdots + m_{k+s})\right| < \sqrt{s+1}.$$

(ii) *It is always true that*

$$m_k - \sum_{i=k+1}^{M} m_i < \sqrt{M-k+1}.$$

Proof. This is similar to the proof of Lemma 4.22; see the proof of [McDuff and Schlenk 2012, Lemma 2.1.8], which generalizes without change to our situation. □

Lemma 4.24. *Assume that $(d; m) \in E$ and $\mu(d; m)(a) > (1+c)\lambda$ for some a with $l(a) = l(m)$. Assume further that $y(a) > 1/q$. Let*

$$v_M = \frac{d}{q(1+c)}\lambda$$

and let $L = l(m)$. Then:

(i) $\left|\sum_i \varepsilon_i\right| \le \sqrt{L}$.

(ii) $v_M > \frac{1}{3}$.

(iii) *Let $\delta = y(a) - 1/q > 0$. Then*

$$d \le \frac{(1+c)\lambda}{\delta}\left(\sqrt{L} - 1\right) \le \frac{(1+c)\lambda}{\delta}\left(\sqrt{q + \lfloor a \rfloor + 2} - 1\right)$$

and $\sqrt{q + \lfloor a \rfloor + 2} \ge 1 + \delta v_M q$.

Proof. Part (i) follows from $\sum_i \varepsilon_i^2 < 1$. Part (ii) follows from the same argument as [McDuff and Schlenk 2012, Lemma 5.1.2]. From [McDuff and Schlenk 2012, Sublemma 5.1.1], we have $q + \lfloor a \rfloor + 2 \ge L$, so Lemma 4.20 implies

$$\sqrt{q + \lfloor a \rfloor + 2} \ge \sqrt{L} \ge 1 + \frac{d}{(1+c)\lambda}\left(y(a) - \frac{1}{q}\right) = 1 + \frac{d}{(1+c)\lambda}\delta = 1 + q v_M \delta.$$

This also shows

$$d \le \frac{(1+c)\lambda}{\delta}\left(\sqrt{q + \lfloor a \rfloor + 2} - 1\right).$$

□

5. Proof of Theorem 1.2, Part III

With the Lemma 4.24 now shown, we can complete the proof of Theorem 1.2. We explain the computation on various intervals separately.

$\left[\frac{1300}{81}, \frac{841}{52}\right]$**.** We now wish to prove that $d\left(a, \frac{13}{2}\right) = \sqrt{a/13}$ for $a \in \left[\frac{1300}{81}, \frac{841}{52}\right]$. Previously, we proved

$$\left(\tfrac{1300}{81}, \tfrac{13}{2}\right) = \tfrac{10}{9} \quad \text{and} \quad d\left(\tfrac{841}{52}, \tfrac{13}{2}\right) = \tfrac{29}{26}.$$

If $d\left(a, \frac{13}{2}\right)$ is not equal to $\sqrt{a/13}$ on the interval $\left[\frac{1300}{81}, \frac{841}{52}\right]$, there exists $(d; m) \in E$ such that

$$\mu(d; m)(a) > 7.5\lambda \quad \text{for some } a \in \left[\tfrac{1300}{81}, \tfrac{841}{52}\right].$$

So, Lemma 4.24 shows that there exists a_0 in $\left[\frac{1300}{81}, \frac{841}{52}\right]$ with $\mu(d; m)(a_0) > 7.5\lambda$ and $l(a_0) = l(m)$. Let $a_0 = p/q = 16 + p'/q$. As $16 < a_0 < 16 + \frac{1}{5}$, we know $q \geq 5$. For $a_0 \in \left[\frac{1300}{81}, \frac{841}{52}\right]$ and $q \geq 5$, we know

$$\delta \geq \frac{1300}{81} + 1 - 15\sqrt{\frac{1300}{81 \cdot 13}} - \frac{1}{q} \geq \frac{31}{81} - \frac{1}{q}.$$

Thus, Lemma 4.24 shows

$$\sqrt{q + 18} \geq 1 + \left(\frac{31}{81} - \frac{1}{q}\right)\frac{q}{3}.$$

Hence, $q \leq 67$.

