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On the tree cover number of a graph

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Given a graph G, the tree cover number of the graph, denoted $T(G)$, is the minimum number of vertex disjoint simple trees occurring as induced subgraphs that cover all the vertices of G. This graph parameter was introduced in 2011 as a tool for studying the maximum positive semidefinite nullity of a graph, and little is known about it. It is conjectured that the tree cover number of a graph is at most the maximum positive semidefinite nullity of the graph.

In this paper, we establish bounds on the tree cover number of a graph, characterize when an edge is required to be in some tree of a minimum tree cover, and show that the tree cover number of the d-dimensional hypercube is 2 for all $d \geq 2$.

1. Introduction

A *simple graph* is a pair $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$ is the vertex set, and E, the edge set, is a set of 2-element subsets (edges) of the vertices. A *multigraph* is a pair $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, and E is a multiset of 2-element subsets of the vertices. That is, a multigraph allows multiple edges between a pair of vertices (note that all simple graphs are multigraphs). Two vertices $u, v \in V(G)$ are said to be *adjacent* if $\{u, v\} \in E(G)$. We say that the edge $\{u, v\} \in E(G)$ is a *simple edge* if $\{u, v\}$ appears in $E(G)$ exactly once. If $\{u, v\}$ appears in $E(G)$ more than once, then it is a *multiedge*. All graphs in this paper are considered to be multigraphs unless otherwise stated.

For a multigraph G, $\mathcal{S}(G)$ denotes *the set of real valued symmetric $n \times n$ matrices* $(a_{i,j})$ satisfying:

(1) $a_{i,j} = 0$ if $i \neq j$ and i, j are nonadjacent,

(2) $a_{i,j} \neq 0$ if $i \neq j$ and i, j are adjacent via one edge, and

(3) $a_{i,j} \in \mathbb{R}$ if $i = j$ or i, j are adjacent via multiple edges.

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The *maximum nullity* of a multigraph G is defined to be

$$M(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G)\}.$$

The maximum nullity of a simple graph G is equivalent to the maximum multiplicity of an eigenvalue among all matrices in $S(G)$. This graph parameter has connections to many other concepts in linear algebra (as can be seen in [Fallat and Hogben 2007; 2014]), and has been given a significant amount of consideration as it is very difficult to compute.

A related and equally important parameter is the maximum positive semidefinite nullity of a graph. A symmetric $n \times n$ real matrix A is said to be *positive semidefinite* if $x^T Ax \geq 0$ for all $x \in R^n$. The *maximum positive semidefinite nullity* of a multigraph G is defined to be

$$M_+(G) = \max\{\text{null}(A) : A \in \mathcal{S}_+(G)\},$$

where $\mathcal{S}_+(G) = \{A \in \mathcal{S}(G) : \text{A is positive semidefinite}\}$. It follows that for a multigraph G, $M_+(G) \leq M(G)$. In some cases, one can use tools such as orthogonal representations (see [Fallat and Hogben 2014]) to compute $M_+(G)$, obtaining a lower bound for $M(G)$.

The tree cover number of a graph was introduced in [Barioli et al. 2011] as another tool for studying the maximum positive semidefinite nullity of a multigraph.

The *(simple) path* on n vertices, denoted P_n, is the graph with vertex set $V(P_n) = \{1, \ldots, n\}$ and edge set $E(P_n) = \{\{i, i+1\} \mid i \in 1, \ldots, n-1\}$. A simple graph $G = (V, E)$ is said to be a *tree* if for every $u, v \in V(G)$, there is exactly one path from u to v.

Given a graph $G = (V, E)$, a *subgraph* $G' = (V', E')$ is a graph such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$, i.e., a subgraph of a graph G can be obtained by deleting edges and vertices (and edges incident to the deleted vertices) of G. A subgraph $G' = (V', E')$ of G is said to be an *induced* subgraph of G if for each edge $uv \in E(G)$ with $u, v \in V(G')$, it follows that $uv \in E(G')$, i.e., an induced subgraph of G can be obtained by only deleting vertices (and any edges incident to the deleted vertices). For a subset $S \subseteq V(G)$, the *graph induced by* S, denoted $G[S]$, is the induced subgraph of G with vertex set S.

