involve
a journal of mathematics

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msp

2017 vol. 10, no. 3

INVOLVE 10:3 (2017)
https://doi.org/10.2140/involve.2017.10.531

Ulrich partitions for two-step flag varieties

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(Communicated by Ravi Vakil)

Ulrich bundles play a central role in singularity theory, liaison theory and Boij–Söderberg theory. It was proved by the first author together with Costa, Huizenga, Miró-Roig and Woolf that Schur bundles on flag varieties of three or more steps are not Ulrich and conjectured a classification of Ulrich Schur bundles on two-step flag varieties. By the Borel–Weil–Bott theorem, the conjecture reduces to classifying integer sequences satisfying certain combinatorial properties. In this paper, we resolve the first instance of this conjecture and show that Schur bundles on $F(k, k+3; n)$ are not Ulrich if $n > 6$ or $k > 2$.

1. Introduction

Let $j, k, l > 0$ be positive integers. Let

$$P = (a_1, \dots, a_k \mid b_1, \dots, b_j \mid c_1, \dots, c_l)$$

be a strictly increasing sequence of integers divided into three nonempty subsequences $a_\bullet$, $b_\bullet$, $c_\bullet$. Let $P(t)$ denote the sequence

$$P(t) = (a_1 + t, \dots, a_k + t \mid b_1, \dots, b_j \mid c_1 - t, \dots, c_l - t)$$

obtained by adding t to each of the entries in the sequence $a_\bullet$ and subtracting t from each of the entries in the subsequence $c_\bullet$. Set $N = kj + kl + jl$.

Definition 1.1. The partition P is called an *Ulrich partition* if the sequences $P(t)$ have exactly two equal entries for $1 \le t \le N$.

Note that $P(t)$ can have repeated entries for at most N values of t. We will refer to $P(t)$ as the time evolution of P at time t. Hence, Ulrich partitions are those for which there are a maximum number of collisions among the entries during their time evolution and these collisions all occur at consecutive times.

MSC2010: primary 14M15; secondary 14J60, 13C14, 13D02, 14F05.
Keywords: flag varieties, Ulrich bundles, Schur bundles.
During the preparation of this article the first author was partially supported by the NSF CAREER grant DMS-0950951535 and NSF grant DMS-1500031; and the second author was partially supported by an NSF RTG grant.

Two partitions P_1 and P_2 are *equivalent* if they differ by adding a constant to all the entries. If P_1 and P_2 are equivalent, then P_1 is Ulrich if and only if P_2 is. We always consider partitions up to equivalence. Our main theorem is the following.

Theorem 1.2. *If $P = (a_1, \dots, a_k \mid b_1, b_2, b_3 \mid c_1, \dots, c_l)$ is an Ulrich partition, then $k + l \leq 3$.*

Given a partition $P = (a_1, \dots, a_k \mid b_1, \dots, b_j \mid c_1, \dots, c_l)$, we obtain a new partition P^s called the *symmetric partition* by multiplying all the entries by -1 and listing the entries in the reverse order:

$$P^s = (-c_l, \dots, -c_1 \mid -b_j, \dots, -b_1 \mid -a_k, \dots, -a_1).$$

The partition P is Ulrich if and only if P^s is Ulrich. Similarly, there is a dual partition P^* obtained by

$$P^* = \bigl(c_1-(N+1)t, \dots, c_l-(N+1)t \mid b_1, \dots, b_j \mid a_1+t(N+1), \dots, a_k+t(N+1)\bigr).$$

This is the partition $P(N+1)$ reordered so that the entries are increasing. By running the time evolution backwards, it is clear that P is Ulrich if and only if P^* is Ulrich (see [Coskun et al. 2017, §3] for more details). We can also form $(P^s)^*$, which is Ulrich if and only if P is.

