involve
a journal of mathematics

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2017 vol. 10, no. 3

INVOLVE 10:3 (2017)
https://doi.org/10.2140/involve.2017.10.361

Dynamics of vertical real rhombic Weierstrass elliptic functions

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(Communicated by Michael E. Zieve)

In this paper, we investigate the dynamics of iterating the Weierstrass elliptic functions on vertical real rhombic lattices. The main result of this paper is to show that these functions can have at most one real attracting or parabolic cycle. If there is no real attracting or parabolic cycle, we prove that the real and imaginary axes, as well as translations of these lines by the lattice, lie in the Julia set. Further, we prove that if there exists a real attracting fixed point, then the intersection of the Julia set with the real axis is a Cantor set. Finally, we apply the theorem to find parameters in every real rhombic shape equivalence class for which the Julia set is the entire sphere.

1. Introduction

There is a rich literature investigating the dynamics of iterating Weierstrass elliptic functions [Hawkins 2006; 2010; 2013; Hawkins and Koss 2002; 2004; 2005; Clemons 2012; Hawkins and McClure 2011; Koss 2014]. Both the lattice shape and its orientation in the plane can affect the dynamical behavior of these functions. Outside of specialized lattice shapes, such as triangular or rhombic square lattices, few results have appeared on the dynamics of elliptic functions on rhombic lattices.

On any lattice, the Weierstrass elliptic function has three distinct critical values. Standard results in the dynamics of meromorphic functions imply that there are at most three different types of periodic Fatou components. Here, we focus on a large subset of real rhombic lattices and show that these lattice shapes force restrictions on the types of periodic Fatou components possible for the Weierstrass elliptic function.

In Section 2, we give background on the Weierstrass elliptic function and describe some results on iterating these functions. In Section 3, we focus on the dynamics of the Weierstrass elliptic function restricted to the real line. In particular, we use the Schwarzian derivative to extend results from [Hawkins 2010; Koss 2014] and show

MSC2010: 54H20, 37F10, 37F20.
Keywords: complex dynamics, meromorphic functions, Julia sets.

that there can be at most one real attracting or parabolic periodic cycle. We use this result in Section 4 to investigate the Weierstrass elliptic function in the complex plane and find examples that satisfy certain interesting dynamical properties. First, we construct examples in Section 4A for which the real and imaginary axes lie in the Julia set. Second, we develop criteria in Section 4B that guarantee that the intersection of the real and imaginary axes with the Julia set is a Cantor set. Finally, we find parameters in every real rhombic shape equivalence class for which the Julia set is the entire sphere in Section 4C.

2. Background

We begin by fixing a lattice Λ defined by

$$\Lambda = [\lambda_1, \lambda_2] = \big\{m\lambda_1 + n\lambda_2 \mid m, n \in \mathbb{Z},\ \lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\},\ \lambda_2/\lambda_1 \notin \mathbb{R}\big\}.$$

We define the Weierstrass elliptic function by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

The Weierstrass elliptic function is an even, meromorphic function that is periodic with respect to the lattice Λ. In the following, we distinguish between iteration and products by using the notation $\wp_\Lambda^n$ or $\wp_\Lambda^n(z)$ to denote iteration and $(\wp_\Lambda)^n$ or $(\wp_\Lambda(z))^n$ to denote products.

The Weierstrass elliptic function can also be defined by the differential equation

$$(\wp_\Lambda'(z))^2 = 4(\wp_\Lambda(z))^3 - g_2\wp_\Lambda(z) - g_3, \tag{1}$$

where $g_2(\Lambda) = 60\sum_{w \in \Lambda \setminus \{0\}} w^{-4}$ and $g_3(\Lambda) = 140\sum_{w \in \Lambda \setminus \{0\}} w^{-6}$. Each pair of complex numbers (g_2, g_3) with $g_2^3 - 27g_3^2 \neq 0$ determines a unique equivalence class of lattices and vice versa, where equivalence means that they generate the same subgroup [Du Val 1973]. We call a lattice Λ with invariants $g_2(\Lambda)$ and $g_3(\Lambda)$ a (g_2, g_3)*-lattice*.

The critical points of $\wp_\Lambda$ are the half-lattice points $\frac{1}{2}\lambda_1 + \Lambda$, $\frac{1}{2}\lambda_2 + \Lambda$ and $\frac{1}{2}\lambda_3 = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 + \Lambda$. By the periodicity of the Weierstrass elliptic function, $\wp_\Lambda$ has exactly three distinct critical values denoted by

$$e_1 = \wp_\Lambda\big(\tfrac{1}{2}\lambda_1\big), \quad e_2 = \wp_\Lambda\big(\tfrac{1}{2}\lambda_2\big), \quad e_3 = \wp_\Lambda\big(\tfrac{1}{2}\lambda_3\big). \tag{2}$$

The critical values of $\wp_\Lambda$ satisfy the equations

$$e_1 + e_2 + e_3 = 0, \quad e_1e_3 + e_2e_3 + e_1e_2 = -\tfrac{1}{4}g_2, \quad e_1e_2e_3 = \tfrac{1}{4}g_3. \tag{3}$$

The second derivative of the Weierstrass elliptic function satisfies the equation

$$\wp_\Lambda''(z) = 6(\wp_\Lambda(z))^2 - \tfrac{1}{2}g_2(\Lambda). \tag{4}$$

If $\Lambda = [\lambda_1, \lambda_2]$, and $k \neq 0$ is any complex number, then $k\Lambda$ is the lattice defined by taking $k\lambda$ for each $\lambda \in \Lambda$, and $k\Lambda$ is said to be *similar* to Λ. Similarity is an equivalence relation between lattices, and an equivalence class of lattices is called a *shape*. Similar lattices give rise to homogeneity properties of the Weierstrass elliptic functions and their invariants:

Lemma 2.1 [Du Val 1973]. *For lattices Λ and Λ' and for $k \in \mathbb{C}\backslash\{0\}$:*

(1) *If $\Lambda' = k\Lambda$ then $g_2(\Lambda') = k^{-4}g_2(\Lambda)$ and $g_3(\Lambda') = k^{-6}g_3(\Lambda)$.*

(2) *If $\Lambda' = k\Lambda$ then $\wp_{\Lambda'}(ku) = k^{-2}\wp_\Lambda(u)$ for all $u \in \mathbb{C}$.*

The Weierstrass elliptic function satisfies a number of algebraic identities. If z is neither a lattice point nor a half-lattice point, then

$$\frac{1}{4}\left(\frac{\wp''_\Lambda(z)}{\wp'_\Lambda(z)}\right)^2 = \wp_\Lambda(2z) + 2\wp_\Lambda(z). \tag{5}$$

A lattice Λ is *real* if $\overline{\Lambda} = \Lambda$. We say $\wp_\Lambda$ is *real* if $z \in \mathbb{R}$ implies $\wp_\Lambda(z) \in \mathbb{R} \cup \{\infty\}$. Real lattices are associated with real lattice invariants.