A *tree cover* is a set of vertex disjoint simple trees occurring as induced subgraphs that cover all the vertices of the graph. The *tree cover number* of a graph G, denoted $T(G)$, is defined as

$$T(G) = \min\{|\mathcal{T}| : \mathcal{T} \text{ is a tree cover of } G\}.$$

Conjecture 1 [Barioli et al. 2011]. $T(G) \leq M_+(G)$.

This bound has been proven to be true for several families of graphs, including outerplanar graphs and chordal graphs [Barioli et al. 2011]. In fact, in the previous work, the authors showed that equality holds for outerplanar graphs (and in fact for all graphs of tree-width at most 2, as observed in [Ekstrand et al. 2012]).

In Section 2 we give bounds on the tree cover number, provide an example in which the tree cover number behaves like the maximum positive semidefinite nullity, and provide an example in which the tree cover number does not behave like the maximum positive semidefinite nullity; see [Barioli et al. 2011; Ekstrand et al. 2012] for definitions of outerplanar and tree-width. In Section 3, we characterize when an edge is required to be in some tree of a minimum tree cover. In Section 4, we prove that the tree cover number of the $d-$dimensional hypercube is 2 for all $d \geq 2$.

1.1. ***More notation and terminology.*** The *cycle* on n vertices, denoted C_n, is the graph with vertex set $V(C_n) = \{1, \dots, n\}$ and edge set

$$E(C_n) = \{\{i, i+1\} \mid i \in 1, \dots, n-1\} \cup \{1, n\}.$$

The *star* $K_{1,n}$ is the graph with vertex set $\{1, \dots, n\}$ and edge set $\{\{1, j\} \mid j \in \{2, \dots, n\}\}$. The *complete graph*, denoted K_n, is the graph on n vertices such that there is an edge between any two vertices.

A graph is said to be *connected* if there is a path from any vertex to any other vertex. If G is not connected, then it is said to be *disconnected*. Given a graph $G = (V, E)$, a *connected component* of G is a subgraph C, where C is connected and no vertex in C is adjacent to any vertex of $V(G) \setminus V(C)$. A graph is said to be a *forest* if each of its connected components is a tree.

If vertices u and v are adjacent, we say that they are *neighbors*. The neighborhood of a vertex v, denoted $N(v)$, is the set of neighbors of v. The degree of v is given by $\deg(v) = |N(v)|$.

For a graph $G = (V, E)$, a *cover* of G is a partition of $V(G)$. An *independent set* S is a subset of $V(G)$ such that no two vertices in S are adjacent. The *independence number* of G, denoted $\alpha(G)$, is defined by

$$\alpha(G) = \max\{|S| : S \text{ is an independent set in } G\}.$$

Given two simple graphs G and H, the *cartesian product* of G and H, denoted $G \times H$, is the graph whose vertex set is the cartesian product $V(G) \times V(H)$, and any two vertices (u, u') and (v, v') are adjacent in $G \times H$ if and only if either $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and u is adjacent to v in G. The union of G and H, denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Throughout this paper, we often denote an edge $\{u, v\}$ by uv. An edge uv is called a *bridge* of G if $C - uv$ is disconnected, where C is the component of G with $uv \in E(C)$ and $C - uv$ denotes the subgraph obtained from C by deleting the edge uv. Note that if $e = uv$ is a bridge, then $e = uv$ is a simple edge.

2. Some bounds for the tree cover number

In this section, we give an upper bound on the tree cover number of a graph using the size of an independent set in the graph. We also provide upper and lower bounds on the tree cover number of a subgraph of G obtained by deleting an edge from G. In addition, we observe that subdividing an edge of a graph does not change the tree cover number.

The following proposition shows that, for a connected simple graph, we are able to bound the tree cover number by the difference between the order of the graph and the size of an independent set of vertices of the graph.

