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A bijective proof of a q-analogue of the sum of cubes using overpartitions

Jacob Forster, Kristina Garrett, Luke Jacobsen and Adam Wood

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We present a q-analogue of the sum of cubes, give an interpretation in terms of overpartitions, and provide a combinatorial proof. In addition, we note a connection between a generating function for overpartitions and the q-Delannoy numbers.

1. Introduction

The formula for the sum of the first n cubes,

$$\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2, \tag{1}$$

is well known and has been proven using various methods. Benjamin and Orrison [2002] gave two combinatorial proofs. More recently, Garrett and Hummel [2004] proved a q-analogue of (1) using integer partitions. (A q-analogue is an expression involving q-binomial coefficients — see Section 2.3 on the next page — and reducing to the given expression when $q \to 1^-$.) In this paper, we give an alternate q-analogue of (1) and provide a bijective proof using overpartitions. The first section is devoted to an introduction to partition theory and establishing necessary notation and facts for our work. Then we state and explain a generating function for overpartitions and relate it to the Delannoy numbers. In the last section we give our q-analogue and provide a combinatorial proof.

2. Background

In this section, we introduce aspects of partition theory that are relevant to our work. For further reading, see [Andrews 1976; Corteel and Lovejoy 2004].

MSC2010: 05A17, 05A19.
Keywords: overpartitions, combinatorial proof, Delannoy numbers, q-analogue.

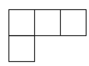
2.1. ***Partitions.***

Definition 1. A partition λ of a positive integer n is a nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i = n$. The λ_i are called the parts of the partition.

As an example, consider $n = 4$. The five distinct partitions of 4 are

$$4,\ 31,\ 22,\ 211,\ 1111.$$

One method of displaying partitions graphically is with Ferrers shapes. A Ferrers shape of a partition $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_k$, where $\lambda_i \geq \lambda_{i+1}$, is a left-justified array of cells with λ_i cells in row i of the shape and $i = 1$ defined as the top row. Below is the Ferrers shape for the partition $\lambda = 31$:



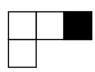
2.2. ***Overpartitions.***

Definition 2. An overpartition λ is a partition $\lambda_1, \lambda_2, \ldots, \lambda_k$ in which the first occurrence of a given part size may be overlined.

Below are the fourteen distinct overpartitions of $n = 4$:

$$4, \bar{4}, 31, \bar{3}1, 3\bar{1}, \bar{3}\bar{1}, 22, \bar{2}2, 211, \bar{2}11, 2\bar{1}1, \bar{2}\bar{1}1, 1111, \bar{1}111.$$

Overpartitions can also be graphically represented using Ferrers shapes by letting the last cell of the rows corresponding to overlined parts be shaded. For example, the Ferrers shape for the overpartition $\lambda = \bar{3}1$ is



2.3. ***Partitions in a k × (n − k) box.*** In order to discuss partitions whose Ferrers shapes fit inside of a $k \times (n-k)$ box, we must first introduce the q-binomial coefficient. The q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{i=n-k+1}^{n}(1-q^i)}{\prod_{i=1}^{k}(1-q^i)},$$

and is a q-analogue of the binomial coefficient. It is well known that

$$g_{n,k}(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

is the generating function for the number of partitions whose Ferrers shapes fit inside of a $k \times (n-k)$ box. This generating function can be easily shown to satisfy the recurrence relation

$$g_{n,k}(q) = q^k g_{n-1,k}(q) + g_{n-1,k-1}(q),$$

which is a q-analogue of the binomial coefficient recurrence. Note that, in this case, the empty partition is included in the set of partitions that fit inside the $k \times (n-k)$ box.

3. Overpartitions in a $2 \times (n-1)$ box

For our main theorem, we are interested in counting the number of overpartitions whose Ferrers shape fits in a $2 \times (n-1)$ box. The generating function for the number of partitions in a $2 \times (n-1)$ box is $\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q$. In this section, we give an analogy of this generating function for overpartitions. We will discuss the recurrence relation for overpartitions that fit in a $k \times (n-k)$ box and then use it to verify a generating function for the number of overpartitions that fit in a $2 \times (n-1)$ box.