Theorem 2.2 [Jones and Singerman 1987]. *The following are equivalent*:

(1) *$\wp_\Lambda$ is real.*

(2) *Λ is a real lattice.*

(3) *$g_2, g_3 \in \mathbb{R}$.*

By Theorem 2.2, we can identify a real lattice Λ with a point (g_2, g_3) in $\mathbb{R}^2$. Further, any point (g_2, g_3) in $\mathbb{R}^2$ with $g_2^3 - 27g_3^2 \neq 0$ gives rise to a real lattice [Du Val 1973].

The following lemma appeared in [Hawkins and Koss 2002] and gives information about $\wp_\Lambda$ on the real line in the case when Λ is real.

Lemma 2.3 [Hawkins and Koss 2002]. *If $\wp_\Lambda$ is real, then it is periodic as a map on $\mathbb{R}$ and has infinitely many real critical points and at least one real critical value. The image of the real critical point is the minimum of $\wp_\Lambda$ on $\mathbb{R}$. In particular, if e_1 denotes the critical value of the real critical points, then $\wp_\Lambda|_\mathbb{R} : \mathbb{R} \to [e_1, \infty]$ is piecewise monotonic and onto.*

2A. ***Properties of lattice shapes.*** The real lattices have distinctive shapes for their period parallelograms. We say $\Lambda = [\lambda_1, \lambda_2]$ is *real rectangular* if there exist generators such that λ_1 is real and λ_2 is purely imaginary. We say $\Lambda = [\lambda_1, \lambda_2]$ is *real rhombic* if there exist generators such that $\lambda_2 = \overline{\lambda}_1$. In each case, the period parallelogram with vertices $0, \lambda_1, \lambda_2$, and $\lambda_1 + \lambda_2$ is rectangular or rhombic, respectively.

Real rhombic and real rectangular lattices lie in regions of the (g_2, g_3)-plane described in the following proposition.

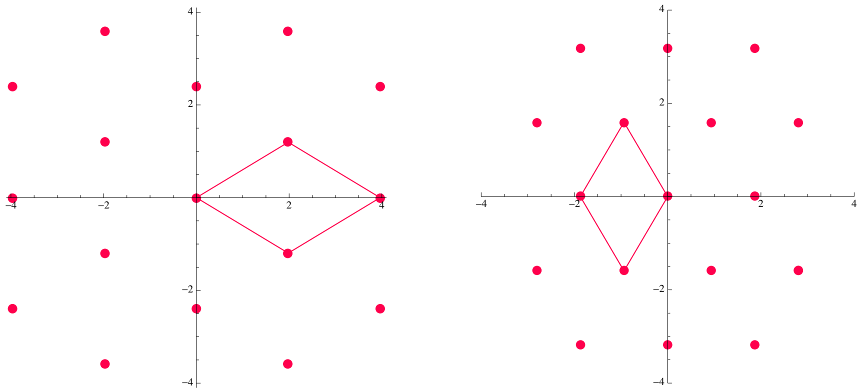

Figure 1. Horizontal (left) and vertical (right) lattices with period parallelograms.

Proposition 2.4 [Du Val 1973]. (1) Λ *is real rhombic if and only if* $g_2^3 - 27g_3^2 < 0$.
(2) Λ *is real rectangular if and only if* $g_2^3 - 27g_3^2 > 0$.

In this paper, we focus primarily on real rhombic lattices Λ, those which have generators of the form $\Lambda = [\lambda, \bar{\lambda}]$. Without loss of generality, we assume that λ lies in quadrant one. Real rhombic lattices always have a real lattice point, which we denote by $\lambda_1 = \lambda + \bar{\lambda} \in \mathbb{R}^+$. Given a real rhombic lattice $\Lambda = [\lambda, \bar{\lambda}]$ with $\lambda = a + ib$, we say Λ is *vertical* if $|b| > |a|$ and *horizontal* if $|b| < |a|$. If $|a| = |b|$, then the lattice is called rhombic square. Figure 1 (left) shows a horizontal lattice and the boundary of one period parallelogram, and Figure 1 (right) shows a vertical lattice and the boundary of one period parallelogram.

Vertical real rhombic lattices have $g_3 > 0$, and horizontal real rhombic lattices have $g_3 < 0$ [Du Val 1973]. Figure 2 shows the location of vertical and horizontal real rhombic lattices in the (g_2, g_3)-plane. The light gray region represents the location of vertical real rhombic lattices, the dark gray region represents horizontal real rhombic lattices, and the white region represents rectangular lattices.

We can use Lemma 2.1 to find all real lattices that are similar to a given real lattice. If Λ is the real lattice corresponding to the invariants (a, b) in the (g_2, g_3)-plane, then parameters that lie on the planar curve $g_3^2 = b^2 g_2^3 / a^3$ represent real lattices similar to Λ. In Figure 2, the orange curve represents the invariants of real lattices that are similar to the lattice Λ with invariants $(g_2, g_3) = (-5, -1)$. In this case, the portion of the curve lying in the upper half-plane represents a vertical real rhombic lattice similarity class, and the portion of the curve lying in the lower half-plane represents horizontal real rhombic lattices in the similarity class.

This paper primarily focuses on real rhombic lattices in the vertical position. The following properties hold true for any such lattice.