Proposition 2. *Let $G = (V, E)$ be a connected simple graph, and let $S \subseteq V(G)$ be an independent set. Then, $T(G) \leq |G| - |S|$. In particular, $T(G) \leq |G| - \alpha(G)$, where $\alpha(G)$ is the independence number of G. Furthermore, this bound is tight.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and suppose that $S = \{v_1, \ldots, v_k\}$ is an independent set. We construct a tree cover of size $n-k$ by the following iterative process: for $i = k+1$, let T_{v_i} be the tree induced by the set of vertices $\{v_{k+1}\} \cup \{N(v_{k+1}) \cap S\}$. For $i = k+2$ to n, let T_{v_i} be the tree induced by the set of vertices in $\{v_i\} \cup \{N(v_i) \cap S\}$ that do not belong to $V(T_{v_j})$ for $k+1 \leq j < i$. Since G is connected, each $s \in S$ has at least one neighbor in $\{v_{k+1}, \ldots, v_n\}$, so this process produces a tree cover of G (where all components are stars) of size $n-k$. Thus, $T(G) \leq n-k$. In particular, $T(G) \leq n - \alpha(G)$. The star $K_{1,n}$ shows that the bound $T(G) \leq |G| - \alpha(G)$ is tight. □

In connection with the conjecture that $\mathrm{T}(G) \leq \mathrm{M}_+(G)$, we show that for some bounds on $\mathrm{M}_+(G)$, analogous bounds hold for $\mathrm{T}(G)$.

For a graph $G = (V, E)$ and $e \in E(G)$, let $G - e$ denoted the graph obtained from G be deleting the edge e. In [Booth et al. 2011], it was shown that

$$\mathrm{M}_+(G) - 1 \leq \mathrm{M}_+(G - e) \leq \mathrm{M}_+(G) + 1,$$

when G is a simple graph. We show that an analogous bound holds for the tree cover number of a multigraph G.

Theorem 3. *For a graph $G = (V, E)$ and $e \in E(G)$,*

$$\mathrm{T}(G) - 1 \leq \mathrm{T}(G - e) \leq T(G) + 1.$$

Proof. Let $u, v \in V(G)$ such that $e = uv$. Consider the graph $G - e$ obtained from G by deleting e (note that e could be a multiedge). Let $\mathcal{T}$ be a minimum tree cover of $G - e$. If u and v are in disjoint trees in $\mathcal{T}$, then $\mathcal{T}$ is a tree cover of G. So, $T(G) \leq T(G - e)$. If u and v are in the same tree in $\mathcal{T}$, denoted by T_{uv}, then the graph induced by the vertices of T_{uv} contains a cycle in G, so $\mathcal{T}$ is not a tree cover of G. However, we may partition the vertices of T_{uv} into two sets

A and B, such that the tree induced by the vertices in A contains u and the tree induced by the vertices in B contains v. Denote these trees by T_A and T_B. Then, $(\mathcal{T} \setminus T_{uv}) \cup T_A \cup T_B$ is a tree cover of G of size $T(G-e)+1$. This shows that $\mathrm{T}(G)-1 \leq \mathrm{T}(G-e)$.

We now show that $T(G-e) \leq T(G)+1$. Suppose there is a minimum tree cover $\mathcal{T}$ of G such that u and v are in separate trees. Then $\mathcal{T}$ is a tree cover of $G-e$, so $T(G-e) \leq T(G)$. Otherwise, let $\mathcal{T}$ be a minimum tree cover of G that uses the edge e (so e is a simple edge by the definition of a tree cover), and let T_e be the tree in $\mathcal{T}$ that contains e. By deleting e from T_e, we produce a tree cover of $G-e$ of size $T(G)+1$. This shows that $T(G-e) \leq T(G)+1$, which completes the proof. $\square$

The next theorem gives a bound that holds for the positive semidefinite maximum nullity of a graph, but the example that follows demonstrates that the analogous bound for the tree cover number fails.

A *2-separation* of a graph $G = (V, E)$ is a pair of subgraphs (G_1, G_2) such that $V(G_1) \cup V(G_2) = V$, $|V(G_1) \cap V(G_2)| = 2$, $E(G_1) \cup E(G_2) = E$, and $E(G_1) \cap E(G_2) = \varnothing$.

Theorem 4 [van der Holst 2009, Theorem 2.8]. *Let (G_1, G_2) be a 2-separation of a graph $G = (V, E)$, and let H_1 and H_2 be obtained from $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, by adding an edge between the vertices of $R = \{r_1, r_2\} = V_1 \cap V_2$. Then*

$$\mathrm{M}_+(G) = \max\{\mathrm{M}_+(G_1) + \mathrm{M}_+(G_2) - 2, \mathrm{M}_+(H_1) + \mathrm{M}_+(H_2) - 2\}.$$

The analogous bound does not hold for the tree cover number. The next example provides a counterexample.