3.1. ***Recurrence relation for overpartitions.*** We will first discuss the general case of overpartitions in a $k \times (n-k)$ box and then consider the case of a $2 \times (n-1)$ box. Let $\bar{p}_{n,k}$ denote the number of overpartitions that can fit in a $k \times (n-k)$ box. Then, $\bar{p}_{n,k}$ satisfies the recurrence relation

$$\bar{p}_{n,k} = \bar{p}_{n-1,k} + \bar{p}_{n-1,k-1} + \bar{p}_{n-2,k-1}. \tag{2}$$

We will explain each term in the recurrence relation. Note that, given a $k \times (n-k)$ box, this recurrence relation indicates that there are three possible disjoint ways of transforming the $k \times (n-k)$ box which, when taken together, describe all possible overpartitions that can fit inside a $k \times (n-k)$ box. These disjoint cases can be easily seen by considering the largest part of an overpartition, λ, in a $k \times (n-k)$ box and are as follows:

(i) $\mathrm{lp}(\lambda) < n-k$. Then the other parts of the overpartition must be less than or equal to $\mathrm{lp}(\lambda)$. This situation describes the number of overpartitions in a $k \times (n-k-1)$ box.

(ii) $\mathrm{lp}(\lambda) = n-k$ and $\mathrm{lp}(\lambda)$ is not overlined. Then the other parts of the overpartition are less than or equal to $n-k$. Thus, this collection of overpartitions is equivalent to the number of overpartitions in a $(k-1) \times (n-k)$ box.

(iii) $\mathrm{lp}(\lambda) = n-k$ and $\mathrm{lp}(\lambda)$ is overlined. Then the other parts of the overpartition must be less than $(n-k)$. Hence, this case is equivalent to the number of overpartitions that fit inside of a $(k-1) \times (n-k-1)$ box.

These cases are shown in Figure 1.

Hence, the three disjoint cases of the recurrence relation cover all possible cases of overpartitions that can fit in a $k \times (n-k)$ box.

To be useful when verifying the generating function in question, (2) must be written in terms of q. That is, let $G(n, k, q)$ be the generating function for the

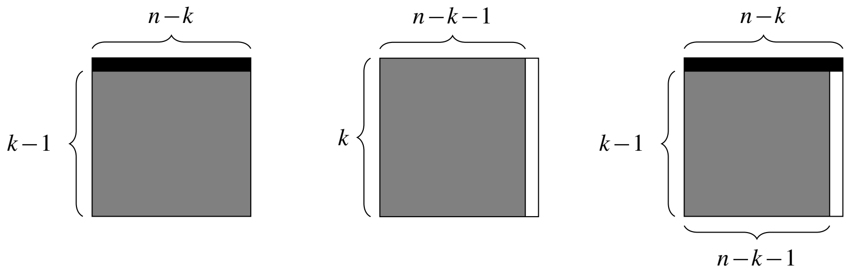


Figure 1. Illustration of the three cases for the recurrence relation. The dimensions of all of the boxes are $k \times (n-k)$. Black denotes a fixed part, gray denotes that the portion can be filled with all possible overpartitions, and white corresponds to empty space. Left: $\mathrm{lp}(\lambda) < n-k$. Middle: $\mathrm{lp}(\lambda) = n-k$ and $\mathrm{lp}(\lambda)$ is not overlined. Right: $\mathrm{lp}(\lambda) = n-k$ and $\mathrm{lp}(\lambda)$ is overlined.

number of overpartitions that fit in a $k \times (n-k)$ box. Then,

$$G(n,k,q) = G(n-1,k,q) + q^{n-k} G(n-1,k-1,q) + q^{n-k} G(n-2,k-1,q).$$

In the case of overpartitions in $2 \times (n-1)$ box, we have the recurrence relation

$$G(n+1,2,q) = G(n,2,q) + q^{n-1} G(n,1,q) + q^{n-1} G(n-1,1,q). \tag{3}$$

We now give the generating function.

Lemma 3. *Let n be a positive integer and $|q| < 1$. Then $f(q) = (2q+2q^2)\begin{bmatrix} n \\ 2 \end{bmatrix}_q + 1$ is the generating function for overpartitions that fit inside of a $2 \times (n-1)$ box.*

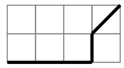
It can be shown that $f(q)$ satisfies (3); therefore, Lemma 3 holds.

3.2. ***The q-analogue of Delannoy numbers.*** Now that we have verified our generating function for the number of overpartitions in a $2 \times (n-1)$ box, we will draw a connection between Lemma 3 and the Delannoy numbers.

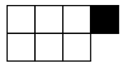
Definition 4. Let m, n be positive integers. The Delannoy numbers $D(m,n)$ are the number of lattice paths from $(0,0)$ to (m,n) in which only east, north, and northeast steps are allowed.

It is easy to see that when we consider the cells above the path drawn from $(0,0)$ to (m,n) as a Ferrers shape, the Delannoy numbers are equal to the number of overpartitions that fit inside of a $m \times n$ box. Note that in this model, the northeast steps correspond to overlined, and thus shaded, cells.

Example 5. Consider a 2×4 box. The following is a lattice path in this box:

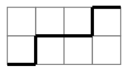


This lattice path corresponds to the Ferrers shape

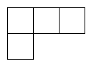


and thus the overpartition $\bar{4}3$.