We also note that for $\frac{1300}{81} < a_0 < \frac{841}{52}$ and $q \geq 5$, we have

$$\lambda \leq \sqrt{\frac{841}{52 \cdot 13}} = \frac{29}{26} \quad \text{and} \quad \delta \geq \frac{31}{81} - \frac{1}{q} \geq \frac{74}{405}.$$

Thus, Lemma 4.24 shows

$$d \leq \frac{7.5 \cdot \frac{29}{26}}{\frac{74}{405}}(\sqrt{85} - 1) < 377.$$

Using Mathematica we can reduce the possibilities for $(d; m)$ to 38 candidates. We can then use Lemma 4.23 to reduce these 38 cases to 11 possible candidates, which can easily verified to not be obstructive by simple calculations.

$\left[\frac{15028}{841}, \frac{961}{52}\right]$**.** We now will show $d\left(a, \frac{13}{2}\right) = \sqrt{a/13}$ for $a \in \left[\frac{15028}{841}, \frac{961}{52}\right]$. Previously, we proved

$$d\left(\tfrac{15028}{841}, \tfrac{13}{2}\right) = \tfrac{34}{29} \quad \text{and} \quad d\left(\tfrac{961}{52}, \tfrac{13}{2}\right) = \tfrac{31}{26}.$$

If $d\left(a, \frac{13}{2}\right)$ is not equal to $\sqrt{a/13}$ on the interval $\left[\frac{15028}{841}, \frac{961}{52}\right]$, then there exists $(d; m) \in E$ such that

$$\mu(d; m)(a) > 7.5\lambda \quad \text{for some } a \in \left[\tfrac{15028}{841}, \tfrac{961}{52}\right].$$

Then Lemma 4.24 shows that there exists $a_0 \in \left[\frac{15028}{841}, \frac{961}{52}\right]$ with $\mu(d, m)(a_0) > 7.5\lambda$ and $l(a_0) = l(m)$. Let $a_0 = p/q$ with $\gcd(p, q) = 1$. For $a_0 \in \left[\frac{15028}{841}, \frac{961}{52}\right]$, we know

$$\delta \geq \frac{15028}{841} + 1 - 15\sqrt{\frac{15028}{841 \cdot 13}} - \frac{1}{q} = \frac{1079}{841} - \frac{1}{q}.$$

Thus, Lemma 4.24 shows

$$\sqrt{q+19} \geq 1 + \left(\frac{1079}{841} - \frac{1}{q}\right)\frac{q}{3}.$$

Hence, $q \leq 11$. We can then verify these cases directly using Mathematica, which by simple calculations can be verified not to be obstructive.

$\left[\frac{18772}{961}, \frac{1089}{52}\right]$. We will now show $d\left(a, \frac{13}{2}\right) = \sqrt{a/13}$ for $a \in \left[\frac{18772}{961}, \frac{1089}{52}\right]$. Previously, we proved

$$d\left(\tfrac{18772}{961}, \tfrac{13}{2}\right) = \tfrac{38}{31} \quad \text{and} \quad d\left(\tfrac{1089}{52}, \tfrac{13}{2}\right) = \tfrac{33}{26}.$$

If $d\left(a, \frac{13}{2}\right)$ is not equal to $\sqrt{a/13}$ on the interval $\left[\frac{18772}{961}, \frac{1089}{52}\right]$, then there exists $(d; m) \in E$ such that

$$\mu(d; m)(a) > 7.5\lambda \quad \text{for some } a \in \left[\tfrac{18772}{961}, \tfrac{1089}{52}\right].$$

