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Leverage centrality of knight's graphs and Cartesian products of regular graphs and path powers

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In 2010, Joyce et al. defined the leverage centrality of a graph as a means to analyze connections within the brain. In this paper we investigate this property from a mathematical perspective and determine the leverage centrality for knight's graphs, path powers, and Cartesian products.

1. Introduction

We recall that the degree of a vertex v is the number of edges incident to v and is denoted $\deg v$. Joyce, Laurienti, Burdette, and Hayasaka [Joyce et al. 2010] defined the property of leverage centrality based on vertex degrees.

Definition 1. Leverage centrality is a measure of the relationship between the degree of a given node v and the degree of each of its neighbors v_i, averaged over all neighbors of v, denoted N_v, and is defined as

$$l(v) = \frac{1}{\deg v} \sum_{v_i \in N_v} \frac{\deg v - \deg v_i}{\deg v + \deg v_i}.$$

This property was used by Joyce et al. [2010] in the analysis of functional magnetic resonance imaging (fMRI) data and has been used to analyze real-world networks including airline connections, electrical power grids, and coauthorship collaborations [Li et al. 2015]. The leverage centralities of complete multipartite graphs and the Cartesian product of paths were investigated by Sharma, Vargas, Waldron, Flórez, and Narayan [Sharma et al. 2017]. Bounds on leverage centrality were determined by Li, Li, Van Mieghem, Stanley, and Wang [Li et al. 2015]. We restate one of their results as our first theorem.

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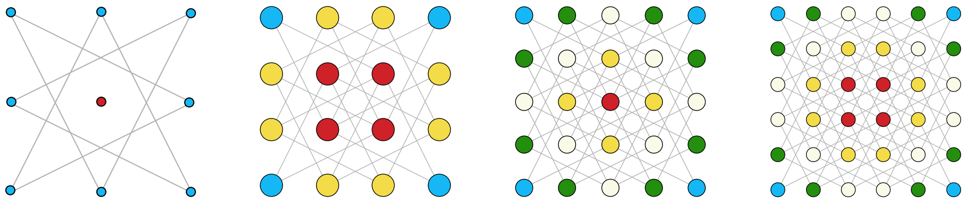

Figure 1. The 3×3, 4×4, 5×5 and 6×6 knight's graphs.

Theorem 2. *For any vertex v, we have $|l(v)| \le 1 - \frac{2}{n}$. Furthermore, these bounds are tight in the cases of stars and complete graphs.*

We note that the bounds are also tight for regular graphs with degree $r > 1$.

In this paper we investigate leverage centrality for various families of graphs including the knight's graphs, path powers, and the Cartesian products of graphs.

2. Leverage centrality of a knight's graph

We define an $n \times n$ *knight's graph* to be the graph with n^2 vertices in which every vertex represents a square in an $n \times n$ chessboard. The vertices on the $n \times n$ chessboard can be placed in an $n \times n$ table where two vertices v_i and v_j are adjacent if they are exactly four entries apart (including the entries of v_i and v_j) and they form an "L" shape. We give examples of knight's graphs of small order in Figure 1, where in each graph all of the vertices of same degree are the same color.

We next state the leverage centrality of each vertex in the $n \times n$ knight's graph. We use t_j to denote the j-th triangular number $\frac{1}{2}j(j+1)$.

Theorem 3. *Let G_n be the $n \times n$* knight's graph.

(1) *The leverage centrality of every vertex of G_3 is zero.*

(2) *If $n = 4$, 6, or 8, then G_n has exactly $t_{n/2}$ distinct leverage centralities.*

(3) *If $n = 5$ or 7, then G_n has exactly $t_{(n+1)/2} - 1$ distinct leverage centralities.*