Example 5. For the graphs G, G_1, G_2, H_1, H_2 given in Figure 1, we have that $M_+(G_i) = 2$, $M_+(H_i) = 3$, and $T(G_i) = T(H_i) = 2$ for $i \in \{1, 2\}$. So by Theorem 4, $M_+(G) = 4$. However,

$$3 = \mathrm{T}(G) > \max\{\mathrm{T}(G_1) + \mathrm{T}(G_2) - 2, \mathrm{T}(H_1) + \mathrm{T}(H_2) - 2\} = 2.$$

3. Characterizing edges required in a minimum tree cover

Proposition 6. *Let $G = (V, E)$ be a graph such that $uv \in E(G)$ is a bridge. Then uv is in a tree in every minimum tree cover of G.*

Proof. Note that there is no path from u to v that does not include uv. Therefore, for any tree cover that does not include uv, it must be the case that u and v are in separate trees. These two trees can be consolidated into one tree by adding the edge uv. $\square$

We then ask the question: if an edge is required in every minimum tree cover, must it be a bridge? Figure 2 shows that such an edge is not necessarily a bridge.

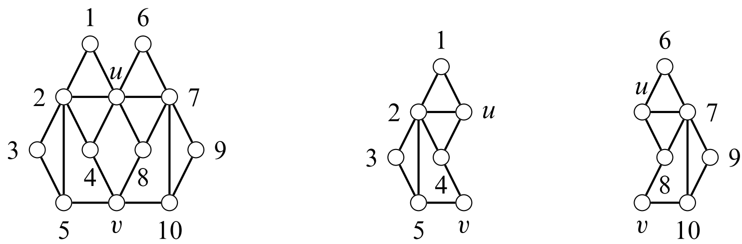


Figure 1. Graphs of G (top left), G_1 (top middle), G_2 (top right), H_1 (bottom left), and H_2 (bottom right).

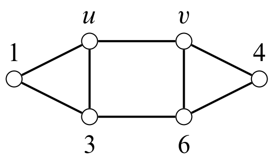


Figure 2. Graph for Example 7.

Example 7. Figure 2 gives a graph whose tree cover number is 2. However, although uv is not a bridge, any tree cover that does not include uv is of size at least 3.

The next lemma gives us a way to determine if an edge is required in every minimum tree cover, given that we are able to compute the necessary tree cover numbers.

Lemma 8. *Let G be a graph, $u, v \in V(G)$, and uv is a simple edge in $E(G)$. Let H be the graph obtained from G by adding a vertex such that $V(H) = V(G) \cup \{w\}$ and $E(H) = E(G) \cup \{uw, vw\}$, where uw and vw are simple edges. Then, uv is required in every minimum tree cover of G if and only if* $\mathrm{T}(H) = \mathrm{T}(G) + 1$.

Proof. First observe that $T(H) \leq T(G) + 1$ since any tree cover of G together with $\{w\}$ is a tree cover for H. Let $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$ be a minimum tree cover of H such that $w \in T_i$ for some i. Since w, u, and v cannot all be in the same tree, then either w is a leaf in T_i or $T_i = \{w\}$. If w is a leaf in T_i, then $T_1, T_2, \ldots, T_i - w, T_{i+1}, \ldots, T_k$ is a tree cover of G, so $T(G) \leq T(H)$. If $T_i = \{w\}$, then $\mathcal{T} \setminus T_i$ is a tree cover for G, so $T(G) \leq T(H) - 1$. This shows that $T(H) = T(G)$ or $T(H) = T(G) + 1$.

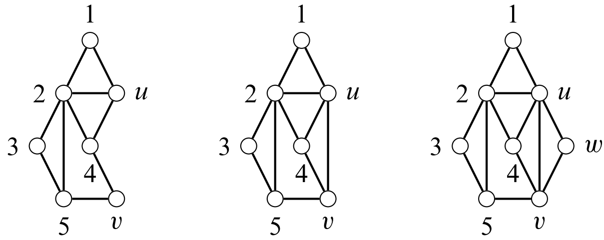


Figure 3. Graphs of G, H, and $\widehat{H}$ for Example 9.

Suppose that uv is required in every minimum tree cover of G. If w is a leaf in T_i, then $T_1, T_2, \ldots, T_i - w, T_{i+1}, \ldots, T_k$ is a tree cover of G with u and v in separate trees, so it follows that $T(H) = T(G) + 1$. If $T_i = \{w\}$, then we also have that $T(H) = T(G) + 1$.