Then Lemma 4.24 shows that there exists $a_0 \in \left[\frac{18772}{961}, \frac{1089}{52}\right]$ with $\mu(d, m)(a_0) > 7.5\lambda$ and $l(a_0) = l(m)$. Let $a_0 = p/q$ with $\gcd(p, q) = 1$. For $a_0 \in \left[\frac{18772}{961}, \frac{1089}{52}\right]$, we know

$$\delta \geq \frac{18772}{961} + 1 - 15\sqrt{\frac{18772}{961 \cdot 13}} - \frac{1}{q} = \frac{2063}{961} - \frac{1}{q}.$$

Thus, Lemma 4.24 shows

$$\sqrt{q+21} \geq 1 + \left(\frac{2063}{961} - \frac{1}{q}\right)\frac{q}{3}.$$

Hence, $q \leq 6$. We can then verify these cases directly using Mathematica to check these cases and we find no obstructions.

$\left[\frac{2548}{121}, 27\right]$. For $a \in \left[\frac{2548}{121}, 27\right]$, we have

$$\sqrt{q+29} \geq \sqrt{q + \lfloor a \rfloor + 2} \quad \text{and} \quad \delta \geq 21 - 15\sqrt{\tfrac{21}{13}}.$$

Hence, Lemma 4.24 implies

$$\sqrt{q+29} \geq 1 + \left(21 - 15\sqrt{\frac{21}{13}}\right)\frac{q}{3},$$

which implies $q < 8$. We can then verify these cases directly using Mathematica to check these cases and we find no obstructions.

$[27, \infty)$. We will apply Remark 2.2. As

$$\sqrt{27} \geq \frac{7.5}{\sqrt{13}}\left(2+\frac{7.5}{d}\right) \quad \text{for } d \geq 18,$$

Remark 2.2 implies we only need to verify $N_k(1, a^2) \leq (a/13) M_k(1, 6.5)$ for all

$$k \leq \frac{18^2}{26}+\frac{7.5 \cdot 18}{13}+\frac{6.5^2-13+1}{26}<25.$$

For $a^2 \geq 27$ and $k \leq 25$, we have

$$N_k(1, a^2)=k \leq \sqrt{\frac{27}{13}} M_k(1, 6.5) \leq \frac{a}{\sqrt{13}} M_k(1, 6.5).$$

This completes the proof that $d(a, b)=\sqrt{a/13}$ for $a \in [27, \infty)$.

6. Conjectures

We now present some conjectures concerning exactly when an ellipsoid embeds into a polydisc.

Extensions of Theorem 1.1. To consider an interesting refinement of Theorem 1.1, define

$$V(b)=\inf\left\{A : d(a, b)=\sqrt{\frac{a}{2b}} \quad \text{for } a \geq A\right\}.$$

Theorem 1.1 implies $V(b) \leq \frac{9}{2}(b+2+1/b)$.

Proposition 6.1. *For $b \geq 1$,*

$$V(b) \geq 2b\left(\frac{2\lfloor b\rfloor+2\lceil\sqrt{2b}+\{b\}\rceil-1}{b+\lfloor b\rfloor+\lceil\sqrt{2b}+\{b\}\rceil-1}\right)^2.$$

Proof.

$$\begin{aligned} d\big(2\lfloor b\rfloor+2\lceil\sqrt{2b}+\{b\}\rceil-1, b\big) &\geq \frac{N_{2\lfloor b\rfloor+2\lceil\sqrt{2b}+\{b\}\rceil-1}\big(1, 2\lfloor b\rfloor+2\lceil\sqrt{2b}+\{b\}\rceil-1\big)}{M_{2\lfloor b\rfloor+2\lceil\sqrt{2b}+\{b\}\rceil-1}(1, b)} \\ &= \frac{2\lfloor b\rfloor+2\lceil\sqrt{2b}+\{b\}\rceil-1}{b+\lfloor b\rfloor+\lceil\sqrt{2b}+\{b\}\rceil-1} \\ &> \sqrt{\frac{2\lfloor b\rfloor+2\lceil\sqrt{2b}+\{b\}\rceil-1}{2b}}. \end{aligned}$$

This implies

$$V(b) \geq 2b\left(\frac{2\lfloor b\rfloor+2\lceil\sqrt{2b}+\{b\}\rceil-1}{b+\lfloor b\rfloor+\lceil\sqrt{2b}+\{b\}\rceil-1}\right)^2. \qquad \square$$

Experimental evidence seems to suggest that for $b > 1$ this bound is sharp.