(4) *If $n \ge 9$, then G_n has exactly* 15 *distinct leverage centralities.*

Proof. We first find the degree of each vertex in the knight's graph on an $n \times n$ chessboard, where $n \ge 3$. To describe the degree of each vertex in the graph G_n, we will arrange the vertices of G_n in an $n\times n$ table. The vertices corresponding to entries $(1, 1)$, $(1, n)$, $(n, 1)$, and (n, n) have degree 2. Those corresponding to entries $(1, 2)$, $(1, n-1)$, $(2, 1)$, $(2, n)$, $(n-1, 1)$, $(n, 2)$, $(n-1, n)$, and $(n, n-1)$ have degree 3. Those corresponding to entries $(2, 2)$, $(2, n-1)$, $(n-1, 2)$, $(n-1, n-1)$ and $(1, i)$, $(i, 1)$, (n, i), and (i, n), where $i = 3, 4, \ldots, n-2$, have degree 4. Those corresponding to entries $(2, i)$, $(i, 2)$, $(n-1, i)$, and $(i, n-1)$, where $i = 3, 4, \ldots, n-2$, have degree 6. Vertices corresponding to entries (i, j), where $i = 3, 4, \ldots, n-3$ and $j = 3, 4, \ldots, n-2$, have degree 8; see, for example, Figure 2 (left).

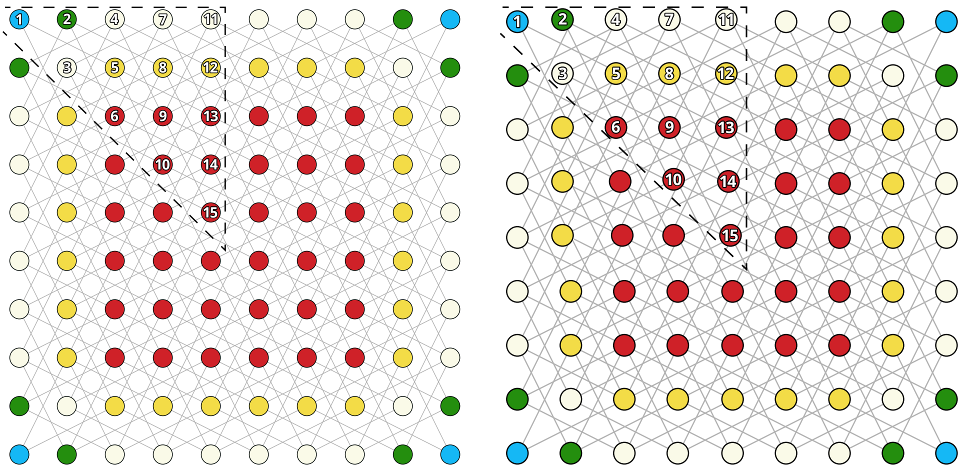

Figure 2. The 10×10 knight's graph (left) and the 9×9 knight's graph (right).

If n is even we subdivide the knight's graph's vertical and horizontal axes and the two diagonals to obtain eight regions. Each region forms a right triangle where the legs have $\frac{1}{2}n$ vertices; see, for example, Figure 2 (left). Using symmetry we can calculate the leverage centrality of all vertices by only analyzing a single triangle.

If n is odd, as in Figure 2 (right), we do the same subdivision; however, in this case two adjacent triangles will overlap — the legs of the right triangle will have $\frac{1}{2}(n+1)$ vertices.

We choose the triangle with vertices

$$\begin{aligned} v_1 &= (1,1), & v_2 &= (1,2), & v_3 &= (2,2), & v_4 &= (1,3), & v_5 &= (2,3), \\ v_6 &= (3,3), & v_7 &= (1,4), & v_8 &= (2,4), & v_9 &= (3,4), & v_{10} &= (4,4), \\ v_{11} &= (1,5), & v_{12} &= (2,5), & v_{13} &= (3,5), & v_{14} &= (4,5), & v_{15} &= (5,5). \end{aligned} \tag{2-1}$$

Note that if $n < 10$, we take triangles with vertices v_i for $i = 1, 2, \ldots, k$, where $k = \frac{1}{2}n$ if $n = 2(k)$ or $k = \frac{1}{2}(n+1)$ if $n = 2(k) - 1$.

Proof of (1). Since G_3 is regular, the leverage centrality of all of its vertices is 0.