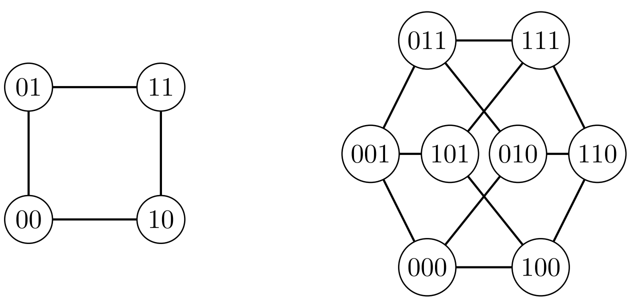
Suppose that there exists a minimum tree cover $\mathscr{T} = \{T_1, T_2, \ldots, T_k\}$ of G such that u and v are in different trees. If $u \in T_i$, we can create a tree cover of H of size k by adding the edge uw to $E(T_i)$. In this case, $T(G) = T(H)$. □

One might think that if H is a graph obtained from G by adding the edge uv, and uv is required in every minimum tree cover of H, then $T(G) = T(H) + 1$. However, this is not true. Example 9 provides a counterexample.

Example 9. It is easy to see that $T(G) = T(H) = 2$ (for H, take the set $\{1, u, v, 5\}$ and $\{2, 3, 4\}$ for example). It can also be verified that $T(\widehat{H}) = 3$. By Lemma 8, it follows that the edge uv is required in every minimum tree cover of H.

4. Tree cover number of the hypercube

The d-dimensional hypercube, denoted Q_d, is the simple graph with vertex set $\{0, 1\}^d$ where two vertices are adjacent if and only if they differ in exactly one position. For example, the 2-dimensional hypercube is a square (see left figure below) and the 3-dimensional hypercube is a cube (right figure). Equivalently,



hypercubes can be inductively defined as the cartesian product of d copies of the complete graph K_2. Hypercubes are a particular case of a larger family of graphs called Hamming graphs. The d-dimensional Hamming graph, denoted $H(d,q)$, is the graph with vertex set $\{0,\dots,n-1\}^d$ where two vertices are adjacent if and only if they differ in exactly one position. Hamming graphs are of use in many areas including error-correcting codes, modeling heat diffusion, and association schemes in statistics. In this section, we show that the tree cover number of the d-dimensional hypercube is 2 for all $d \geq 2$.

Theorem 10. *Let Q_d be the d-dimensional hypercube graph. For all $d \geq 2$, $T(Q_d) = 2$.*

Proof. We first list explicit sets which induce a tree cover of size 2 for Q_d, for $d \in \{2,3,4,5\}$:

$$T_{1_2} = \{(00),(01)\}, \qquad T_{2_2} = \{(11),(10)\}.$$

$$T_{1_3} = \{(010),(000),(001),(110)\}, \qquad T_{2_3} = \{(111),(011),(100),(101)\}.$$

$$T_{1_4} = \{(0011),(0010),(0000),(0110),(1111),(1011),(1100),(1101)\},$$
$$T_{2_4} = \{(0001),(0111),(0100),(0101),(1010),(1000),(1001),(1110)\}.$$

$$\begin{aligned} T_{1_5} = \{&(00100),(00011),(00000),(00110),(01111),(01011),(01100),(01101),\\ &(10001),(10111),(10100),(10101),(11010),(11000),(11001),(11110)\},\end{aligned}$$

$$\begin{aligned} T_{2_5} = \{&(00010),(00001),(00111),(00101),(01010),(01000),(01001),(01110),\\ &(10011),(10010),(10000),(10110),(11111),(11011),(11100),(11101)\}.\end{aligned}$$

Other values of d are handled by induction. Throughout the proof, the sets $\widehat{T}_{1_j}$ and $\widehat{T}_{2_j}$ are covers that will be used as preliminary steps to obtain the sets T_{1_j} and T_{2_j} that will induce a tree cover of size two for Q_j. The proof proceeds as follows: first we give a cover and a tree cover of size two for Q_6; due to the volume of data this information is presented in an online-only supplement. Then, using this tree cover, we construct a cover and a tree cover of size two for Q_7, which again appears in the supplement. We then inductively show that for $d \geq 8$ we can systematically construct a tree cover of size two using the covers and tree covers constructed for Q_{d-1} and Q_{d-2}.