Conjecture 6.2. *For $b > 1$,*

$$V(b) = 2b\left(\frac{2\lfloor b\rfloor + 2\lceil\sqrt{2b} + \{b\}\rceil - 1}{b + \lfloor b\rfloor + \lceil\sqrt{2b} + \{b\}\rceil - 1}\right)^2.$$

Generalizations of Theorem 1.2. The methods used to compute the graph of $d(a, 6.5)$ should extend for the most part to any b. In light of those techniques, experimental evidence, and a conjecture regarding $d(a, b)$ for $b \in \mathbb{Z}$ by David Frenkel and Felix Schlenk relayed to us by Daniel Cristofaro-Gardiner, we offer a conjecture regarding the graph of $d(a, b)$ for $b \geq 6$; see Figure 2.

Conjecture 6.3. *For $b \geq 6$, we have $d(a, b) = \sqrt{a/(2b)}$ with the exception that*

$$d(a, b) = 1 \quad \text{for } a \in \big[1, b + \lfloor b\rfloor\big].$$

For $k \in \mathbb{Z}$, with $0 \leq k < \sqrt{2b} + \{b\}$, we have

$$d(a, b) = \frac{a}{b + \lfloor b\rfloor + k} \quad \text{for } a \in \big[\alpha_k, 2(\lfloor b\rfloor + k) + 1\big],$$

$$d(a, b) = \frac{2(\lfloor b\rfloor + k) + 1}{b + \lfloor b\rfloor + k} \quad \text{for } a \in \big[2(\lfloor b\rfloor + k) + 1, \beta_k\big],$$

where

$$\alpha_0 = b + \lfloor b\rfloor, \quad \alpha_1 = \beta_0 = \frac{(b + \lfloor b\rfloor + 1)(2\lfloor b\rfloor + 1)}{b + \lfloor b\rfloor},$$

$$\alpha_k = \frac{(b + \lfloor b\rfloor + k)^2}{2b} \quad \text{for } k \geq 2, \qquad \beta_k = 2b\left(\frac{2(\lfloor b\rfloor + k) + 1}{b + \lfloor b\rfloor + k}\right)^2 \quad \text{for } k \geq 1.$$

For integers m, if

$$b \in \left[m - \frac{m}{(m+1)^2}, m + \frac{1}{2+m}\right],$$

let $b = m + \varepsilon$. Then

$$d(a, b) = \frac{ma + 1}{2m^2 + (2 + \varepsilon)m + \varepsilon} \quad \text{for } a \in [\alpha^*, 2m + 4],$$

$$d(a, b) = \frac{m(2m + 4) + 1}{2m^2 + (2 + \varepsilon)m + \varepsilon} \quad \text{for } a \in [2m + 4, \beta^*],$$

where

$$\alpha^* = \frac{1}{2(2m^3 + 2m^2\varepsilon)}\Big(8m^3 + 4m^2 + 8m^2\varepsilon + 4m^3\varepsilon + \varepsilon^2 + 2m\varepsilon^2 + b^2\varepsilon^2 - (1+m)(2m+\varepsilon) \times\sqrt{-4m^2 + 8m^3 + 4m^4 - 4m\varepsilon + 8m^2\varepsilon + 4m^3\varepsilon + \varepsilon^2 + 2m\varepsilon^2 + m^2\varepsilon^2}\Big),$$

$$\beta^* = \frac{2(\varepsilon + m + 8m\varepsilon + 8m^2 + 20m^2\varepsilon + 16m^3\varepsilon + 16m^4 + 4m^4\varepsilon + 4m^5)}{(1+m)^2(2m+\varepsilon)^2}.$$