Proof of (2). For case $n = 4$, it is easy to see that $l(v_1) = -\frac{1}{3}$, $l(v_2) = -\frac{1}{21}$, and $l(v_3) = \frac{5}{21}$.

Now consider the cases $n = 6$ and $n = 8$. From the above analysis we only need to calculate the leverage centrality for a triangle with legs that have $\frac{1}{2}n$ vertices (see Figure 2 (left) for an example of those triangles). Thus, to calculate the leverage centrality of these special cases, we consider the triangle with vertices $v_1, \ldots, v_{t_i}$, where t_i is the i-th triangular number where $i = 1, 2, \ldots, \frac{1}{2}n$, and then use Tables 1 and 3, respectively.

vertex v_i	deg v_i	AD(v_i)	$l(v_i)$
v_1	2	6, 6	$-1/2$
v_2	3	4, 6, 8	$-215/693$
v_3	4	4, 4, 8, 8	$-1/6$
v_4	4	3, 6, 4, 8	$-41/420$
v_5	6	2, 3, 4, 6, 6, 8	$187/1260$
v_6	8	3, 3, 4, 4, 4, 4, 6, 6	$73/231$

Table 1. Leverage centrality when $n = 6$. Here AD(v_i) denotes the degrees of vertices adjacent to v_i.

vertex v_i	deg v_i	AD(v_i)	$l(v_i)$
v_1	2	6, 6	$-1/2$
v_2	3	4, 6, 8	$-215/693$
v_3	4	4, 4, 8, 8	$-1/6$
v_4	4	3, 6, 6, 8	$-31/210$
v_5	6	2, 4, 4, 6, 8, 8	$43/420$
v_6	8	3, 3, 4, 4, 6, 6, 8, 8	$215/924$
v_7	4	4, 4, 8, 8	$-1/6$
v_8	6	3, 3, 6, 6, 8, 8	$4/63$
v_9	8	4, 4, 4, 4, 6, 6, 8, 8	$17/84$
v_{10}	8	6, 6, 6, 6, 6, 6, 6, 6	$1/7$

Table 2. Leverage centrality when $n = 7$. Here AD(v_i) denotes the degrees of vertices adjacent to v_i.

Proof of (3). First consider the case $n = 5$. From the above analysis we only need to calculate the leverage centrality for a triangle with legs that have three vertices. It is easy to see that $l(v_1) = -\frac{1}{2}$, $l(v_2) = -\frac{19}{77}$, $l(v_3) = -\frac{1}{35}$, $l(v_4) = -\frac{1}{35}$, $l(v_5) = \frac{3}{10}$, and $l(v_6) = \frac{5}{11}$. This shows that there are only five distinct leverage centralities in G_5.

Now consider the case $n = 7$. From the above analysis we only need to calculate the leverage centrality for a triangle with legs that have four vertices. From Table 2 we can see that G_7 has only nine distinct leverage centralities.

Proof of (4). For $n = 8$ and $n = 10$, the proof is similar to those of parts (2) and (3).

We now suppose $n > 10$. Consider the 15 vertices in the triangle given in (2-1) and their relevant data, given in Table 3.

The analysis for the remaining vertices in the triangle is as follows. From the definition of the knight's graph we know that if two vertices v_i and v_j are adjacent, then they are four entries apart (including the entries of v_i and v_j) and they form an "L" shape. This implies that if $n \geq 11$ then the leverage centrality of every vertex located

n	vertex v_i	$\deg v_i$	$\mathrm{AD}(v_i)$	$l(v_i)$
8, 9, 10	v_1	2	6, 6	−1/2
8, 9, 10	v_2	3	4, 6, 8	−215/693
8, 9, 10	v_3	4	4, 4, 8, 8	−1/6
8, 9, 10	v_4	4	3, 6, 6, 8	−31/210
8, 9, 10	v_5	6	2, 4, 4, 6, 8, 8	43/420
8, 9, 10	v_6	8	3, 3, 4, 4, 6, 6, 8, 8	215/924
8, 9, 10	v_7	4	4, 6, 8, 8	−13/60
8, 9, 10	v_8	6	3, 4, 6, 8, 8, 8	11/630
8, 9, 10	v_9	8	4, 4, 4, 6, 6, 8, 8, 8	9/56
8, 9, 10	v_{10}	8	6, 6, 6, 6, 8, 8, 8, 8	1/14
9, 10	v_{11}	4	6, 6, 8, 8	−4/45
9, 10	v_{12}	6	4, 4, 8, 8, 8, 8	−1/35
9, 10	v_{13}	8	4, 4, 6, 6, 8, 8, 8, 8	5/42
9, 10	v_{14}	8	6, 6, 8, 8, 8, 8, 8, 8	1/28
9, 10	v_{15}	8	8, 8, 8, 8, 8, 8, 8, 8	0