As a consequence of the proof, we obtain a complete classification of Ulrich partitions where the $b_\bullet$ subsequence has length 3. Up to equivalence and these symmetries, they are

$$(0 \mid 1, 2, 3 \mid 8),\ (-8, 0 \mid 1, 2, 3 \mid 8),\ (0 \mid 1, 2, 5 \mid 8),\ (-1 \mid 1, 2, 6 \mid 7),\ (0 \mid 1, 3, 6 \mid 8).$$

We now explain the significance of Ulrich partitions. Let $X \subset \mathbb{P}^m$ be an arithmetically Cohen–Macaulay projective variety of dimension d. A vector bundle $\mathcal{E}$ on X is called an *Ulrich bundle* if $H^i(X, \mathcal{E}(-i)) = 0$ for $i > 0$ and $H^j(X, \mathcal{E}(-j-1)) = 0$ for $j < d$ (see [Herzog et al. 1991; Brennan et al. 1987; Eisenbud et al. 2003]). These are the bundles whose Hilbert polynomials have d zeros at the first d negative integers. They play a central role in singularity theory, liaison theory and Boij–Söderberg theory. For example, if X admits an Ulrich bundle, then the cone of cohomology tables of X coincides with that of $\mathbb{P}^m$ [Eisenbud and Schreyer 2011]. Thus, classifying Ulrich bundles on projective varieties is an important problem in commutative algebra and algebraic geometry, as discussed by E. Coskun et al. [2013], I. Coskun et al. [2017], and Faenzi [2008], who also give further references. In particular, it is interesting to decide when representation theoretic bundles on flag varieties are Ulrich.

Let $0 < k_1 < k_2 < n$ be three positive integers. Set $k_0 = 0$ and $k_3 = n$. Let V be an n-dimensional vector space. The two-step partial flag variety $F(k_1, k_2; n)$ parameterizes partial flags $W_1 \subset W_2 \subset V$, where W_i has dimension k_i. The variety

$F(k_1, k_2; n)$ has a minimal embedding in projective space corresponding to the ample line bundle with class the sum of the two Schubert divisors. We will always consider $F(k_1, k_2; n)$ in this embedding and $\mathcal{O}(1)$ will refer to the hyperplane bundle in this embedding.

The variety $F(k_1, k_2; n)$ has a collection of tautological bundles

$$0 = T_0 \subset T_1 \subset T_2 \subset T_3 = \underline{V} = V \otimes \mathcal{O}_{F(k_1,k_2;n)},$$

where $\underline{V}$ is the trivial bundle of rank n and T_i, for $i = 1$ or 2, is the subbundle of $\underline{V}$ of rank k_i which associates to a point $[W_1 \subset W_2]$ the subspace W_i. Let $U_i = T_i/T_{i-1}$. Given $\lambda = (\lambda_1 \mid \lambda_2 \mid \lambda_3)$ a concatenation of partitions λ_i of length $k_i - k_{i-1}$, the Schur bundle E_λ is defined by

$$E_\lambda = \mathbb{S}^{\lambda_1} U_1^* \otimes \mathbb{S}^{\lambda_2} U_2^* \otimes \mathbb{S}^{\lambda_3} U_3^*,$$

where $\mathbb{S}^\lambda$ is the Schur functor of type λ.

Costa and Miró-Roig [2015] initiated the study of determining when Schur bundles are Ulrich. They showed every Grassmannian admits Ulrich Schur bundles and classified these bundles. Coskun et al. [2017] showed that Schur bundles on flag varieties with three or more steps are never Ulrich for their minimal embedding. They also constructed several infinite families of Ulrich Schur bundles on specific two-step flag varieties and showed that many two-step flag varieties do not admit Ulrich Schur bundles. They conjectured a complete classification of Ulrich Schur bundles on two-step flag varieties.

Conjecture 1.3 [Coskun et al. 2017, Conjecture 5.9]. *A two-step flag variety $F(k_1, k_2; n)$ does not admit an Ulrich Schur bundle with respect to $\mathcal{O}(1)$ if $k_2 \geq 3$ and $n - k_2 \geq 3$.*

The Borel–Weil–Bott theorem computes the cohomology of Schur bundles and allows one to determine whether a Schur bundle is Ulrich. There is a bijective correspondence between equivalence classes of Ulrich partitions of type $(n - k_2, k_2 - k_1, k_1)$ and Schur bundles E_λ on $F(k_1, k_2; n)$ which are Ulrich [Coskun et al. 2017, Proposition 3.5]. Hence, classifying Ulrich Schur bundles is equivalent to classifying Ulrich partitions. Consequently, as a corollary of Theorem 1.2, we resolve the first case of Section 1.

Theorem 1.4. *The flag variety $F(k, k+3; n)$ does not admit an Ulrich Schur bundle with respect to $\mathcal{O}(1)$ if $n > 6$ or $k > 2$.*

In particular, the only two step flag varieties of the form $F(k, k+3; n)$ that admit Ulrich Schur bundles are $F(1, 4; 5)$, $F(1, 4; 6)$ and $F(2, 5; 6)$. All the Ulrich Schur bundles on these varieties have been classified in [Coskun et al. 2017]. There has been work on classifying Ulrich Schur bundles on other homogeneous varieties using the same strategy (see [Fonarev 2016]).

