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Discrete dynamics of contractions on graphs

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We study the dynamical behavior of functions on vertices of a graph that are contractions in the graph metric. We show that the fixed point set of such functions must be convex. If a function has no fixed points and the graph is a tree, we prove that every dynamical cycle must have an even period and the function behaves eventually like a symmetry.

1. Introduction

This work was inspired by dynamics of analytic functions on the unit disk. The key property of such functions is the point-invariant Schwarz lemma, i.e., that analytic functions are contractions in the hyperbolic metric of the disk. This property allows the proof of various results about iteration of analytic functions; see, for example, the survey paper [Poggi-Corradini 2011].

Our purpose is to study dynamics of contractions in a discrete setting. In particular, we study dynamics on finite graphs (in most cases, trees). A connected graph can be considered as a discrete metric space of vertices with the *graph metric*. Let $G = (V, E)$ be a finite, connected, simple graph with the set of vertices V and the set of edges E. Then for all vertices $x, y \in V$, we say the distance between x and y, denoted $d(x, y)$, is the number of edges in the shortest path connecting x to y. Such path is called a *geodesic*. Note that trees as metric spaces are 0-hyperbolic [Anderson 1999], so we expect them to have some similar properties to the unit disk with hyperbolic metric.

We wish to study contractions (in the graph metric) on the vertices of a graph. Let f be a function on the vertices of G to the vertices of G. We say f is a *contraction* if, for all vertices $x, y \in V$, we have $d(f(x), f(y)) \le d(x, y)$. We will need some terminology from dynamics. Let f be a function. We denote by $f^{\circ n}(x) = f \circ f \circ f \circ \cdots \circ f(x)$ (n terms) the n-th *iterate* of f. If for some point x and some positive integer n, we have $f^{\circ n}(x) = x$, then we say x is a periodic point, x lies on a *dynamical cycle* of f of period n, or that x lies on a *dynamical n-cycle*

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of f. If $f(x) = x$, we say x is a *fixed point* of f. We use the term *dynamical cycle* to distinguish these cycles from the graph cycles.

It is easy to show by induction that, given a contraction f, the map $f^{\circ n}$ is also a contraction for any positive integer n. Dynamical cycles and fixed points will be the main focus of our study.

2. Fixed point sets

Our goal is to characterize the set of fixed points of a contraction on graph vertices. Note that in the general case, the fixed point set can be empty:

Example 2.1. Let G_1 be a graph with four vertices x, y, z, w. Let f be a function on the vertices of G_1 defined by $f(x) = y$, $f(y) = z$, $f(z) = w$, and $f(w) = x$. Then $\{x, y, z, w\}$ forms a dynamical 4-cycle of f (see Figure 1). The map f is clearly a contraction since for all $a, b \in \{x, y, w, z\}$, we have $d(f(a), f(b)) = d(a, b)$. In this case, the set of fixed points of f is empty.

Example 2.2. Let G_2 be a graph with vertices x_0, x_1, x_2, y_0, y_1, z_0 and z_1 as shown in Figure 2. Let f be a contraction on the vertices of G_2 such that x_0, x_1, x_2 are fixed by f, and $\{y_0, y_1\}$ and $\{z_0, z_1\}$ are dynamical 2-cycles of f.

Note that one main difference between the two examples is that for any two vertices in G_2, the geodesic connecting them is unique, whereas this is not the case with G_1. Notice also that for any two fixed points in G_2, the geodesic connecting them contains only fixed points.

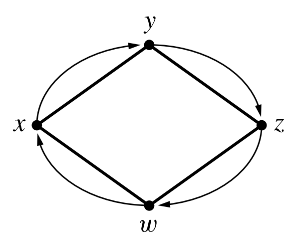

Figure 1. Dynamical 4-cycle.

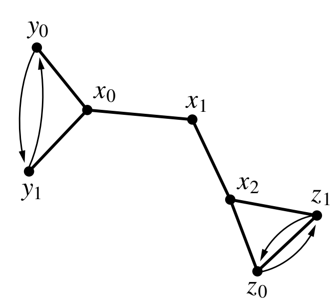

Figure 2. Cycles and fixed points.

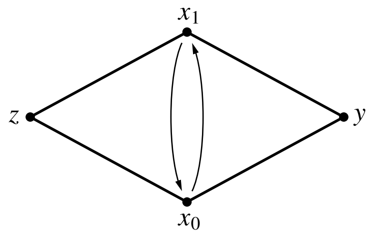

Figure 3. The graph G_3 with nonunique geodesics.