Table 3. Leverage centrality with $n = 8, 9, 10$. Here $\mathrm{AD}(v_i)$ denotes the degrees of vertices adjacent to v_i.

in entries (r, t) is zero for $r = 5, 6, \dots, k$ and $t = 5, 6, \dots, k$, where $k = \frac{1}{2}n$ if $n = 2k$ or $k = \frac{1}{2}(n+1)$ if $n = 2k-1$. Moreover, every vertex located in position (i, j) will have the same leverage centrality as the vertices located in entries $(i, 6)$, where $i = 1, 2, \dots, k$ and $j = 7, \dots, k$, where $k = \frac{1}{2}n$ if $n = 2k$ or $k = \frac{1}{2}(n+1)$ if $n = 2k-1$. □

3. Leverage centralities of P_n^k

Let P_n^k be the graph with vertices $v_1, v_2, \dots, v_n$ and edges (v_i, v_j) when $1 \le |i-j| \le k \le n-1$. In this paper we will assume $n > 1$. This family contains both paths (when $k = 1$) and complete graphs (when $k = n-1$). Note that $\deg v_i = \min\{i+k-1, 2k\}$. The neighbors of v_i are $v_{i-1}, v_{i-2}, \dots, v_{i-s}$ and $v_{i+1}, v_{i+2}, \dots, v_{i+t}$, where $s = \min\{k, i-1\}$ and $t = \min\{k, n-i\}$. The above conditions can be combined in the next lemma to give the leverage centrality of any vertex in P_n^k.

Lemma 4. *Suppose the vertex $v_i \in V(P_n^k)$ has neighbors $v_{i-1}, v_{i-2}, \dots, v_{i-s}$ and $v_{i+1}, v_{i+2}, \dots, v_{i+t}$, where $s = \min\{k, i-1\}$ and $t = \min\{k, n-i\}$. Then*

$$l(v_i) = \frac{1}{\delta_i} \sum_{i-s \le j \le i+t} \frac{\delta_i - \delta_j}{\delta_i + \delta_j},$$

where $\delta_x = \min\{x+k-1, 2k\}$ for $x = i, j$.

We begin by determining the leverage centrality of vertices in a path P_n, where $n \ge 2$. We note that by symmetry $l(v_i) = l(v_{n+1-i})$ for all $1 \le i \le n$. We start with small values of n. When $n = 2$, both vertices have a leverage centrality of zero. When $n = 3$, the two vertices of degree 1 have leverage centrality $\frac{1-2}{1+2} = -\frac{1}{3}$ and the vertex of degree 2 has leverage centrality $\frac{2-1}{1+2} = \frac{1}{3}$. When $n = 4$, the two vertices of degree 1 have leverage centrality $\frac{1-2}{1+2} = -\frac{1}{3}$ and the two vertices of degree 2 have leverage centrality $\frac{1}{2}\left(\frac{2-1}{1+2} + \frac{2-2}{2+2}\right) = \frac{1}{6}$.

Next, we use the operation of edge subdivision to handle cases where $n \geq 5$. Recall that in an edge subdivision an edge $u - v$ is replaced by a path on three vertices $u - w - v$. We note that if we extend the length of a path by subdividing the edge between vertices c and d, the new vertices will have a leverage centrality of zero. Further subdivision of an edge connecting two vertices with degree 2 will include a new vertex with leverage centrality zero. Hence, there will be exactly three distinct leverage centralities in any path with five or more vertices. The general result follows.