2. The proof of the main theorem

Theorem 2.1. *There are no Ulrich partitions* $(a_1, \dots, a_k \mid b_1, b_2, b_3 \mid c_1, \dots, c_l)$ *with* $k + l > 3$.

We begin with the following simple observation, which is a special case of [Coskun et al. 2017, Lemma 4.3].

Lemma 2.2. *If* $P = (a_1, \dots, a_l \mid b_1, \dots, b_j \mid c_1, \dots, c_k)$ *is an Ulrich partition, then all the entries in the sequences* $a_\bullet$ *and* $c_\bullet$ *are equal modulo* 2.

Proof. If P is Ulrich, the a_p and c_q entries of $P(t_{pq})$ must be equal at some time t_{pq}. From now on, we will express this by saying a_p and c_q collide at time $t = t_{pq}$. Hence $a_p + t_{pq} = c_q - t_{pq}$ or, equivalently, $c_q - a_p = 2t_{pq}$. Consequently, a_p and c_q are equal modulo 2. Since this holds for each $1 \le p \le l$ and $1 \le q \le k$, we conclude that all the entries in the sequences $a_\bullet$ and $c_\bullet$ have the same parity. Furthermore, their parities remain equal in $P(t)$ for all t. □

Let $P = (a_1, \dots, a_k \mid b_1, b_2, b_3 \mid c_1, \dots, c_l)$ be an Ulrich partition. Recall that we always assume $k, l > 0$. Up to symmetry and duality, there are three possibilities:

(1) The sequence b_1, b_2, b_3 may be consecutive.

(2) Only the entries b_1, b_2 may be consecutive.

(3) Finally, no two of the entries in $b_\bullet$ are consecutive.

We will analyze each of these cases separately.

The $b_\bullet$ sequence is consecutive. In this case, we will see that $k + l \le 3$ and up to symmetry and duality the two possible partitions are $(0 \mid 1, 2, 3 \mid 8)$ or $(-8, 0 \mid 1, 2, 3 \mid 8)$. In fact, we can analyze sequences where the $b_\bullet$ sequence is consecutive more generally.

Proposition 2.3. *Let* P *be an Ulrich partition of the form* $(a_1, \dots, a_k \mid 1, 2, \dots, r \mid c_1, \dots, c_l)$, *where the* $b_\bullet$ *sequence consists of* r *consecutive integers. Assume that* $r \ge 3$. *Then* $k + l \le 3$.

Proof. Without loss of generality, we may assume that at $t = 1$, the collision is $a_k b_1$. Then for $1 \le t \le r$, the collision is $a_k b_t$. We claim that at $t = r + 1$, the collision must be $a_k c_1$. The collision must be either $a_{k-1} b_1$ or $a_k c_1$. If r is odd, then it cannot be $a_{k-1} b_1$ since otherwise a_{k-1} and a_k would have different parities. If r is even and the collision is $a_{k-1} b_1$, we obtain a contradiction as follows. Let t_0 be the time of the collision $a_k c_1$. Until that time all the collisions must be between an entry from $a_\bullet$ and an entry from $b_\bullet$. We conclude that $t_0 = ir + 1$ for some i. At time $t = t_0 + 1$, the collision cannot be $a_k c_2$. Otherwise, we would have $c_2 - c_1 = 2$ and the collisions $c_1 b_1$ and $c_2 b_3$ would occur at the same time. If $i > 1$, the collision at $t = t_0 + 1$ cannot be $b_r c_1$. Hence, at $t = t_0 + 1$, the collision must be $a_{k-i} b_1$. This violates parity since a_k is even while a_{k-i} is odd. We conclude that at $t = r+1$, the collision is $a_k c_1$.