Consider the sets $\widehat{T}_{1_6}$ and $\widehat{T}_{2_6}$ given in the supplement. Note that $\{\widehat{T}_{1_6}, \widehat{T}_{2_6}\}$ is a cover for Q_6, and that $Q_6[\widehat{T}_{1_6}]$ and $Q_6[\widehat{T}_{1_7}]$ are both forests, each consisting of two disjoint trees. Let $x_{1_6} = (001101)$, $x_{2_6} = (110010)$, $y_{1_6} = (001001)$, $y_{2_6} = (110100)$. Then x_{1_6} and x_{2_6} are in $\widehat{T}_{1_6}$, and they are not in the same tree in $Q_6[\widehat{T}_{1_6}]$. Similarly, y_{1_6} and y_{2_6} are in $\widehat{T}_{2_6}$, and they are not in the same tree in $Q_6[\widehat{T}_{2_6}]$. By swapping x_{1_6} and y_{1_6}, the resulting sets T_{1_6} and T_{2_6} (listed in the supplement) induce a tree cover for Q_6 of size two.

To obtain a tree cover of size two for Q_7, we begin by adding a 0 to the beginning of each element in T_{1_6}, and a 1 to the beginning of each element in T_{2_6}. Denote these sets by $T_{1_{6,0}}$ and $T_{2_{6,1}}$, respectively, and let $\widehat{T}_{1_7} := T_{1_{6,0}} \cup T_{2_{6,1}}$. Similarly, we construct the sets $T_{1_{6,1}}$ and $T_{2_{6,0}}$, and let $\widehat{T}_{2_7} := T_{1_{6,1}} \cup T_{2_{6,0}}$ (see supplement). Then, both $Q_7[\widehat{T}_{1_7}]$ and $Q_7[\widehat{T}_{2_7}]$ are forests consisting of two disjoint trees. By swapping $0x_{2_6}$ and $0y_{2_6}$, the resulting sets T_{1_7} and T_{2_7} (given in supplement) induce a tree cover of size two for Q_7.

We proceed by induction to prove the claim for Q_d with $d \geq 8$. Suppose that we have constructed the sets $\widehat{T}_{1_{d-2}} = \{x_1, x_2, \ldots, x_n\}$ and $\widehat{T}_{2_{d-2}} = \{y_1, y_2, \ldots, y_n\}$ such that $\{\widehat{T}_{1_{d-2}}, \widehat{T}_{2_{d-2}}\}$ gives a cover for Q_{d-2} satisfying the following conditions:

(1) $Q_{d-2}[\widehat{T}_{1_{d-2}}]$ and $Q_{d-2}[\widehat{T}_{2_{d-2}}]$ are forests composed of two disjoint trees.

(2) Swapping x_1 and y_1 results in sets

$$T_{1_{d-2}} = \{y_1, x_2, \ldots, x_n\}, \quad T_{2_{d-2}} = \{x_1, y_2, \ldots, y_n\},$$

that induce a tree cover of Q_{d-2} of size two.

(3) For the cover

$$\begin{aligned}\widehat{T}_{1_{d-1}} &= T_{1_{d-2,0}} \cup T_{2_{d-2,1}} = \{0y_1, 0x_2, 0x_3, \ldots, 0x_n, 1x_1, 1y_2, \ldots, 1y_n\},\\ \widehat{T}_{2_{d-1}} &= T_{2_{d-2,0}} \cup T_{1_{d-2,1}} = \{0x_1, 0y_2, 0y_3, \ldots, 0y_n, 1y_1, 1x_2, \ldots, 1x_n\},\end{aligned}$$

of Q_{d-1}, swapping $0x_2 \in \widehat{T}_{1_{d-1}}$ and $0y_2 \in \widehat{T}_{2_{d-1}}$ results in sets

$$\begin{aligned}T_{1_{d-1}} &= \{0y_1, 0y_2, 0x_3, \ldots, 0x_n, 1x_1, 1y_2, \ldots, 1y_n\},\\ T_{2_{d-1}} &= \{0x_1, 0x_2, 0y_3, \ldots, 0y_n, 1y_1, 1x_2, \ldots, 1x_n\},\end{aligned}$$

for Q_{d-1} that induced a tree cover of Q_{d-1} of size two.