Example 6. Consider another lattice path in a 2×4 box:



This lattice path corresponds to the Ferrers shape



and thus the partition 31.

Lemma 7. *Let n be a positive integer and $|q| < 1$. The generating function for overpartitions that fit inside a $2 \times (n-1)$ box, $g(q) = (2q + 2q^2)\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q + 1$, is a q-analogue of the Delannoy numbers, $D(2, n-1)$.*

Proof. As per the definition of a q-analogue, we first take the limit as $q \to 1^-$ of $g(q)$ to find the expression that our generating function generalizes in terms of q. Therefore, we see

$$\lim_{q \to 1^-} (2q + 2q^2)\begin{bmatrix} n \\ 2 \end{bmatrix}_q + 1 = 4\binom{n}{2} + 1 = 2n(n-1) + 1.$$

Next, we must show that this result, $2n(n-1)+1$, is indeed the expression for the Delannoy numbers $D(2, n-1)$. According to [Pan 2015], a formula for the Delannoy numbers is

$$D(n,k) = \sum_{d=0}^{n} 2^d \binom{k}{d}\binom{n}{d}.$$

Therefore, in this case, we have

$$D(2, n-1) = \sum_{d=0}^{2} \binom{n-1}{d}\binom{2}{d},$$

which readily simplifies to

$$D(2, n-1) = 2n(n-1) + 1.$$

Ergo, we have equality and the generating function $g(q) = (2q + 2q^2)\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q + 1$ is a q-analogue of the Delannoy numbers $D(2, n-1)$. □

4. A q-analogue of the sum of cubes

In [Garrett and Hummel 2004], the authors give a q-analogue of the sum of cubes and a bijective proof using partitions. We give another q-analogue of the sum of cubes and provide a bijective proof with a similar method, but using overpartitions.

Theorem 8. *Let n be a positive integer and let $|q| < 1$. Then,*

$$\sum_{i=1}^{n} 2q^{i-1}\left(\frac{1-q^{i-1}}{1-q}\right)^2\left(\left(\frac{1-q^{i-2}}{1-q}\right)+\left(\frac{1-q^{i}}{1-q}\right)\right) = (2q+2q^2)\begin{bmatrix} n \\ 2 \end{bmatrix}_q^2. \tag{4}$$

Note that, taking the limit as $q \to 1^-$, we obtain

$$\sum_{i=1}^{n} i^3 = \binom{n+1}{2}^2.$$

Thus, the above theorem is a q-analogue of the sum of cubes.

Bijective proof. We will prove Theorem 8 by interpreting the terms combinatorially and finding a weight-preserving bijection between two sets of overpartitions. Let R be a set of pairs of overpartitions, (λ, μ), where λ is a nonempty overpartition that fits inside a $2 \times (n-1)$ box and μ is a partition that fits inside a $2 \times (n-2)$ box. It follows that $f(q) = \sum_{(\lambda,\mu)\in R} q^{|\lambda|+|\mu|}$ is a generating function for R and is equal to the right-hand side of (4).

Given a positive integer n, let L be a set of tuples, $(v, a, b) \cup (v, a, b')$, where the allowed values of v, a, b, and b' are:

- v is an overpartition into two parts, where the largest part is equal to at most $n-1$ and can be overlined and the smallest part is at most $n-2$ and cannot be overlined.
- $0 \le a \le n-2$.
- $0' \le b' \le (n-3)'$.
- $0 \le b \le n-1$.

Let $\ell = (v, a, b) \in L$. Then $g(q) = \sum_{\ell\in L} q^{|\ell|}$, where $|\ell| = |v| + a + b$ is a generating function for L and is equal to the left-hand side.

We will now define a bijection between the finite sets R and L. Then, we can show that $f(q) = g(q)$; therefore, (4) holds. So, let $\phi : R \to L$, where $\phi(\lambda, \mu) = (v, a, b)$ and define ϕ in cases:

<u>Case 1</u>: $\lambda_1 > \mu_1$.

(a) $\lambda_2 \neq 0$, and λ_2 is not overlined.

(i) If λ_1 is not overlined, then $\phi(\lambda, \mu) = ((\lambda_1)(\lambda_2 - 1), \mu_2, \mu_1 + 1)$.

(ii) If λ_1 is overlined, then $\phi(\lambda, \mu) = ((\overline{\lambda_1})(\lambda_2 - 1), \mu_2, \mu_1 + 1)$.

(b) λ_2 is overlined or $\lambda_2 = 0$.

(i) If λ_1 is not overlined, then $\phi(\lambda, \mu) = ((\lambda_1)(\lambda_2), \mu_1, \mu_2)$.

(ii) If λ_1 is overlined, then $\phi(\lambda, \mu) = ((\overline{\lambda_1})(\lambda_2), \mu_1, \mu_2)$.