Hence, for $t = r+1+i$ with $1 \leq i \leq r$, the collisions are $b_{r+1-i}c_1$. If the progression stops at time $t = 2r+1$, we obtain the Ulrich partition $(0 \mid 1, 2, \ldots, r \mid 2r+2)$. Else, at time $t = 2r+2$, the collision must be $a_{k-1}c_1$. Otherwise, the collision would have to be $a_k c_2$. At time $t = 2r+3$, since the collision could not be $a_k c_3$, the collision would have to be $a_{k-1}c_1$. Then at time $t = 3r+3$, the values a_{k-1}, b_r and c_2 would collide simultaneously. This contradiction shows that the collision at $t = 2r+2$ must be $a_{k-1}c_1$. Hence, for times $t = 2r+2+i$ with $1 \leq i \leq r$, the collisions must be $a_{k-1}b_i$. If the progression stops at $t = 3r+2$, we obtain the Ulrich partition $(-2r-2, 0 \mid 1, 2, \ldots, r \mid 2r+2)$.

Otherwise, at time $t = 3r+3$, the collision must either be $a_k c_2$ or $a_{k-2}c_1$. Then at time $t = 3r+4$, the only possible collisions are $a_{k-2}c_1$ or $a_k c_2$, respectively, since the distance between consecutive entries in $a_\bullet$ or $c_\bullet$ has to be at least $r > 2$. If the order is $a_k c_2$ and $a_{k-2}c_1$, then at time $t = 3r+4$ the entry c_2 is $3r+2$ and a_{k-2} is $-r-2$. The entries a_{k-2}, b_r and c_2 collide simultaneously at time $t = 5r+5$. Hence, the order of collisions must be $a_{k-2}c_1$ at time $t = 3r+3$ and $a_k c_2$ at time $3r+4$. If $r \geq 5$, then at time $t = 3r+5$, there cannot be any collisions. If $3 \leq r \leq 4$, the only possible collision at time $t = 3r+5$ is $a_{k-3}c_1$. But then a_{k-3}, b_r and c_2 collide simultaneously at time $t = 5r+8$. This is a contradiction. Hence, the time evolution must stop at time $t = 3r+2$ and we conclude the proposition. □

In particular, we conclude that up to equivalence and symmetries, the only Ulrich partitions where the $b_\bullet$ sequence consists of three or more consecutive integers are $(0 \mid 1, 2, \ldots, r \mid 2r+2)$ and $(-2r-2, 0 \mid 1, 2, \ldots, r \mid 2r+2)$.

Exactly two of the $b_\bullet$ entries are consecutive. Up to symmetry and duality, we may assume that b_1 and b_2 are consecutive.

Lemma 2.4. *Assume that b_1 and b_2 are the only two consecutive entries in the $b_\bullet$ sequence and $P = (a_1, \ldots, a_k \mid b_1, b_2, b_3 \mid c_1, \ldots, c_l)$ is Ulrich. Then the $b_\bullet$ sequence up to equivalence and symmetry must be* 1, 2, 5 *or* 1, 2, 6. *In the first case, at time $t = 1$ the collision is $a_k b_1$. In the second case, at time $t = 1$ the collision is $b_3 c_1$.*

Proof. At time $t = 1$, the collision is either $a_k b_1$ or $b_3 c_1$. First, assume that at time $t = 1$ the collision is $b_3 c_1$. Since b_2 and b_3 are not consecutive, the collision at time $t = 2$ cannot be $c_1 b_2$. By parity, the collision cannot be $b_3 c_2$. Consequently, at time $t = 2$ the collision must be $a_k b_1$. Hence, at time $t = 3$, the collision is $a_k b_2$. If at time $t = 4$ the collision is $a_k c_1$, then the $b_\bullet$ sequence is 1, 2, 6. Otherwise, the only possible collision is $a_{k-1}b_1$ since $a_k b_3$ or $b_2 c_1$ cannot occur before $a_k c_1$ and $b_3 c_2$ is excluded by parity. Moreover, $\mid b_3 - b_2 \mid \geq 8$ and $a_k - a_{k-1} = 2$.

The last collision at time $t = N$ is either $a_1 b_3$ or $b_1 c_l$. If it is $b_1 c_l$, then the collisions at time $t = N-1$ and $t = N-2$ must be $b_2 c_l$ and $a_l b_3$, respectively. Note that at time $t = N - 2$, the collision cannot be b_1c_{l-1}. Otherwise, $c_l - c_{l-1} = 2$ and c_l would collide with a_k at the same time as c_{l-1} collides with a_{k-1}. Then at time $t = N-3$, the collision cannot be $a_{k-1}b_3$ or $c_{l-1}b_1$ by parity. Since $b_3-b_2 \geq 8$, the collision cannot be a_1c_l. We conclude that at $t = N - 3$ there are no possible collisions. This is a contradiction.