Theorem 5. *Let P_n be a path where $n \geq 5$. Then $l(v_1) = l(v_n) = -\frac{1}{3}$, $l(v_2) = l(v_{n-1}) = \frac{1}{6}$, and for all $3 \leq i \leq n-2$, we have $l(v_i) = 0$.*

3.1. *Leverage centralities of P_n^2.* We now calculate the leverage centralities for paths P_n^2. Again by symmetry, we have $l(v_i) = l(v_{n+1-i})$ for all $1 \leq i \leq n$.

- $n = 3$: For $1 \leq i \leq 3$, we have $l(v_i) = 0$.
- $n = 4$: $l(v_1) = -\frac{4}{15}$ and $l(v_2) = \frac{2}{15}$.
- $n = 5$: $l(v_1) = -\frac{4}{15}$, $l(v_2) = \frac{2}{105}$ and $l(v_3) = \frac{5}{21}$.
- $n = 6$: $l(v_1) = -\frac{4}{15}$, $l(v_2) = -\frac{1}{35}$ and $l(v_3) = \frac{13}{84}$.
- $n = 7$: $l(v_1) = -\frac{4}{15}$, $l(v_2) = -\frac{1}{35}$ $l(v_3) = \frac{5}{42}$ and $l(v_4) = \frac{3}{28}$.
- $n = 8$: $l(v_1) = -\frac{4}{15}$, $l(v_2) = -\frac{1}{35}$, $l(v_3) = \frac{5}{42}$ and $l(v_4) = \frac{1}{28}$.
- $n \geq 9$: $l(v_1) = -\frac{4}{15}$, $l(v_2) = -\frac{1}{35}$, $l(v_3) = \frac{5}{42}$ $l(v_4) = \frac{1}{28}$ and for all $5 \leq i \leq n-4$, $l(v_i) = 0$.

It is clear that to calculate the leverage centralities of all vertices in P_n^k for all k in this manner would require lengthy computation. However by noticing that the leverage centralities become fixed when n becomes large enough ($n \geq 4k+1$), we can compute the leverage centralities in a more formal manner.

First we give an elementary result with the leverage centralities for the first vertex in any path power.

Proposition 6. *If $v_1 \in V(P_n^k)$, then*

$$l(v_1) = \sum_{i=1}^{k} \frac{-i}{2k+i}.$$

Proof. The vertex v_1 has k neighbors, with degrees $k+1, k+2, \ldots, 2k$. Then

$$l(v_1) = \frac{1}{k}\left(k \sum_{i=1}^{k} \frac{-i}{2k+i}\right) = \sum_{i=1}^{k} \frac{-i}{2k+i}. \qquad \square$$

We continue with three lemmas which will help us determine the relationships between the leverage centralities of different vertices in P_n^k.

Lemma 7. *If i is an integer and $1 < \frac{1}{2}a \le i < a$ then we have*

$$\frac{1}{a}\left(\frac{a-i}{a+i}\right) > \frac{1}{a-1}\left(\frac{a-1-i}{a-1+i}\right).$$

Proof. Let $\frac{1}{2}a \le i < a$. This implies

$$\begin{aligned}
2ia - a^2 + (a-i) > 0 &\Rightarrow -a^2 + a + 2ia - 1 - i > 0 \\
&\Rightarrow a^3 - 2a^2 + (2+i)a - 1 - i > a^3 - a^2 + (1-i)a \\
&\Rightarrow (a-i)(a-1)(a-1+i) > a(a+i)(a-1-i) \\
&\Rightarrow \frac{1}{a}\left(\frac{a-i}{a+i}\right) > \frac{1}{a-1}\left(\frac{a-1-i}{a-1+i}\right). \qquad \square
\end{aligned}$$

Lemma 8. *For all $1 \le a \le 2k$, we have*

$$\frac{1}{a}\left(\frac{-1}{2a+1}\right) > \frac{-1}{(a-1)a}.$$

Proof. We first note that when $a = 2k$,

$$\frac{1}{a}\left(\frac{a-(a+1)}{a+a+1}\right) > \frac{1}{a-1}\left(\frac{a-1-(a+1)}{a-1+a+1}\right)$$

is clear since the left side is positive and the right side is negative.