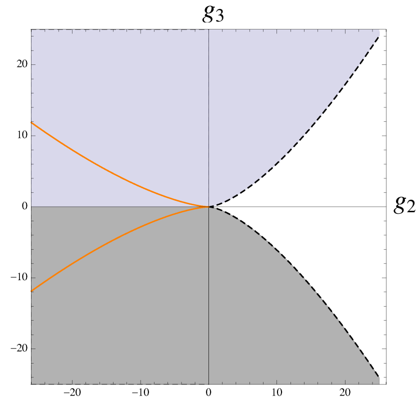

Figure 2. Real rhombic lattices are located in the shaded region of the (g_2, g_3)-plane. Light gray shading shows the location of vertical real rhombic lattices, and dark gray shading shows the location of horizontal real rhombic lattices. Rectangular lattices lie in the white region. All points colored orange represent real lattices similar to the $(-5, -1)$-lattice.

Proposition 2.5 [Hawkins and Koss 2004; 2005; Du Val 1973]. *If Λ is a vertical real rhombic lattice, then all of the following properties hold true*:

(1) $g_3 > 0$.

(2) $e_1 > 0$, *where e_1 is the image of the real critical point.*

(3) $e_2 = \bar{e}_3$.

(4) $\mathrm{Re}(e_2) = \mathrm{Re}(e_3) = -\frac{1}{2}e_1$.

(5) *If z lies on a vertical line passing through any real lattice point or any real half-lattice point, then $\wp_\Lambda(z) \in [-\infty, e_1)$.*

Although our focus in this paper is rhombic lattices, the following proposition about rectangular lattices will be used in Section 4A.

Proposition 2.6 [Hawkins and Koss 2004; 2005; Du Val 1973]. *If Λ is a real rectangular lattice, then all of the following properties hold true*:

(1) $e_1, e_2, e_3 \in \mathbb{R}$.

(2) *If $g_3 > 0$, then $e_2 < e_3 < 0 < e_1$.*

(3) *If $g_3 < 0$, then $e_2 < 0 < e_3 < e_1$.*

(4) *$\wp_\Lambda$ maps the imaginary axis to $[-\infty, e_2)$.*

It will be helpful to identify a specified lattice within each shape equivalence class. We define the *standard lattice* within any real (rhombic or rectangular) equivalence

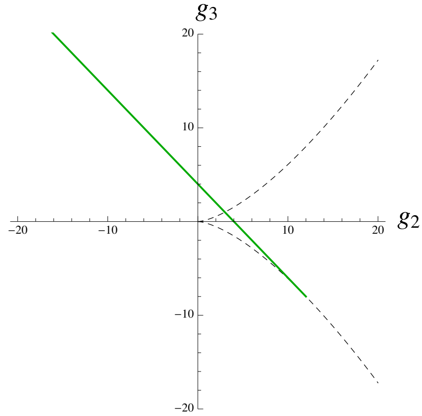

Figure 3. Standard lattices are shown in green.

class as the lattice Γ for which the real half-lattice point $\frac{1}{2}\gamma_1$ satisfies $\wp_\Gamma\left(\frac{1}{2}\gamma_1\right) = 1$. Using the equations appearing in (3) with $e_1 = 1$, we obtain

$$1 + e_2 + e_3 = 0, \quad e_2 + e_3 + e_2 e_3 = -\tfrac{1}{4} g_2, \quad e_2 e_3 = \tfrac{1}{4} g_3.$$

All real rhombic standard lattices lie on the line segment $g_3 = -g_2 + 4$ with $g_2 < 3$ in real lattice space. All real rectangular standard lattices lie on the line segment $g_3 = -g_2 + 4$ with $3 < g_2 < 12$ in real lattice space. (The ray when $g_2 > 12$ represents real rectangular lattices for which one of the nonreal critical points gives rise to the critical value of 1.) Figure 3 shows the location of real standard lattices in the (g_2, g_3)-plane in green.

Given any standard lattice, we can use the homogeneity property to find infinitely many similar lattices for which the real critical points land on a pole in one iteration. The following lemma is a result of the homogeneity property in Lemma 2.1.

Lemma 2.7 [Hawkins and Koss 2004]. *Let Γ be a standard real lattice, where γ_1 is chosen to be the smallest real positive lattice point. If m is any positive integer and $k = \sqrt[3]{1/(m\gamma_1)}$, then the lattice $\Lambda = k\Gamma$ has $\wp_\Lambda\left(\frac{1}{2}\lambda_1\right) = m\lambda_1$, and thus $\wp_\Lambda\left(\frac{1}{2}\lambda_1\right)$ is a pole.*

2B. ***Iterating elliptic functions.*** We give a brief overview of the dynamics of meromorphic functions; more details can be found in [Baker et al. 1992; Bergweiler 1993; Rippon and Stallard 1999]. Let $f : \mathbb{C} \to \mathbb{C}_\infty$ be a meromorphic function. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}_\infty$ such that $\{f^n \mid n \in \mathbb{N}\}$ is defined and normal in some neighborhood of z. The *Julia set* is the complement of the Fatou set on the sphere, $J(f) = \mathbb{C}_\infty \backslash F(f)$. A point z_0 is *periodic* of period p if there exists a $p \geq 1$ such that $f^p(z_0) = z_0$. We also call the set $\{z_0, f(z_0), \ldots, f^{p-1}(z_0)\}$ a *p-cycle*. The *multiplier* of a point z_0 of period p is the derivative $(f^p)'(z_0)$. A periodic point z_0 is classified as *attracting, repelling,* or *neutral* if $|(f^p)'(z_0)|$ is less than, greater than, or equal to 1 respectively. If $|(f^p)'(z_0)| = 0$ then z_0 is called a *superattracting* periodic point.

Suppose U is a connected component of the Fatou set. We say U is *preperiodic* if there exists $n > m \geq 0$ such that $f^n(U) = f^m(U)$, and the minimum of $n - m = p$ for all such n, m is the *period* of the cycle. Elliptic functions have a finite number of critical values, and thus it turns out that the classification of periodic components of the Fatou set is no more complicated than that of rational maps of the sphere. Periodic components of the Fatou set of elliptic functions may be attracting domains, parabolic domains, Siegel disks, or Herman rings. In particular, elliptic functions have no wandering domains or Baker domains [Baker et al. 1992; Hawkins and Koss 2002; Rippon and Stallard 1999].