If the last collision is a_1b_3, then the two previous collisions must be b_1c_l and b_2c_l by parity. At time $t = N - 3$, the collision cannot be b_1c_{l-1} since $c_l - c_{l-1}$ cannot be 2. The collision cannot be a_2b_3 by parity. It cannot be a_1c_l since $b_3 - b_2 \geq 8$. We obtain a contradiction. We conclude that if at $t = 1$ the collision is b_3c_1, then at $t = 4$ the collision must be a_kc_1 and the $b_\bullet$ sequence is up to equivalence $1, 2, 6$.

Next assume that the collision at $t = 1$ is a_kb_1. Let $t = 2j + 1$ be the first odd time when the collision is not of the form a_ib_1. If $j = 1$, since the entries in $b_\bullet$ are not consecutive, at time $t = 3$ the collision must be b_3c_1. Then at time $t = 4$, by parity, the only possible collision is a_kc_1. Therefore, the $b_\bullet$ sequence is $1, 2, 5$. If $j > 1$, then $a_k - a_{k-1} = 2$. The collision at time $t = 2j + 1$ must be b_3c_1. Otherwise, the collision would have to be a_kb_3. Then at time $t = 2j+2$, by parity the collision would have to be a_kc_1. Then the collisions $a_{k-1}b_3$ and b_3c_1 would happen at the same time at $t = 2j + 3$. We conclude that at time $t = 2j + 1$ the collision is b_3c_1. At time $t = 2j + 2$, by parity we cannot have a collision of the form a_ib_1 or b_3c_{l-1}. We conclude that the collision must be a_kc_1. If $j > 1$, then at time $r = 2j + 2$ the collisions $a_{k-1}c_1$ and a_kb_3 occur at the same time leading to a contradiction. We conclude that $j = 1$ and the $b_\bullet$ sequence is $1, 2, 5$. $\square$

We thus obtain two standard Ulrich partitions of type $(1, 3, 1)$ given by $(0 \mid 1, 2, 5 \mid 8)$ and $(-1 \mid 1, 2, 6 \mid 7)$. To conclude the analysis in this case, we argue that these Ulrich partitions cannot be extended to longer Ulrich partitions.

Lemma 2.5. *The only Ulrich partition of the form*

$$(a_1, \dots, a_{k-1}, a_k = 0 \mid b_1 = 1, b_2 = 2, b_3 = 5 \mid c_1 = 8, c_2, \dots, c_l)$$

is $(0 \mid 1, 2, 5 \mid 8)$. *The only Ulrich partition of the form*

$$(a_1, \dots, a_{k-1}, a_k = -1 \mid b_1 = 1, b_2 = 2, b_3 = 6 \mid c_1 = 7, c_2, \dots, c_l)$$

is $(-1 \mid 1, 2, 6 \mid 7)$.

Proof. Suppose there exists an Ulrich partition of the form

$$(a_1, \dots, a_{k-1}, 0 \mid 1, 2, 5 \mid 8, c_2, \dots, c_l)$$

with k or l bigger than 1. Then the last collision at time $t = N$ must be either a_1b_3 or b_1c_l. If the collision is a_1b_3, then by parity the collision at time $t = N - 1$ must be b_1c_l. Then a_1 and c_l have different parities and can never collide. We obtain a contradiction. We conclude that at $t = N$ the collision must be b_1c_l. Hence, at time $t = N - 1$ the collision is b_2c_l. If the collision at $t = N - 2$ is a_1b_3, then the distance between a_1 and a_k (which is equal to $N-7$) is equal to the distance between c_1 and c_l. Hence, these pairs collide simultaneously leading to a contradiction. We conclude that at time $t = N - 2$, the collision must be b_1c_{l-1}. Hence the collisions at times $t = N - 3, N - 4$ must be b_2c_{l-1} and b_3c_l, respectively. However, at time $t = N - 5$ there are no possible collisions. The collision cannot be b_1c_{l-2} by parity. There are no collisions between c_{l-1}, c_l and any entries in the $b_\bullet$ sequence. On the other hand, if a_1 collides with c_l, then at time $t = N - 4$ the a_1b_3 collision coincides with the b_2c_{l-1} collision. This contradiction shows that $k = l = 1$.