Let $1 \le a$. Then

$$\begin{aligned}
2a^2 + a > a^2 + 1 &\Rightarrow \frac{-1}{2a^2+a} > \frac{-1}{a^2+1} \\
&\Rightarrow \frac{1}{a}\left(\frac{-1}{2a+1}\right) > \frac{1}{a+1}\left(\frac{-2}{2a}\right) \\
&\Rightarrow \frac{1}{a}\left(\frac{a-(a+1)}{a+a+1}\right) > \frac{1}{a-1}\left(\frac{a-1-(a+1)}{a-1+a+1}\right) \\
&\Rightarrow \frac{1}{a}\left(\frac{-1}{a+a+1}\right) > \frac{1}{a-1}\left(\frac{-2}{a-1+a+1}\right). \qquad \square
\end{aligned}$$

Lemma 9. *Let $2 \le i \le k-1$. Then*

$$\frac{1}{k+1}\left(\frac{1-i}{2k+1+i}\right) > \frac{1}{k}\left(\frac{-i}{2k+i}\right).$$

Proof. Note that

$$0 > \frac{1-i}{2k+i+1} > \frac{-i}{2k+i+1} > \frac{-i}{2k+i}.$$

Hence

$$0 > \frac{k+1-(k+i)}{k+1+k+i} > \frac{k-(k+i)}{k+k+i}.$$

Since $\frac{1}{k+1} < \frac{1}{k}$, we have

$$0 > \frac{1}{k+1}\left(\frac{1-i}{k+1+k+i}\right) > \frac{1}{k}\left(\frac{-i}{k+k+i}\right). \qquad \square$$

Proposition 6 and Lemmas 7, 8, and 9 can be combined as follows.

Proposition 10. *Let $G = P_n^k$, where $n \geq 4k+1$. Then:*

(i) $l(v_i) = l(v_{n+1-i})$.

(ii) *For all* $0 \leq j \leq k-1$,

$$l(v_{k+j+1}) = \frac{1}{2k}\left(\sum_{i=k+j}^{2k-1} \frac{2k-i}{2k+i}\right).$$

(iii) *For all* $0 \leq j \leq k-1$,

$$l(v_{k-j}) = \frac{1}{2k-j-1}\sum_{i=k}^{2k-1} \frac{2k-j-i}{2k-j+i} + \frac{k-j}{2k-1-j}\left(\frac{2k-j-1-2k}{2k-j-1+2k}\right).$$

(iv) *For all* $2k+1 \leq j \leq n-2k$, *we have* $l(v_j) = 0$.

This leads to the following theorem.

Theorem 11. *Let $G = P_n^k$, where $n \geq 4k+1$. Then the vertex with the largest leverage centrality in G is v_{k+1}, and furthermore $l(v_{k+1}) > l(v_k) > \cdots > l(v_1)$ and $l(v_{k+1}) > l(v_{k+2}) > \cdots > l(v_{2k+1})$.*

Proof. For the first part, we recall that

$$l(v_{k+1}) = \frac{1}{2k}\sum_{i=k}^{2k-1}\left(\frac{2k-i}{2k+i}\right) + \frac{k}{2k}\left(\frac{2k-2k}{2k+2k}\right),$$

and for $0 \leq j \leq k-1$,

$$l(v_{k-j}) = \frac{1}{2k-j-1}\sum_{i=k}^{2k-1} \frac{2k-j-i}{2k-j+i} + \frac{k-j}{2k-1-j}\left(\frac{2k-j-1-2k}{2k-j-1+2k}\right).$$

We seek to show that $l(v_{k+1}) > l(v_k) > \cdots > l(v_1)$. When comparing terms from $l(v_r)$ with $l(v_{r-1})$ for a fixed i, five cases are needed to show that the i-th term of $l(v_r)$ is larger than the i-th term of $l(v_{r-1})$.