Let $C = \{U_0, U_1, \ldots, U_{p-1}\}$ be a periodic cycle of components of $F(f)$. If C is a cycle of immediate attractive basins or parabolic domains, then $U_j \cap \mathrm{Crit}(f) \neq \varnothing$ for some $0 \leq j \leq p-1$. If C is a cycle of Siegel disks or Herman rings, then $\partial U_j \subset \overline{\bigcup_{n\geq 0} f^n(\mathrm{Crit}(f))}$ for all $0 \leq j \leq p-1$. In particular, any periodic component of an elliptic function has an associated critical point.

Although the Weierstrass elliptic function can have three distinct postcritical orbits, there are restrictions on the possible types of distinct Fatou cycles. Rectangular lattices have been investigated in [Hawkins and Koss 2002; 2004; 2005; Koss 2014]. The following proposition, proved in [Hawkins and Koss 2005], describes the possible postcritical behavior of the Weierstrass elliptic function on real rhombic lattices.

Proposition 2.8 [Hawkins and Koss 2005]. *For any real rhombic lattice Λ one of the following must occur*:

(1) $J(\wp_\Lambda) = \mathbb{C}_\infty$.

(2) *There exist one real postcritical orbit and two conjugate postcritical orbits*; *therefore*, *there are at most two different types of periodic Fatou components. If the nonreal critical values lie in the Fatou set*, *then they are associated with cycles with the same period and multiplier.*

The periodicity of $\wp_\Lambda$ on any real lattice Λ gives rise to many forms of symmetry in the Fatou and Julia sets, as well as restrictions on possible Fatou behavior.

Proposition 2.9 [Hawkins and Koss 2002; 2004]. *For any real lattice* Λ:

(1) $J(\wp_\Lambda) + \Lambda = J(\wp_\Lambda)$ *and* $F(\wp_\Lambda) + \Lambda = F(\wp_\Lambda)$.

(2) $(-1)J(\wp_\Lambda) = J(\wp_\Lambda)$ *and* $(-1)F(\wp_\Lambda) = F(\wp_\Lambda)$.

(3) $\overline{J(\wp_\Lambda)} = J(\wp_\Lambda)$ *and* $\overline{F(\wp_\Lambda)} = F(\wp_\Lambda)$.

(4) $J(\wp_\Lambda)$ *and* $F(\wp_\Lambda)$ *are symmetric with respect to any critical point. That is, if c is any critical point of $\wp_\Lambda$, then $c + z \in F(\wp_\Lambda)$ if and only if $c - z \in F(\wp_\Lambda)$.*

(5) $\wp_\Lambda$ *has no cycle of Herman rings.*

3. Dynamics on the real line

In this section, we focus on the dynamics of $\wp_\Lambda$ restricted to the real line. When $g_3 = 0$ and $g_2 < 0$, Hawkins [2010] showed that $\wp_\Lambda$ has a Julia set that is equal to the entire sphere. The proof of this result involved showing that $\wp_\Lambda$ has no attracting or parabolic cycles because the Schwarzian derivative of $\wp_\Lambda$ was negative, extending work of Singer [1978] on interval maps to the case of elliptic functions. Koss [2014] showed that when Λ is real rectangular, the Schwarzian of $\wp_\Lambda$ is negative, so there can be at most one attracting fixed point. The techniques used in these proofs relied on special properties of the critical values for these lattices: for real rhombic square lattices, the critical values lie on the imaginary axis, and for real rectangular lattices, the critical values are all real. As such, these methods cannot be applied in the case of real rhombic lattices because the critical values are located elsewhere in the plane. In this section, we present a different proof that when Λ is a vertical real rhombic lattice, the Schwarzian is negative.

We begin with the definition of the Schwarzian derivative.

Definition 3.1. If x is not a critical point or pole of a meromorphic function F, the *Schwarzian derivative* is defined to be

$$S_F(x) = \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2.$$

When $F = \wp_\Lambda$, we have $S_{\wp_\Lambda}$ restricted to $\mathbb{R}$ is a real-valued, even elliptic function with poles at lattice points and half-lattice points [Hawkins 2010].

Using Lemma 2.3, we know $\wp_\Lambda(\mathbb{R}) \subset \mathbb{R} \cup \{\infty\}$. For any p-cycle

$$S = \{x_0, \wp_\Lambda(x_0), \dots, \wp_\Lambda^{p-1}(x_0)\} \subset \mathbb{R},$$

we associate to it a set

$$B(S) = \{x \in \mathbb{R} \mid \wp_\Lambda^k(x) \to S \text{ as } k \to \infty\}.$$

The set S is *topologically attracting* if $B(S)$ contains an open interval, and in this case we call $B(S)$ the *real attracting basin* of S. The *real immediate attracting basin* of S is the union of components of $B(S)$ in $\mathbb{R}$ that contain points from S, and we denote this set by $B_0(S)$. Using Lemma 2.3, if $|(\wp_\Lambda^p)'(z_0)| < 1$, then $S \subset [e_1, \infty)$ and $B(S) \neq \varnothing$, so S is topologically attracting.

Hawkins [2010] proved that when $g_3 = 0$ and $g_2 < 0$, the Weierstrass elliptic function satisfied a minimum principle. The proof relied on special properties of the lattice shape, but the result was extended to all real rectangular lattices in [Koss 2014]. The proof in [Koss 2014] carries over identically for real rhombic lattices, and we state the result here without proof.

Lemma 3.2 (minimum principle). *Assume that Λ is a real lattice. Suppose we have a closed interval $I \subset \mathbb{R}$ with endpoints $l < r$, not containing any poles or critical points of $f_{n,\Lambda,b}$. Then*

$$|f'_{n,\Lambda,b}(x)| > \min\{|f'_{n,\Lambda,b}(l)|, |f'_{n,\Lambda,b}(r)|\} \quad \forall x \in (l, r).$$

In [Hawkins 2010], the minimum principle was used to extend Singer's theorem on interval maps to the setting of the Weierstrass elliptic function on a real square lattice. The extension of Singer's theorem given in [Hawkins 2010] relied only on the minimum principle and generic properties of elliptic functions on real lattices; the proof for our setting follows identically, so we do not provide it.