Suppose there exists an Ulrich partition of the form

$$(a_1, \dots, a_{k-1}, -1 \mid 1, 2, 6 \mid 7, c_2, \dots, c_l)$$

with k or l bigger than 1. The argument is almost identical to the previous case. The last collision at time $t = N$ cannot be a_1b_3. Otherwise, at time $t = N - 1$ the collision would have to be b_1c_l and the distance between a_1 and a_k would be equal to the distance between c_1 and c_l. We conclude that the collision at time $t = N$ is b_1c_l. Hence, at time $t = N - 1$ the collision is b_2c_l. At time $t = N - 2$, the collision cannot be a_1b_3, otherwise at that time c_l would be at position 3 and would have different parity from a_1. We conclude that at time $t = N - 2$ the collision must be b_1c_{l-1}. This determines the collisions at $t = N - 3, N - 4$ which must be b_2c_{l-1} and b_3c_l. Then, as in the previous case, at time $t = N - 5$, there cannot be any collisions leading to a contradiction. This shows that $k = l = 1$. □

None of the $b_\bullet$ entries are consecutive. In this case, we have the following lemma.

Lemma 2.6. *Let $(a_1, \dots, a_k \mid b_1, b_2, b_3 \mid c_1, \dots, c_l)$ be an Ulrich partition with $k, l > 0$ and such that no entries in the $b_\bullet$ sequence are consecutive. Then up to equivalence and symmetry the $b_\bullet$ sequence is* 1, 3, 6.

Proof. Without loss of generality, we may assume that at $t = 1$ the collision is a_kb_1. By parity and the fact that $b_2 - b_1 > 1$, we conclude that at $t = 2$ the collision must be b_3c_1. Similarly, by parity and the fact that $b_3 - b_2 > 1$, at time $t = 3$ the collision is either a_kb_2 or $a_{k-1}b_1$. If the collision is a_kb_2, then the collision at $t = 4$ has to be a_kc_1. By parity, it cannot be $a_{k-1}b_1$. It cannot be b_3c_2, otherwise the collisions b_1c_1 and b_2c_2 would occur at the same time. We conclude that at time $t = 0$ the $b_\bullet$ sequence must be 1, 3, 6 and $a_k = 0$ and $c_1 = 8$.

If the collision at time $t = 3$ is $a_{k-1}b_1$, then by parity the collision at $t = 4$ may only be one of a_kb_2, b_2c_1 or b_3c_2. It cannot be b_2c_1, otherwise a_kb_3 and $a_{k-1}b_2$ would occur at the same time since both a_{k-1}, a_k and b_2, b_3 would be 2 apart. Similarly, it cannot be b_3c_2, otherwise a_kc_2 and $a_{k-1}c_1$ would occur at the same time. We conclude that at $t = 4$, the collision is a_kb_2. At time $t = 5$, the collision cannot be b_3c_2 by parity. Hence, it is either $a_{k-2}b_1$ or a_kc_1. It cannot be $a_k c_1$, otherwise at time $t = 6$ all three a_{k-1}, b_2 and c_1 collide. Hence, at $t = 5$ the collision is $a_{k-2}b_1$. In this case, we have $b_3 - b_2 \geq 5$. Now consider the last two collisions at $t = N$ and $N - 1$. They are either $a_1 b_3$ at $t = N$ and $b_1 c_l$ at $t = N - 1$, or $b_1 c_l$ at $t = N$ and $a_1 b_3$ at $t = N - 1$. Notice that it cannot be the latter. Otherwise, the distance between a_1 and a_k would be equal to the distance between c_1 and c_l and the pair would collide simultaneously. We conclude that the collisions at $t = N$ and $N - 1$ must be $a_1 b_3$ and $b_1 c_l$, respectively. Then at time $t = N - 3$, the collision cannot be $a_2 b_3$ by parity. It cannot be $a_1 b_2$ or $b_2 c_l$ because of the distances between the entries in the $b_\bullet$ sequence. Finally, it cannot be $b_1 c_{l-1}$ since otherwise the distance between c_l and c_{l-1} would be 2 and they would collide with the pair a_k and a_{k-1} simultaneously. We conclude that this case is not possible. This concludes the proof of the lemma. □

We thus get the standard Ulrich partition of type $(1, 3, 1)$ given by $(0 \mid 1, 3, 6 \mid 8)$. To conclude the analysis in this case, we argue that this Ulrich partition cannot be extended to longer Ulrich partitions.