(4) x_1 and x_2 are not in the same induced tree in $\widehat{T}_{1_{d-2}}$.

(5) y_1 and y_2 are not in the same induced tree in $\widehat{T}_{2_{d-2}}$.

We are also assuming that $x_i \neq x_j,\ y_i \neq y_j$ for $i \neq j$, and $x_i \neq y_j$ for all i, j.

Then we can construct a cover for Q_d such that swapping two of the elements in the cover will result in a tree cover of size two for Q_d. Furthermore, we show that the constructed cover and tree cover for Q_d, together with the constructed cover and tree cover for Q_{d-1}, still satisfy the above hypotheses, which proves the claim for all $d \geq 8$.

We first construct a cover $\{\widehat{T}_{1_d}, \widehat{T}_{2_d}\}$ for Q_d in the following way:

$$\begin{aligned}\widehat{T}_{1_d} &= T_{1_{d-1,0}} \cup T_{2_{d-1,1}}\\ &= \{00y_1, 00y_2, 00x_3, \ldots, 00x_n, 01x_1, 01y_2, 01y_3, \ldots, 01y_n,\\ &\qquad 10x_1, 10x_2, 10y_3, \ldots, 10y_n, 11y_1, 11x_2, 11x_3, \ldots, 11x_n\}.\end{aligned}$$

$$\begin{aligned}\widehat{T}_{2_d} &= T_{2_{d-1,0}} \cup T_{1_{d-1,1}} \\ &= \{00x_1, 00x_2, 00y_3, \ldots, 00y_n, 01y_1, 01x_2, 01x_3, \ldots, 01x_n, \\ &\qquad 10y_1, 10y_2, 10x_3, \ldots, 10x_n, 11x_1, 11y_2, 11y_3, \ldots, 11y_n\}.\end{aligned}$$

Note that since $Q_{d-1}[T_{1_{d-1}}]$ and $Q_{d-1}[T_{2_{d-1}}]$ are two disjoint trees, it follows that $Q_d[\widehat{T}_{1_d}]$ is a forest consisting of two disjoint trees. Similarly, $Q_d[\widehat{T}_{2_d}]$ is a forest consisting of two disjoint trees. By swapping $01x_1$ and $01y_1$, we obtain the sets

$$\begin{aligned}T_{1_d} &= \{00y_1, 00y_2, 00x_3, \ldots, 00x_n, 01y_1, 01y_2, 01y_3, \ldots, 01y_n, \\ &\qquad 10x_1, 10x_2, 10y_3, \ldots, 10y_n, 11y_1, 11x_2, 11x_3, \ldots, 11x_n\}, \\ T_{2_d} &= \{00x_1, 00x_2, 00y_3, \ldots, 00y_n, 01x_1, 01x_2, 01x_3, \ldots, 01x_n, \\ &\qquad 10y_1, 10y_2, 10x_3, \ldots, 10x_n, 11x_1, 11y_2, 11y_3, \ldots, 11y_n\}.\end{aligned}$$

We now show that $\{Q_d[T_{1_d}], Q_d[T_{2_d}]\}$ is a tree cover for Q_d of size two by showing:

(1) $Q_d[T_{1_d}]$ and $Q_d[T_{2_d}]$ are forests (i.e., there are no cycles in each of $Q_d[T_{1_d}]$ and $Q_d[T_{2_d}]$).

(2) Both $Q_d[T_{1_d}]$ and $Q_d[T_{2_d}]$ are connected graphs.

We show that $Q_d[T_{1_d}]$ is a forest (a similar argument shows that $Q_d[T_{2_d}]$ is a forest). From our construction $Q_d[\widehat{T}_{1_d}]$ is a forest composed of 2 trees, denoted $Q_d[\widehat{A}]$ and $Q_d[B]$, where

$$\begin{aligned}\widehat{A} &:= \{00y_1, 00y_2, 00x_3, \ldots, 00x_n, 01x_1, 01y_2, 01y_3, \ldots, 01y_n\}, \\ B &:= \{10x_1, 10x_2, 10y_3, \ldots, 10y_n, 11y_1, 11x_2, 11x_3, \ldots, 11x_n\}.\end{aligned}$$