Theorem 3.3. *If Λ is a real rhombic lattice and $S_{\wp_\Lambda} < 0$, then*:

(1) *The real immediate basin of attraction of a topologically attracting periodic orbit of $\wp_\Lambda$ contains a real critical point.*

(2) *If $y \in \mathbb{R}$ is in a rationally neutral p-cycle for $\wp_\Lambda$, then it is topologically attracting*; *i.e., there exists an open interval I such that for every $x \in I$, $\lim_{k\to\infty} \wp_\Lambda^{kp}(x) = y$.*

Next, we show that the Schwarzian of $\wp_\Lambda$ is negative on any vertical real rhombic lattice.

Theorem 3.4. *If Λ is a vertical real rhombic lattice, then $S_{\wp_\Lambda} < 0$.*

Proof. Suppose $g_3 > 0$ and $g_2^3 - 27g_3^2 \neq 0$. From Lemma 2.3, $g_3 > 0$ implies that the critical value e_1 is also positive. Further, e_1 is the absolute minimum of $\wp_\Lambda$ by Lemma 2.3, so it follows that $\wp_\Lambda(x) > 0$ for all x except lattice points when $g_3 > 0$. Rearranging (5) to

$$\left(\frac{\wp''_\Lambda(x)}{\wp'_\Lambda(x)}\right)^2 = \wp_\Lambda(2x) + 2\wp_\Lambda(x),$$

we find that

$$S_{\wp_\Lambda}(x) = \frac{\wp'''_\Lambda(x)}{\wp'_\Lambda(x)} - 6\wp_\Lambda(2x) - 12\wp_\Lambda(x). \tag{6}$$

Twice differentiating (1) yields $\wp'''_\Lambda = 12\wp_\Lambda \wp'_\Lambda$, and substituting into (6), we find

$$S_{\wp_\Lambda}(x) = -6\wp_\Lambda(2x).$$

Because $\wp_\Lambda(x) > 0$, we have $S_{\wp_\Lambda}(x) = -6\wp_\Lambda(2x) < 0$. □

We show graphs of $\wp_\Lambda$ and $S_{\wp_\Lambda}$ on the lattice with invariants $g_2(\Lambda) = 4$ and $g_3(\Lambda) = 2$ in Figure 4. Note that the proof of Theorem 3.4 did not rely on our assumption that the lattice was rhombic but instead relied on the critical value being positive. Thus it provides a new and much simpler proof that the Schwarzian is negative for real rectangular lattices when $g_3 > 0$.

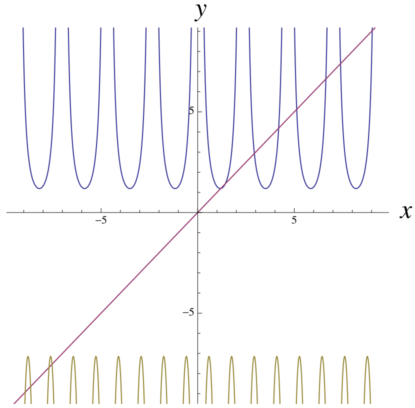

Figure 4. Graph of $\wp_\Lambda$ shown in blue and $S_{\wp_\Lambda}$ shown in yellow for $g_2 = 4$ and $g_3 = 2$. The line $y = x$ is shown in red.

The corollary below follows immediately from Theorems 3.3 and 3.4.

Corollary 3.5. *If* Λ *is a vertical real rhombic lattice*, *then* $\wp$ *has either* 0 *or* 1 *real attracting or parabolic cycles. If there exists a real nonrepelling cycle*, *then a real critical point is contained in the cycle of Fatou components.*

The assumption that Λ is a vertical real rhombic lattice is far from trivial. In fact, it seems to be necessary because the next example, found experimentally, shows that horizontal lattices might violate Corollary 3.5.

Example 3.6. Consider $\wp_\Lambda(x)$ with $g_2 = 27$ and $g_3 = -27.07$, shown in Figure 5. In this case, Λ is a horizontal real lattice since $g_3 < 0$. This function has only one

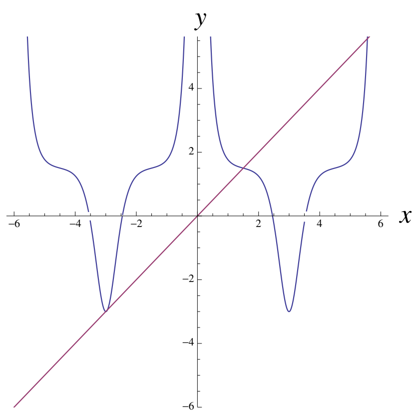

Figure 5. The graph of $\wp_\Lambda$ as described in Example 3.6, which has a real attracting fixed point that does not attract a real critical point. The function $\wp_\Lambda$ is shown in blue, and the line $y = x$ is shown in red.

real critical value, but it has two real attracting fixed points. The critical value at $x \approx -2.997$ is attracted to the attracting fixed point at $x \approx -3.0006$. There is an additional attracting fixed point at $x \approx 1.50037$ that does not attract any real critical points.

4. Dynamics on the complex plane

Connectivity properties of Julia sets of Weierstrass elliptic functions are not well understood except for the most regular cases where the lattice is square or triangular [Clemons 2012; Hawkins 2006; 2010; Hawkins and Look 2006; Hawkins and Koss 2002; 2004; 2005]. It is not clear whether the Julia set of an arbitrary real rhombic lattice or real rectangular lattice is connected, Cantor, or infinitely disconnected but not Cantor.

In this section, we examine what the results of Section 3 imply for the complex function $\wp_\Lambda$. In Sections 4A and 4B, we prove some results about the Julia set restricted to the real axis. In Section 4C, we find invariants for which the Julia set is the entire sphere.

4A. ***The real axis lies in the Julia set.*** Here, we present conditions under which the entire real axis must lie in the Julia set. We begin with a theorem, proved in [Hawkins and Koss 2005]. It required knowing that the real critical value belonged to the Julia set, as well as information about the orbits of the complex critical points.

Theorem 4.1 [Hawkins and Koss 2005]. *If Λ is a real rhombic lattice such that the complex critical values are associated with nonrepelling complex cycles and the real critical value is in $J(\wp_\Lambda)$, then the real and imaginary axes are contained in $J(\wp_\Lambda)$.*

The hypotheses about the orbits of the complex critical values are necessary because of behavior such as that which occurs in Example 3.6: it may be possible that the real critical value lies in $J(\wp_\Lambda)$ but there is a real attracting cycle that attracts the complex critical values. However, the results in Section 3 enable us to remove the hypotheses relating to the orbits of the complex critical values for $\wp_\Lambda$ on a vertical real rhombic lattice.