Definition 2.3. Let $G = (V, E)$ be a graph and let $H \subset V$. We say H is *convex* if for any two vertices in H, the geodesic connecting them contains only vertices in H. (See, for example, [Gross and Yellen 2006].)

Thus in Example 2.2, the set of fixed points of f is convex. In fact, this is true in general:

Theorem 2.4. *Let $G = (V, E)$ be a graph such that the geodesic between any two vertices is unique. Let f be a contraction on the vertices of G. Then the set of fixed points of f is convex.*

Proof. Let $x, y \in V$ be fixed by f. Let L be the unique geodesic connecting them. Let $z \in L$. We need to show that $f(z) = z$. We will first show that $f(z) \in L$ and it will follow that $f(z) = z$.

By way of contradiction, suppose $f(z) \notin L$. Then there exist unique geodesics connecting x to $f(z)$ and y to $f(z)$, respectively. We can concatenate these geodesics to construct a walk K connecting x to y, so the length of K is $d(x, f(z))+d(f(z), y)$ and the length of L is $d(x, z)+d(z, y)$. Since f is a contraction and x and y are fixed points, we have $d(f(z), x) \le d(z, x)$ and $d(f(z), y) \le d(z, y)$. Then it follows that $d(x, f(z)) + d(f(z), y) \le d(x, z) + d(z, y)$. If $d(x, f(z)) + d(f(z), y) = d(x, z) + d(z, y)$, then L is not a unique geodesic between x and y, a contradiction. If $d(x, f(z)) + d(f(z), y) < d(x, z) + d(z, y)$, then K is shorter than L, which is also a contradiction. Thus it must be that $z \in L$.

Now we will show $f(z) = z$. Suppose $f(z) \ne z$. Since $f(z)$ lies on the geodesic L connecting x to y, we have $d(x, z) + d(z, y) = d(x, f(z)) + d(f(z), y) = d(x, y)$. We can assume without loss of generality that $d(x, f(z)) < d(x, z)$, in which case we obtain $d(y, f(z)) = d(x, y) - d(x, f(z)) > d(x, y) - d(x, z) = d(y, z)$, contradicting the fact that f is a contraction. Thus we conclude that $f(z) = z$. □

Note that if for any two points in G the geodesic connecting them is not unique then the conclusion of Theorem 2.4 does not necessarily hold, as can be seen in the following counterexample.

Example 2.5. Let G_3 be a graph with vertices x_0, x_1, y and z as shown in Figure 3.

Let f be a contraction such that the vertices z and y are fixed and the points x_0 and x_1 form a dynamical 2-cycle. Note that the geodesic connecting z to y is not unique, since the path from z to y through x_0 is the same length as the path through x_1. Despite the fact that z and y are fixed and that x_0, x_1 lie on the geodesics connecting them, x_0 and x_1 are clearly not fixed. Thus the conclusion of Theorem 2.4 does not hold in this case.

Corollary 2.6. *Let $G = (V, E)$ be a graph such that for any two vertices in G the geodesic connecting them is unique. Let f be a contraction on V. Suppose f has a dynamical cycle J of period k. Let z be a point which lies on the geodesic connecting two consecutive points in J. Then z lies on a dynamical cycle whose period divides k.*

Proof. Let $x, y \in J$. Let $z \in V$ such that z lies on the geodesic between x and y. Since J is a dynamical cycle of period k, we know $f^{\circ k}(x) = x$ and $f^{\circ k}(y) = y$. Thus x and y are fixed by the k-th iterate of f. Since f is a contraction, any iterate of f is also a contraction. Thus Theorem 2.4 applied to $f^{\circ k}$ implies $f^{\circ k}(z) = z$. So z must lie on a dynamical cycle whose period divides k. $\square$

Now we will consider a particular case when the graph is a tree. For any tree, a path connecting any two points is unique, hence geodesics are unique, so Theorem 2.4 holds. But the converse is also true for trees: any convex set of vertices will be a fixed point set for some contraction.

We will need the following property of a tree structure: in a tree, a concatenation of two geodesics from x to y and from y to z is either a geodesic from x to z or a walk that follows the geodesic connecting x to y until the first common point of the two geodesics, y', then follows the geodesic from y' to y, then goes back to y' along the same geodesic and finally follows the geodesic from y' to z. Note that concatenation of geodesics from x to y' and from y' to z will form a geodesic that connects x to z.