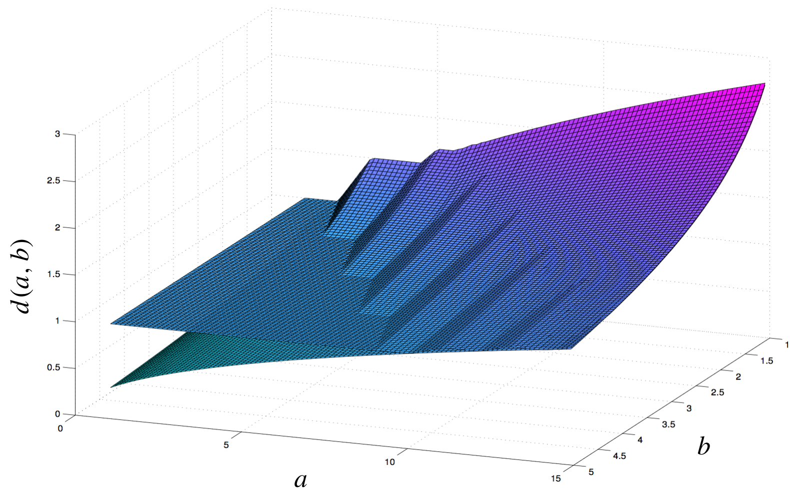

Figure 2. Approximate plot of the graph of $d(a, b)$.

We note that Conjecture 6.3 implies Conjecture 6.2 for $b \geq 6$. Furthermore, we prove that the conjecture is a lower bound for $d(a, b)$.

Proposition 6.4. *For $b \geq 6$, we have $d(a, b) \geq \sqrt{a/(2b)}$ and*

$$d(a, b) \geq 1 \quad \textit{for } a \in \big[1, b + \lfloor b \rfloor\big].$$

For $k \in \mathbb{Z}$, with $0 \leq k < \sqrt{2b} + \{b\}$, we have

$$d(a, b) \geq \frac{a}{b + \lfloor b \rfloor + k} \quad \textit{for } a \in \big[\alpha_k, 2(\lfloor b \rfloor + k) + 1\big],$$

$$d(a, b) \geq \frac{2(\lfloor b \rfloor + k) + 1}{b + \lfloor b \rfloor + k} \quad \textit{for } a \in \big[2(\lfloor b \rfloor + k) + 1, \beta_k\big],$$

where $\alpha_k, \beta_k, \alpha^*, \beta^*$ are as in Conjecture 6.3. For integers m, if

$$b \in \left[m - \frac{m}{(m+1)^2}, m + \frac{1}{2+m}\right],$$

let $b = m + \varepsilon$. Then

$$d(a, b) \geq \frac{ma + 1}{2m^2 + (2 + \varepsilon)m + \varepsilon} \quad \textit{for } a \in [\alpha*, 2m + 4],$$

$$d(a, b) \geq \frac{m(2m + 4) + 1}{2m^2 + (2 + \varepsilon)m + \varepsilon} \quad \textit{for } a \in [2m + 4, \beta*].$$

Proof. We know that $d(a, b) \geq \sqrt{a/(2b)}$ because symplectic embeddings are volume preserving. We also have

$$d(a, b) \geq \frac{N_1(1, a)}{M_1(a, b)} = \frac{1}{1} = 1.$$

Additionally, for $k \in \mathbb{Z}$, with $\le k < \sqrt{2b} + \{b\}$, and $a \in \big[2(\lfloor b \rfloor + k), 2(\lfloor b \rfloor + k) + 1\big]$, we have

$$d(a, b) \ge \frac{N_{2(\lfloor b \rfloor + k)+1}(1, a)}{M_{2(\lfloor b \rfloor + k)+1}(1, b)} = \frac{a}{b + \lfloor b \rfloor + k}.$$