Theorem 4.2. *If Λ is a vertical real rhombic lattice and the real critical value is in $J(\wp_\Lambda)$, then the real and imaginary axes are contained in $J(\wp_\Lambda)$.*

Proof. By Proposition 2.9(5), $\wp_\Lambda$ has no Herman rings. By Theorem 2.2, $\wp_\Lambda$ is real. No interval in $\mathbb{R}$ can lie within a Siegel disk component because $\wp_\Lambda : \mathbb{R} \to \mathbb{R}$, which would contradict that $\wp_\Lambda^n$ is conjugate to an irrational rotation of the unit disk. Since the real critical value lies in $J(\wp_\Lambda)$, Corollary 3.5 implies that there are no attracting or parabolic cycles on the real axis. Therefore, the entire real axis must lie in $J(\wp_\Lambda)$. Proposition 2.5(5) implies that the imaginary axis maps to the

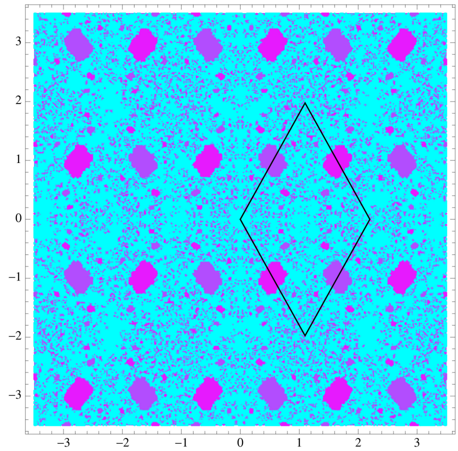

Figure 6. The function $\wp_\Lambda$ for $g_2 = 1$ and $g_3 = 6.5$ has the real and imaginary axes lying in the Julia set.

real axis, and thus the imaginary axis must lie in $J(\wp_\Lambda)$ by the invariance of the Julia set. □

Figure 6 illustrates an example, found experimentally, of a function $\wp_\Lambda$ for which the real axis lies in the Julia set. The lattice Λ in this example has invariants $g_2 = 1$ and $g_3 = 6.5$. The function has two attracting fixed points at $-0.607 \pm 0.942i$, each of which attracts a complex critical value. All points colored pink iterate to $-0.607 + 0.942i$, and all points colored purple iterate to $-0.607 - 0.942i$. Points colored blue lie in the Julia set.

4B. ***The Julia set restricted to the real axis is Cantor.*** Next, we move to a discussion about conditions which imply that the Julia set restricted to the real axis is Cantor. In [Hawkins and Koss 2005], any real rectangular square lattice or real triangular lattice with an attracting fixed point was shown to have a Cantor Julia set on the real axis. Here we extend the result to all real rectangular and all vertical real rhombic lattices.

The following proposition is an amalgamation of results from [Hawkins and Koss 2002; 2005].

Proposition 4.3. *Assume that Λ is a real rectangular or vertical real rhombic lattice with real period λ_1 such that $\wp_\Lambda$ has an attracting fixed point. If $\Lambda_{n_0} = [n_0\lambda_1, (n_0+1)\lambda_1]$ is the interval containing e_1, then the attracting fixed point t_{n_0} is in Λ_{n_0}. Further, there is a repelling fixed point $P_{n_0} \in \Lambda_{n_0}$ where $P_{n_0} = c_{n_0} + q$ for the critical point $c_{n_0} \in \Lambda_{n_0}$ and $0 < q < \frac{1}{2}\lambda_1$. Then $B = (P'_{n_0}, P_{n_0})$ is the immediate basin of attraction for t_{n_0}, where $P'_{n_0} = c_{n_0} - q$.*

In addition, we need the following lemma in the proof.

Lemma 4.4. *If Λ is a real rectangular or vertical real rhombic lattice then $\wp'_\Lambda(x)$ is monotone increasing.*

Proof. We begin with (4),

$$\wp''_\Lambda(x) = (6\wp_\Lambda(x))^2 - \tfrac{1}{2}g_2.$$

If $g_2 < 0$, then $\wp''_\Lambda(x) > 0$ at all x except lattice points.

Next, suppose $g_2 > 0$ and $g_3 > 0$. Using Lemma 2.3, e_1 is the minimum of $\wp_\Lambda$ on the real axis, so

$$\wp''_\Lambda(x) = (6\wp_\Lambda(x))^2 - \tfrac{1}{2}g_2 \geq 6e_1^2 - \tfrac{1}{2}g_2. \tag{7}$$

Using the second property from (3),

$$6e_1^2 - \tfrac{1}{2}g_2 = 6e_1^2 + 2(e_1e_3 + e_2e_3 + e_1e_2) = 6e_1^2 + 2e_1(e_2 + e_3) + 2e_2e_3.$$

Using the first property from (3),

$$6e_1^2 + 2e_1(e_2 + e_3) + 2e_2e_3 = 6e_1^2 - 2e_1^2 + 2e_2e_3 = 4e_1^2 + 2e_2e_3. \tag{8}$$

Finally, we apply the third property from (3) to obtain

$$4e_1^2 + 2e_2e_3 = 4e_1^2 + \frac{g_3}{2e_1}.$$

Since $g_3 > 0$ and $e_1 > 0$ by Propositions 2.5(2) and 2.6(2), we have $\wp''_\Lambda(x) > 0$.

Finally, if $g_3 < 0$ and Λ is real rectangular, Proposition 2.6(3) implies that $e_2 < 0 < e_3 < e_1$, and e_1 is the minimum of $\wp_\Lambda$ on the real axis. Using the argument from (7) to (8) in the previous paragraph, we have

$$\wp''_\Lambda(x) > 4e_1^2 + 2e_2e_3.$$

Using the first property from (3),

$$\begin{aligned} 4e_1^2 + 2e_2e_3 &= 4e_1^2 + 2e_3(-e_1 - e_3) \\ &= 2(e_1^2 - e_1e_3) + 2(e_1^2 - e_3^2). \end{aligned}$$

Since $0 < e_3 < e_1$, both terms are positive, so $\wp''_\Lambda(x) > 0$.