Case (i): $2k - j - i > 1$. Use Lemma 7.

Case (ii): $2k - j - i = 1$. In the i-th term, the numerator is positive for the j-th term and the numerator is zero for $(j+1)$-th term.

Case (iii): $2k - j - i = 0$. In the i-th term, the numerator is zero for the j-th term and the numerator is negative for $(j+1)$-th term.

Case (iv): $2k - j - i < 0$. Use Lemma 8.

Case (v):

$$\frac{k-j}{2k-1-j}\left(\frac{2k-j-1-2k}{2k-j-1+2k}\right) > \frac{k-(j+1)}{2k-1-(j+1)}\left(\frac{2k-(j+1)-1-2k}{2k-(j+1)-1+2k}\right).$$

Use Lemma 8.

The combination of these five cases yields $l(v_{k+1}) > l(v_k) > \cdots > l(v_1)$. For the second part we note that for $0 \le r \le k-1$, we have $l(v_{k+r}) > l(v_{k+r+1})$ as terms with positive value are replaced by zeros in each successive case. Hence, $l(v_{k+1}) > l(v_{k+2}) > \cdots > l(v_{2k+1})$. We note that we have not obtained a linear ordering, but two separate linear orderings both starting with the largest leverage centrality $l(v_{k+1})$. □

4. Cartesian product of graphs

In this next section we give some general results about the leverage centrality of the Cartesian product of graphs. These build upon results by Sharma et al. [2017].

Definition 12. Given a graph F with vertex set $V(F)$ and edge set $E(F)$, and a graph H with vertex set $V(H)$ and edge set $E(H)$, we let G define the Cartesian product of F and H to be the graph $G = F \times H$, which is defined as

$$\begin{aligned} V(G) &= \{(u, v) \mid u \in V(F) \text{ and } v \in V(H)\}, \\ E(G) &= \Big\{(u_1, v_1), (u_2, v_2) \;\Big|\; u_1 = u_2 \text{ and } (v_1, v_2) \in E(H), \\ &\qquad\qquad\qquad\qquad \text{or } v_1 = v_2 \text{ and } (u_1, u_2) \in E(F)\Big\}. \end{aligned}$$

We next present an elementary result from graph theory.

Lemma 13. *If $G = F \times H$, then the degree of a vertex (u, v) in G is the sum of the degrees of vertices u and v, where $u \in V(F)$ and $v \in V(H)$.*

Theorem 14. *Let G be a graph and let RG_r be a regular graph where each vertex has degree r. Let $u \in V(RG_r)$ and let v_i and v_j be vertices in G with degrees k_i and k_j respectively. For each vertex $(u, v_i) \in V(RG_r \times G)$ we have*

$$l(u, v_i) = \frac{1}{r + k_i} \sum_{j \neq i} \frac{k_i - k_j}{2r + k_i + k_j}.$$

Proof. Consider a vertex $(u, v_i) \in V(RG_r \times G)$. We note that

$$\deg(u, v_i) = \deg u + \deg v_i = r + k_i.$$

Then

$$l(u, v_i) = \frac{1}{r + k_i} \sum_{j \neq i} \frac{k_i - k_j}{2r + k_i + k_j}. \qquad \square$$

We conclude by posing the following problem where the graphs may not be regular.

Problem 15. Given graphs F and H where the leverage centralities are known for all vertices in F and H, determine the leverage centralities for all vertices in $F \times H$.

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References

[Joyce et al. 2010] K. E. Joyce, P. J. Laurienti, J. H. Burdette, and S. Hayasaka, "A new measure of centrality for brain networks", *PLOS ONE* **5**:8 (2010), art. id. e12200.

[Li et al. 2015] C. Li, Q. Li, P. Van Mieghem, H. E. Stanley, and H. Wang, "Correlation between centrality metrics and their application to the opinion model", *Eur. Phys. J. B* **88**:3 (2015), art. id. 65. MR

[Sharma et al. 2017] A. Sharma, R. Vargas, A. Waldron, R. Florez, and D. A. Narayan, "Leverage centrality of graphs", preprint, 2017. arXiv

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