Proposition 2.7. *Let $T = (V, E)$ be a tree and $H \subset V$ be convex set. Then there exists a contraction f such that H is the fixed point set of f.*

Proof. Given H, we define the desired contraction f as follows: for all $x \in V$, $f(x) = y$, where $y \in H$ is the closest vertex to x in H; see Figure 4. Note that such a y is unique. Indeed, suppose $y_1, y_2 \in H$ are at the same shortest distance from $x \notin H$. Apply the property mentioned above the proposition to the concatenation of geodesics connecting y_1 to x and x to y_2. If it is a geodesic, then $x \in H$, which is a contradiction. If instead there is a common point y', then $y' \in H$ and it is closer to x than y_1 and y_2 are, again a contradiction. Thus the point y is unique and the function f is well-defined. Also, H is clearly fixed point set of f.

Now we need to show that f is a contraction. Let $f(x_1) = y_1$ and $f(x_2) = y_2$. Consider a walk following the geodesic from x_1 to y_1, then from y_1 to y_2. If there is a common point of these geodesics other than y_1, then this point is in H and within a shorter distance to x_1 than y_1, which contradicts the construction of y_1. So the

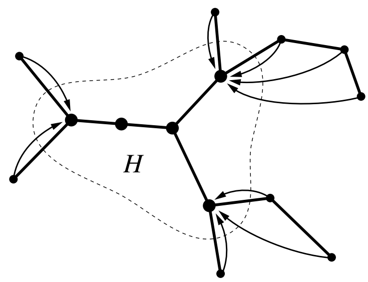

Figure 4. Constructing the contraction f for a given convex subset of vertices H.

concatenation of these two geodesics is the geodesic from x_1 to y_2. Similarly, the geodesic from y_1 to x_2 passes through y_2, and finally, the geodesic from x_1 to x_2 is just a concatenation of those from x_1 to y_1, y_1 to y_2 and y_2 to x_1. So we have

$$d(x_1, x_2) = d(x_1, y_1) + d(y_1, y_2) + d(y_2, x_2) \geq d(y_1, y_2),$$

and f is a contraction. □

3. Contractions with no fixed points

In the previous section, we characterized the set of fixed points of a contraction on the vertices of a graph with unique geodesics, in particular a tree. Next we want to consider the case when a contraction has no fixed points. Then there must exist a dynamical cycle. We will use the following property of periodic points:

Lemma 3.1. *Let G be a finite graph, and f be a contraction on vertices of G. If x and y are two periodic points of f (not necessarily from the same dynamical cycle), then $d(f(x), f(y)) = d(x, y)$.*

Proof. Assume x belongs to a dynamical m-cycle and y belongs to a dynamical n-cycle. Let K be a common multiple of m and n. Then we have

$$d(x, y) \geq d(f(x), f(y)) \geq \cdots \geq d(f^{\circ K}(x), f^{\circ K}(y)) = d(x, y).$$

So all inequalities must be, in fact, equalities and in particular, $d(f(x), f(y)) = d(x, y)$. □

Now let us introduce some notation. Let $G = (V, E)$ be a graph and f a contraction on V. Let $J \subset V$ be a dynamical cycle of f. Then we denote by J' the set of all vertices which lie on geodesics connecting consecutive points in J, together with the vertices in J.

Theorem 3.2. *Let T be a finite tree. Let f be a contraction on the vertices of T. If f has no fixed points, then f has a dynamical 2-cycle such that the points in the cycle are connected by an edge. Moreover, such a cycle is unique.*

Proof. Suppose f has no fixed points. Since the number of vertices of T is finite, every vertex of T either lies on a dynamical cycle of period greater than 1 or is eventually mapped into one. Let k be the smallest period of all dynamical cycles of f. Let J be a dynamical cycle of period k such that the distance between consecutive points in J is least among all dynamical cycles of f of period k. We want to show that $k = 2$.

We claim that for $k > 2$ there must exist two geodesics connecting consecutive points in J that intersect at a point other than their endpoints. If not, the points in J' would form a graph cycle, which is a contradiction since T is a tree. Thus there must exist two geodesics which intersect at a point which is not one of their endpoints.