Thus, $\wp'_\Lambda(x)$ is monotone increasing over the intervals on which it is defined. □

The concavity can be observed in Figure 4. We are now ready to prove the main result of this section.

Proposition 4.5. *If Λ is a real rectangular or vertical real rhombic lattice for which $\wp_\Lambda$ has a real attracting fixed point, we have the following for any $\lambda \in \Lambda$:*

(1) $J(\wp_\Lambda) \cap (\mathbb{R} + \lambda)$ *is a Cantor set.*

(2) $J(\wp_\Lambda) \cap \big(\{z \mid z = iy,\ y \in \mathbb{R}\} + \lambda\big)$ *is a Cantor set.*

Proof. Using the notation of Proposition 4.3, let $B = (P'_{n_0}, P_{n_0})$ be the immediate basin of attraction for the attracting fixed point t_{n_0} lying in the interval Λ_{n_0}. By Lemma 2.3 and Proposition 2.9, if $\Lambda_j = [j\lambda_1, (j+1)\lambda_1]$, then $\wp_\Lambda(\Lambda_j) = [e_1, \infty]$ for all j. Further, $\wp_\Lambda^{-1}(B)$ consists of infinitely many disjoint open intervals, one in each Λ_j, each of which is a translation of B by a real lattice point. We label those intervals $B_j \subset \Lambda_j$. We label the intervals complementary to the B_j using C_j such that C_o contains the pole at the origin; then C_{n_0+1} contains the repelling fixed point P_{n_0}.

Let $s = |\wp'_\Lambda(P_{n_0})| > 1$ and $\alpha = \frac{1}{2}(1+s)$. Since $\wp'_\Lambda$ is strictly monotonic by Lemma 4.4,

$$\wp_\Lambda : C_j \to [e_1, \infty] \setminus (B \cap [e_1, \infty])$$

for each j, and $|\wp'_\Lambda(x)| > \alpha > 1$ for each $x \in C_j$.

Defining

$$JR = \left\{ x \in \mathbb{R} \;\middle|\; \wp_\Lambda^m(x) \in \textstyle\bigcup_{j \geq n_0} C_j \text{ for all } m \text{ for which } \wp_\Lambda^m \text{ is analytic} \right\},$$

we have

$$z \in JR \iff z \in J(\wp_\Lambda) \cap \mathbb{R}.$$

Since $|\wp'_\Lambda(z)| > \alpha > 1$ for all $z \in C_j$, we know $\operatorname{diam}(\wp_\Lambda^{-m} C_j) \to 0$ as $m \to \infty$. Standard arguments imply that JR is a Cantor set (see [Devaney and Keen 1988]).

To show that the intersection of the Julia set with the imaginary axis is Cantor, we need to investigate rectangular lattices and vertical real rhombic lattices separately. First, if Λ is vertical real rhombic, then by Proposition 2.5(5) and the evenness of $\wp_\Lambda$, every point $x \in (-\infty, e_1)$ has infinitely many pairs of preimages of the form $\pm ia$, $a \in \mathbb{R}$, that lie on the imaginary axis. More precisely, let $\lambda' = \lambda - \bar{\lambda}$ denote the period of $\wp_\Lambda$ along the imaginary axis, and set

$$\Lambda'_j = \left\{ z \in \mathbb{C} \;\middle|\; \operatorname{Re}(z) = 0 \text{ and } j\lambda' < \operatorname{Im}(z) < (j+1)\lambda' \right\};$$

we can think of Λ'_j as a set of line segments along the imaginary axis on which $\wp_\Lambda$ is periodic. Then in each Λ'_j, $j \in \mathbb{Z}$, there are exactly two preimages of every point $x \in (-\infty, e_1)$. Therefore, for each j, the set

$$\wp_\Lambda^{-1}(JR) \cap \Lambda'_j = J(\wp_\Lambda) \cap \Lambda'_j$$

is a homeomorphic image of JR and hence a Cantor set. If we denote by JR_i the set $J(\wp_\Lambda) \cap \{z = iy\}$, then we have shown that JR_i is a Cantor set.

Finally, if Λ is real rectangular, then by Proposition 2.6(4) and the evenness of $\wp_\Lambda$, every point $x \in (-\infty, e_2)$ again has infinitely many pairs of preimages of the form $\pm ia$, $a \in \mathbb{R}$, that lie on the imaginary axis. If $\Lambda = [\lambda_1, \lambda_2]$, where λ_2 is pure imaginary, then let

$$\Lambda'_j = \left\{ z \in \mathbb{C} \;\middle|\; \operatorname{Re}(z) = 0 \text{ and } j\lambda_2 < \operatorname{Im}(z) < (j+1)\lambda_2 \right\}.$$

Proceeding as in the vertical real rhombic case, JR_i is Cantor. □

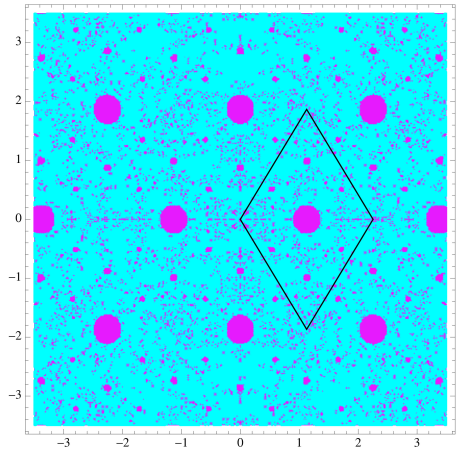

Figure 7. An example where the Julia set restricted to the real axis is Cantor.

We note that finding examples that satisfy the hypothesis of Proposition 4.5 is straightforward using the homogeneity property.

Corollary 4.6. *Let* Γ *be a standard real rectangular or a standard real vertical rhombic lattice*, *where* γ_1 *is chosen to be the smallest positive real lattice point. If* m *is any positive odd integer*, $k = \sqrt[3]{2/(m\gamma_1)}$, *and* $\Lambda = k\Gamma$, *then*

(1) $J(\wp_\Lambda) \cap (\mathbb{R} + \lambda)$ *is a Cantor set*,

(2) $J(\wp_\Lambda) \cap \big(\{z \mid z = iy,\ y \in \mathbb{R}\} + \lambda\big)$ *is a Cantor set*

for any $\lambda \in \Lambda$.