Case 2: $\lambda_1 \leq \mu_1$.

(a) λ_2 is not overlined.

(i) If λ_1 is not overlined, then $\phi(\lambda, \mu) = ((\mu_1 + 1)(\mu_2), \lambda_2, (\lambda_1 - 1)')$.

(ii) If λ_1 is overlined, then $\phi(\lambda, \mu) = ((\overline{\mu_1 + 1})(\mu_2), \lambda_2, (\lambda_1 - 1)')$.

(b) λ_2 is overlined.

(i) If λ_1 is not overlined, then $\phi(\lambda, \mu) = ((\mu_1 + 1)(\mu_2), \lambda_1, (\lambda_2 - 1)')$.

(ii) If λ_1 overlined, then $\phi(\lambda, \mu) = ((\overline{\mu_1 + 1})(\mu_2), \lambda_1, (\lambda_2 - 1)')$.

To prove that ϕ is a bijection, one can show that it is one-to-one and onto. It is easier, however, to construct its inverse. We can define $\phi^{-1} : L \to R$ by the following cases, starting with the case of whether b is primed or not primed.

Case 1: b is not primed.

(a) $a \geq b$.

(i) If v_1 is not overlined, then $\phi^{-1}(v, a, b) = (v, (a)(b))$.

(ii) If v_1 is overlined, then $\phi^{-1}(v, a, b) = (v, (a)(b))$.

(b) $a < b$.

(i) If v_1 is not overlined, then $\phi^{-1}(v, a, b) = ((v_1)(v_2 + 1), (b - 1)(a))$.

(ii) If v_1 is overlined, then $\phi^{-1}(v, a, b) = ((v_1)(v_2 + 1), (b - 1)(a))$.

Case 2: b is primed.

(a) $a \geq b + 2$.

(i) If v_1 is not overlined, then $\phi^{-1}(v, a, b) = ((a)(b + 1), (v_1 - 1)(v_2))$.

(ii) If v_1 is overlined, then $\phi^{-1}(v, a, b) = ((\bar{a})(b + 1), (v_1 - 1)(v_2))$.

(b) $a < b + 2$.

(i) If v_1 is not overlined, then $\phi^{-1}(v, a, b) = ((b + 1)(a), (v_1 - 1)(v_2))$.

(ii) If v_1 is overlined, then $\phi^{-1}(v, a, b) = ((\overline{b + 1})(a), (v_1 - 1)(v_2))$.

The details of verifying that ϕ and ϕ^{-1} are inverses are not hard and are left to the reader. However, we will conclude the combinatorial proof with two examples of ϕ and ϕ^{-1} to help make the bijection clearer.

Example 9. Let $(\lambda, \mu) = (54, 22)$. First, we find i. We have $\lambda_1 > \mu_1$, so $i = 5+1 = 6$. For ϕ, we are in Case 1(a)(i), so $\phi(\lambda, \mu) = ((\lambda_1)(\lambda_2 - 1), \mu_2, \mu_1 + 1)$. Therefore, $\phi(54, 22) = (53, 2, 3)$. Note that $|\lambda| + |\mu| = 9+4 = 13$ and $|v| + a + b = 8+2+3 = 13$. Next, we act on (v, a, b) with the inverse. We are in Case 1(b)(i), so $\phi^{-1}(v, a, b) = ((v_1)(v_2 + 1), (b - 1)(a))$. So, $\phi^{-1}(53, 2, 3) = (54, 22)$.

Example 10. Let $(\lambda, \mu) = (\bar{2}1, 41)$. First, we find i. We have $\lambda_1 \leq \mu_1$, so $i = 4+2=6$. For ϕ, we are in Case 2(a)(ii), so $\phi(\lambda, \mu) = ((\overline{\mu_1+1})(\mu_2), \lambda_2, (\lambda_1-1)')$. Therefore, $\phi(\bar{2}1, 41) = (\bar{5}1, 1, 1')$. Note that $|\lambda|+|\mu| = 3+5 = 8$ and $|v|+a+b = 6+1+1 = 8$. Next, we act on (v, a, b) with the inverse. We are in Case 2(b)(ii), so $\phi^{-1}(v, a, b) = ((\overline{b+1})(a), (v_1-1)(v_2))$. So, $\phi^{-1}(\bar{5}1, 1, 1') = (\bar{2}1, 41)$.

5. Conclusion

Although the specific case of overpartitions whose Ferrers shape fits in a $2 \times (n-1)$ box is central to the proof presented here, extending this idea to the general case of a $k \times (n-k)$ box would be useful. This general work could lead to q-analogues of other expressions. In particular, investigating q-analogues for the sums of other integer powers is a natural extension of our work.

6. Acknowledgments

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