Suppose two nonconsecutive geodesics intersect at some point y. Then we claim that there must exist two consecutive geodesics which intersect at point z which is not one of their endpoints. Indeed, if we start from the point y of intersection of two nonconsecutive geodesics and follow one of the geodesics to the point x_j on the cycle J, then follow the next geodesic to the point $x_{j+1} = f(x_j)$, and so on, we will eventually return to the point y. Since the graph is a tree, the walk constructed this way must go over each edge in this walk at least twice. In particular, there must exist a vertex w which is farthest away from y on this walk and an edge $\{w, z\}$ such that our walk will follow the edge from z to w and then immediately return to z through the same edge. Note that w must be an endpoint of two consecutive geodesics, because one geodesic cannot follow the same edge twice. Then z lies on the intersection of two consecutive geodesics.

Without loss of generality, let x_0, x_1, x_2 be the endpoints of the two consecutive geodesics constructed above. By Corollary 2.6, z must lie on a dynamical cycle whose period divides k, but since k is the smallest possible cycle length, z must lie on a dynamical k-cycle.

Since f is a contraction and x_0, x_1, x_2 are points on a dynamical cycle, f must map the geodesic from x_0 to x_1 bijectively to the geodesic from x_1 to x_2. Since z lies on the geodesic from x_0 to x_1, the point $f(z)$ must lie on the geodesic from x_1 to x_2. Thus both z and $f(z)$ lie on the geodesic from x_1 to x_2 and we have $d(z, f(z)) < d(x_1, x_2) = d(x_0, x_1)$. So we have found a dynamical k-cycle $\{z, f(z), \ldots, f^{\circ(k-1)}(z)\}$ such that the distance between two consecutive points in this cycle is less than $d(x_0, x_1)$.

This contradicts the way we selected J, so k must be equal to 2 and the geodesic from x_0 to x_1, which is the same as the geodesic from x_1 to x_0, must contain no other points. This means there is a dynamical 2-cycle $\{x_0, x_1\}$ and x_0 and x_1 are connected by an edge.

Now we need prove that such a dynamical 2-cycle is unique. Let $\{y_0, y_1\}$ be another such cycle. Without loss of generality assume that the distance a between x_0 and y_0 is the shortest among all distances from a point in $\{x_0, x_1\}$ to a point in $\{y_0, y_1\}$. Now consider x_1; it is connected to x_0 by an edge. If x_1 lies on the geodesic from x_0

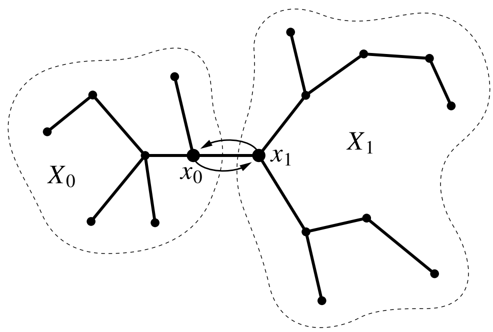

Figure 5. Unique 2-cycle $\{x_0, x_1\}$ and sets X_0 and X_1.

to y_0, then $d(x_1, y_0) < d(x_0, y_0)$, which contradicts the choice of x_0, y_0. Otherwise, the geodesic from y_0 to x_1 follows the geodesic from y_0 to x_0 and then the edge connecting x_0 to x_1, so $d(y_0, x_1) = a + 1$. Similarly, $d(x_0, y_1) = a + 1$, and finally, $d(x_1, y_1) = a + 2$. But then $d(x_1, y_1) = d(f(x_0), f(y_0)) > d(x_0, y_0)$, which contradicts the assumption that f is a contraction. □

It will in fact turn out that every dynamical cycle of a contraction with no fixed points has even period. To prove this, we will need the following corollary to Theorem 3.2. Let us introduce the following notation. Let $\{x_0, x_1\}$ be the points in the 2-cycle constructed in Theorem 3.2. We let X_0 denote the set of all points which are within shorter distance to x_0 than to x_1. Similarly we let X_1 denote the set of all points which are within shorter distance to x_1 than to x_0; see Figure 5.