Proof. Let Γ be a standard real rectangular or a standard real vertical rhombic lattice, where γ_1 is chosen to be the smallest positive real lattice point, m a positive odd integer, $k = \sqrt[3]{2/(m\gamma_1)}$, and $\Lambda = k\Gamma$. Lemma 2.1(2) implies

$$\wp_\Lambda\big(\tfrac{1}{2}\lambda_1\big) = \wp_{k\Gamma}\big(\tfrac{1}{2}k\gamma_1\big) = \frac{1}{k^2}\wp_\Gamma\big(\tfrac{1}{2}\gamma_1\big) = \tfrac{1}{2}m\lambda_1.$$

Thus $\frac{1}{2}m\lambda_1$ is a superattracting fixed point by the periodicity of $\wp_\Lambda$. Proposition 4.5 gives the result. □

Figure 7 illustrates an example constructed through Corollary 4.6 of a function $\wp_\Lambda$ for which the intersection of the real axis and the Julia set is Cantor. We begin with the standard real vertical rhombic lattice Γ with invariants $g_2(\Gamma) = -1$ and $g_3(\Gamma) = 5$ and obtain the lattice Λ with invariants $g_2(\Lambda) \approx -1.269$ and $g_3(\Lambda) \approx 7.148$. The function $\wp_\Lambda$ has a superattracting fixed point at approximately 1.126. Points colored pink in Figure 7 iterate to the superattracting fixed point, and points colored blue lie in the Julia set.

4C. ***Invariants for which the Julia set is everything.*** Lemma 2.7 was used in [Hawkins and Koss 2002; 2004] to find isolated examples for which the Julia set of $\wp_\Lambda$ is the entire sphere by constructing real lattices for which all three critical values were prepoles. These results were broadened to $\wp_\Lambda$ for every real rhombic square lattice Λ in [Hawkins 2010] and a countable number of real rectangular lattices Λ in every similarity class in [Koss 2014]. For real rhombic square or real rectangular lattices, the entire postcritical orbit of $\wp_\Lambda$ is real, except for at most two points. The Schwarzian derivative was used to show that the functions examined in these papers have no Fatou components.

For real rhombic lattices, the postcritical set is not real. However, we can use the results of Section 3 to find parameters in each real rhombic shape equivalence class for which the Fatou set is empty.

Theorem 4.7. *Let Γ be a standard real vertical rhombic lattice, where γ_1 is chosen to be the smallest positive real lattice point. If m is any positive integer and $k = \sqrt[3]{1/(m\gamma_1)}$, then $\wp_\Lambda$ on the lattice $\Lambda = k\Gamma$ has $J(\wp_\Lambda) = \mathbb{C}_\infty$.*

Proof. Let Γ, m, k, and Λ be defined as in the hypothesis. Since m is a positive integer, $k > 0$ and Λ is a real vertical rhombic lattice. By Proposition 2.9(5), $\wp_\Lambda$ has no Herman rings. By Lemma 2.7, $\wp_\Lambda\left(\tfrac{1}{2}\lambda_1\right) = m\lambda_1 = e_1$ is a pole. By Proposition 2.5(4), $\mathrm{Re}(e_2) = \mathrm{Re}(e_3) = -\tfrac{1}{2}m\lambda_1$, so e_2 and e_3 lie on a vertical line passing through a real lattice point or a real half-lattice point. Proposition 2.5(5) implies

$$\wp_\Lambda(e_2) = \wp_\Lambda(e_3) \in \mathbb{R} \cup \{\infty\}.$$

Using Theorem 2.2, the postcritical set is a subset of $\mathbb{R} \cup \{e_1, e_2, \infty\}$. No interval in $\mathbb{R}$ can lie within a Siegel disk component because $\wp_\Lambda : \mathbb{R} \to \mathbb{R}$, which would contradict that $\wp_\Lambda^n$ is conjugate to an irrational rotation of the unit disk. No subset of $\mathbb{R}$ can form the boundary of a Siegel disk since $\wp_\Lambda$ is periodic with respect to Λ. Theorem 3.3 implies that if there were an attracting or parabolic cycle of Fatou components, then the cycle must lie on the real axis and contain a real critical point, a contradiction to the assumption that all real critical points are prepoles. Thus there can be no Fatou components, and $J(\wp_\Lambda) = \mathbb{C}_\infty$. □

We can use Theorem 4.7 and previous results from [Hawkins 2010; Koss 2014] to illustrate parameters in the (g_2, g_3)-plane for which $J(\wp_\Lambda) = \mathbb{C}_\infty$. We show an approximation of the locus of parameters for the cases $m = 1, 2, 3$, and 4 from Theorem 4.7 in this paper and the corresponding Theorem 4.3 in [Koss 2014] in increasingly darker shades in Figure 8: light blue corresponds to $m = 1$, medium blue corresponds to $m = 2$, dark blue corresponds to $m = 3$, black corresponds to $m = 4$. If Λ is a real rhombic square lattice, then $J(\wp_\Lambda) = \mathbb{C}_\infty$ [Hawkins 2010]; these lattices appear in gray in Figure 8 as the negative real axis.

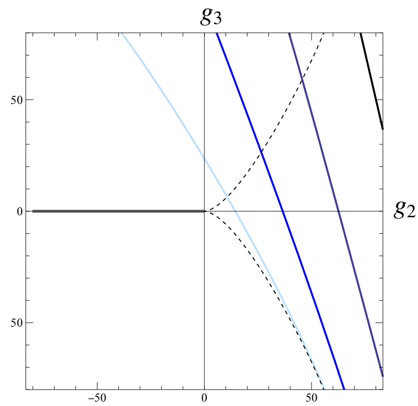

Figure 8. The locus of parameters for which $J(\wp_\Lambda) = \mathbb{C}_\infty$.

Acknowledgement

The authors would like to express their sincere appreciation to the referee, whose comments and suggestions significantly improved the paper.

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Received: 2015-05-15 Revised: 2016-04-22 Accepted: 2016-05-02

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2017 vol. 10 no. 3