Thus,

$$d(a, b) \ge 1 \quad \text{for } a \in \big[b + \lfloor b \rfloor, 2\lfloor b \rfloor + 1\big],\ k = 0,$$

$$d(a, b) \ge \frac{2\lfloor b \rfloor + 1}{b + \lfloor b \rfloor} \quad \text{for } a \in \left[\frac{(b + \lfloor b \rfloor + 1)(2\lfloor b \rfloor + 1)}{b + \lfloor b \rfloor}, 2\lfloor b \rfloor + 3\right],\ k = 1,$$

$$d(a, b) \ge \sqrt{\frac{a}{2b}} \quad \text{for } a \in \big[\alpha_k, 2(\lfloor b \rfloor + k) + 1\big],\ k \ge 2.$$

We also have, for $a \in \big[2(\lfloor b \rfloor + k) + 1, \infty\big)$,

$$d(a, b) \ge \frac{N_{2(\lfloor b \rfloor + k)+1}(1, a)}{M_{2(\lfloor b \rfloor + k)+1}(1, b)} = \frac{2(\lfloor b \rfloor + k) + 1}{b + \lfloor b \rfloor + k}.$$

Thus,

$$d(a, b) \ge \sqrt{\frac{a}{2b}} \quad \text{for } a \in \big[2(\lfloor b \rfloor + k) + 1, \beta_k\big].$$

Furthermore, if

$$b \in \left[m - \frac{m}{(m+1)^2}, m + \frac{1}{2+m}\right]$$

for some $m \in \mathbb{Z}$ and $a \in \big[2m + 4 - 1/m, 2m + 4\big]$, then

$$d(a, b) \ge \frac{N_{(m+1)^3}(1, a)}{M_{(m+1)^3}(1, b)} = \frac{ma + 1}{2m^2 + (2 + \varepsilon)m + \varepsilon},$$

so

$$d(a, b) \ge \sqrt{\frac{a}{2b}} \quad \text{for } a \in [\alpha^*, 2m + 4].$$

We also have, for $a \in [2m + 4m\beta^*]$,

$$d(a, b) \ge \frac{N_{(m+1)^3}(1, a)}{M_{(m+1)^3}(1, b)} = \frac{m(2m + 4) + 1}{2m^2 + (2 + \varepsilon)m + \varepsilon}.$$

Thus,

$$d(a, b) \ge \sqrt{\frac{a}{2b}} \quad \text{for } a \in [2m + 4, \beta^*]. \qquad \square$$

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References

[Beck and Robins 2007] M. Beck and S. Robins, *Computing the continuous discretely: Integer-point enumeration in polyhedra*, Springer, New York, 2007. MR Zbl

[Buse and Hind 2013] O. Buse and R. Hind, "Ellipsoid embeddings and symplectic packing stability", *Compos. Math.* **149**:5 (2013), 889–902. MR Zbl

[Cristofaro-Gardiner and Kleinman 2013] D. Cristofaro-Gardiner and A. Kleinman, "Ehrhart polynomials and symplectic embeddings of ellipsoids", preprint, 2013. arXiv

[Frenkel and Müller 2012] D. Frenkel and D. Müller, "Symplectic embeddings of 4-dimensional ellipsoids into cubes", preprint, 2012. arXiv

[Hutchings 2014] M. Hutchings, "Lecture notes on embedded contact homology", pp. 389–484 in *Contact and symplectic topology*, edited by F. Bourgeois et al., Bolyai Soc. Math. Stud. **26**, János Bolyai Math. Soc., Budapest, 2014. MR Zbl

[Li and Li 2002] B.-H. Li and T.-J. Li, "Symplectic genus, minimal genus and diffeomorphisms", *Asian J. Math.* **6**:1 (2002), 123–144. MR Zbl

[McDuff 2011] D. McDuff, "The Hofer conjecture on embedding symplectic ellipsoids", *J. Differential Geom.* **88**:3 (2011), 519–532. MR Zbl

[McDuff and Schlenk 2012] D. McDuff and F. Schlenk, "The embedding capacity of 4-dimensional symplectic ellipsoids", *Ann. of Math.* (2) **175**:3 (2012), 1191–1282. MR Zbl

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