Lemma 2.7. *The only Ulrich partition of the form*

$$(a_1, \dots, a_{k-1}, a_k = 0 \mid b_1 = 1, b_2 = 3, b_3 = 6 \mid c_1 = 8, c_2, \dots, c_l)$$

is $(0 \mid 1, 3, 6 \mid 8)$.

Proof. Suppose there were a longer Ulrich partition. Then the last two collisions at times $t = N$ and $t = N - 1$ must be $a_1 b_3$ and $b_1 c_l$, respectively. Otherwise, as in the previous cases, the distance between a_1 and a_k would equal the distance between c_1 and c_l. But then at time $t = N - 2$ there cannot be any collisions. The entries c_l and a_k do not collide with any entries in the $b_\bullet$ sequence or with each other by the distribution of the $b_\bullet$ sequence. The collision cannot be $b_1 c_{l-1}$ and it cannot be $a_{k-1} b_3$. Otherwise, the distance between a_k and a_{k-1} would be 2 and the collisions $a_k b_1$ and $a_{k-1} b_2$ would be at the same time. This contradiction concludes the proof. □

Proof of Theorem 1.2. Let $P = (a_1, \dots, a_k \mid b_1, b_2, b_3 \mid, c_1, \dots, c_l)$ be an Ulrich partition. If the $b_\bullet$ sequence is consecutive, then by Proposition 2.3, up to symmetry, duality and equivalence, $P = (-8, 0 \mid 1, 2, 3 \mid 8)$ or $(0 \mid 1, 2, 3 \mid 8)$. If only two entries in the $b_\bullet$ sequence are consecutive, then by Lemmas 2.4 and 2.5, $P = (0 \mid 1, 2, 5 \mid 8)$ or $P = (-1 \mid 1, 2, 6 \mid 7)$. Finally, if none of the entries in the $b_\bullet$ sequence are consecutive, then by Lemmas 2.6 and 2.7, $P = (0 \mid 1, 3, 6 \mid 8)$. In all cases we have $k + l \leq 3$. □

Acknowledgements

We would like to thank Jack Huizenga and Matthew Woolf for many discussions on Ulrich bundles on flag varieties. This paper grew out of Jaskowiak's senior thesis.

References

[Brennan et al. 1987] J. P. Brennan, J. Herzog, and B. Ulrich, "Maximally generated Cohen–Macaulay modules", *Math. Scand.* **61**:2 (1987), 181–203. MR Zbl

[Coskun et al. 2013] E. Coskun, R. S. Kulkarni, and Y. Mustopa, "The geometry of Ulrich bundles on del Pezzo surfaces", *J. Algebra* **375** (2013), 280–301. MR Zbl

[Coskun et al. 2017] I. Coskun, L. Costa, J. Huizenga, R. M. Miró-Roig, and M. Woolf, "Ulrich Schur bundles on flag varieties", *J. Algebra* **474**:1 (2017), 49–96.

[Costa and Miró-Roig 2015] L. Costa and R. M. Miró-Roig, "$\mathrm{GL}(V)$-invariant Ulrich bundles on Grassmannians", *Math. Ann.* **361**:1 (2015), 443–457. MR Zbl

[Eisenbud and Schreyer 2011] D. Eisenbud and F.-O. Schreyer, "Boij–Söderberg theory", pp. 35–48 in *Combinatorial aspects of commutative algebra and algebraic geometry*, edited by G. Fløystad et al., Abel Symp. **6**, Springer, Berlin, 2011. MR Zbl

[Eisenbud et al. 2003] D. Eisenbud, F.-O. Schreyer, and J. Weyman, "Resultants and Chow forms via exterior syzygies", *J. Amer. Math. Soc.* **16**:3 (2003), 537–579. MR Zbl

[Faenzi 2008] D. Faenzi, "Rank 2 arithmetically Cohen–Macaulay bundles on a nonsingular cubic surface", *J. Algebra* **319**:1 (2008), 143–186. MR Zbl

[Fonarev 2016] A. Fonarev, "Irreducible Ulrich bundles on isotropic Grassmannians", preprint, 2016. arXiv

[Herzog et al. 1991] J. Herzog, B. Ulrich, and J. Backelin, "Linear maximal Cohen–Macaulay modules over strict complete intersections", *J. Pure Appl. Algebra* **71**:2–3 (1991), 187–202. MR Zbl

Received: 2016-05-12 Accepted: 2016-06-15

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2017 vol. 10 no. 3