Corollary 3.3. *Let T be a finite tree and f a contraction on the vertices of T such that f has no fixed points. Let $\{x_0, x_1\}$ be the unique dynamical 2-cycle, where x_0 and x_1 are connected by an edge. Then for all vertices z that lie on any dynamical cycle, if $z \in X_0$ (respectively X_1), then $f(z) \in X_1$ (respectively X_0).*

Proof. Let z lie on a dynamical cycle and $z \in X_0$. By way of contradiction, suppose $f(z) \in X_0$. Let $a = d(z, x_0)$; then $d(z, x_1) = a + 1$. By Lemma 3.1, $d(f(z), x_1) = a$, and since $f(z) \in X_0$, we must have $d(f(z), x_0) < d(f(z), x_1) = a$. But by Lemma 3.1 again, $d(f(z), x_0) = d(f(z), f(x_1)) = d(z, x_1) = a + 1$, which is a contradiction. So $f(z) \in X_1$. □

Note that if z is not a periodic point, then the above claim does not hold.

Example 3.4. Let T be a tree with vertices x_0, x_1 and z such that there are edges between x_0 and x_1 and between x_0 and z, and f be a contraction such that $\{x_0, x_1\}$ forms a dynamical 2-cycle and $f(z) = x_0$ (see Figure 6). Then f has no fixed points, and x_0 and x_1 form the unique 2-cycle connected by an edge. Since $f(z) = x_0$, we have $z \in X_0$ and also $f(z) \in X_0$. Thus we see that if a point z is in X_0 but does not lie on a dynamical cycle, it is not necessarily true that $f(z) \in X_1$.

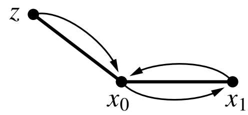

Figure 6

Now we are ready to prove the following:

Theorem 3.5. *Let T be a finite tree and f a contraction on the vertices of T such that f has no fixed points. Then every dynamical cycle of f has even period.*

Proof. Since f has no fixed points, f has a dynamical 2-cycle $\{x_0, x_1\}$ whose points are connected by an edge and sets of vertices X_0 and X_1 as defined above. Let $\{y_0, y_1, \dots, y_{n-1}\}$ be a dynamical n-cycle of f. Without loss of generality, suppose $y_0 \in X_0$. Then by Corollary 3.3, we have $y_1 \in X_1$, and in general, $y_{2k} \in X_0$ and $y_{2k+1} \in X_1$. If n is odd, then $y_0 = f(y_{n-1}) \in X_1$, which is a contradiction to $y_0 \in X_0$. Hence every dynamical cycle of f has even period. □

If a contraction f on the vertices of a tree T has no fixed points, then f eventually behaves like a symmetry. More precisely:

Theorem 3.6. *Let $T = (V, E)$ be a finite tree and f a contraction on V without fixed points. Then there exists a subset H of V and a nonnegative integer N such that $f^{\circ N}(V) = H$ and f is a symmetry on the connected subgraph induced by H. In particular, there is an edge in the subgraph such that two connected components obtained by removing this edge are isomorphic graphs and f is an isomorphism.*

Proof. Since T is finite and has no fixed points, each vertex of T will be mapped eventually to a point on a dynamical cycle. Thus there exists N such that $f^{\circ N}(V) = H$ contains only periodic points of f. Note that by Corollary 2.6, the subgraph induced by H is connected. Let $\{x_0, x_1\}$ be the unique dynamical 2-cycle whose points are connected by an edge. Then by Corollary 3.3, for all $z \in H \cap X_0$, we have $f(z) \in H \cap X_1$ and for all $z \in H \cap X_1$, we have $f(z) \in H \cap X_0$. Moreover, since all points in H are periodic, f bijectively maps $H \cap X_0$ to $H \cap X_1$. Now we need to show that any two vertices y, z in $H \cap X_0$ are connected by an edge if and only if $f(y)$ and $f(z)$ are connected by an edge. But being connected by an edge is equivalent to $d(y, z) = 1$, and since by Lemma 3.1, $d(y, z) = d(f(y), f(z))$, the required conclusion follows. □

4. Conclusion

Note that in the classical case of the unit disk in the complex plane, any analytic self-map of the disk always has a fixed point in the closed disk. This is the consequence of the classical Denjoy–Wolff theorem (see, for example, [Abate 1989] and references therein). In our study, a contraction without fixed points must behave like a symmetry. Symmetries are contractions in the unit disk, but they are not analytic (in fact, they are anticonformal, i.e., they preserve the value of angles, but change their orientation). So we can say that our result agrees with the classical case.

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