By definition $T_{1_d} = (\widehat{T}_{1_d} \setminus \{01x_1\}) \cup \{01y1\}$. By removing $01x_1$ from $\widehat{T}_{1_d}$, B is not affected, and $Q_d[\widehat{A}\setminus\{01x_1\}]$ is now the union of $\deg(01x_1)$ disjoint trees. We now show that by adding $01y_1$ to $\widehat{T}_{1_d} \setminus \{01x_1\}$, no cycles are created in $Q_d[\, T_{1_d}]$. Define $A = \{00y_1, 00y_2, 00x_3, \ldots, 00x_n, 01y_1, 01y_2, 01y_3, \ldots, 01y_n\}$ (note that $T_{1_d} = A \cup B$). Between A and B, the only vertices that are adjacent are $01y_1$ and $11y_1$ (everything else differs in more than one position). Hence, if there is a cycle in $Q_d[T_{1_d}]$, it must be in $Q_d[A]$. Since $Q_d[A \setminus \{01y_1\}]$ (which equals $Q_d[\hat{A}\setminus\{01x_1\}]$) is a forest composed of $\deg(01x_1)$ trees, if there is a cycle in $Q_d[A]$ it must involve $01y_1$. We will now show that it is not possible to have a cycle involving $01y_1$, hence no cycle is possible in $Q_d[T_{1_d}]$.

Note that there is an edge between $00y_1$ and $01y_1$, and that there are no edges between $01y_1$ and any of $00y_2, 00x_3, \ldots, 00x_n$. Thus, the neighbors of $01y_1$ in $Q_d[A]$ are $00y_1$ and a subset of $\{01y_3, 01y_4, \ldots, 01y_n\}$ (since y_1 is not adjacent to y_2 by condition (5) above, then $01y_1$ is not adjacent to $01y_2$). Let $01y_i$ and $01y_j$, $i \neq j$, be arbitrary neighbors of $01y_1$. We show that:

(a) There is no path from $01y_i$ to $01y_j$ in $Q_d[A]$ for $i, j \in \{3, 4, \ldots, n\}$ that does not include $01y_1$.

(b) There is no path from $00y_1$ to $01y_i$ in $Q_d[A]$ that does not include $01y_1$.

To see (a), note that from condition (1), $Q_{d-2}[\{y_1, \ldots, y_n\}]$ is a forest of two disjoint trees. This implies that $Q_d[\{01y_1, \ldots, 01y_n\}]$ is a forest of two disjoint trees. Then, within $Q_d[\{01y_1, \ldots, 01y_n\}]$ there is no path from $01y_i$ to $01y_j$ that does not include $01y_1$. Note that vertices of $\{01y_3, 01y_4, \ldots, 01y_n\}$ are not adjacent to any vertices in A except for possibly each other and $01y_1$ and $01y_2$. Thus, any path from $01y_i$ to $01y_j$ not including $01y_1$ must include $01y_2$. By condition (1), y_1 and y_2 are not in the same induced tree of $Q_{d-2}[\{y_1, \ldots, y_n\}]$, so $01y_1$ and $01y_2$ are not in the same induced tree of $Q_d[\{01y_1, \ldots, 01y_n\}]$. Since $01y_i$ and $01y_j$ are neighbors of $01y_1$, and $01y_1$ is not in the same induced tree as $01y_2$ in $Q_d[\{01y_1, \ldots, 01y_n\}]$, then $01y_i$ and $01y_j$ are not in the same induced tree as $01y_2$. Thus, the only path from $01y_i$ to $01y_j$ is $(01y_i, 01y_1, 01y_j)$.

For (b), we have that the vertices in the set $\{01y_3, 01y_4, \ldots, 01y_n\}$ are not connected in $Q_d[A]$ to any vertices in A except for possibly each other and $01y_1$. We also have that $01y_i$ is not adjacent to $00y_1$ in $Q_d[A]$. So any path from $01y_i$ to $00y_1$ must include $01y_1$.

Next we show that $Q_d[T_{1_d}]$ is connected (a similar argument shows that $Q_d[T_{2_d}]$ is connected). Recall from the hypotheses that

$$Q_{d-2}[\widehat{T}_{2_{d-2}}] = Q_{d-2}[\{y_1, y_2, \ldots, y_n\}]$$

is a forest consisting of two disjoint trees, and

$$Q_{d-2}[T_{2_{d-2}}] = Q_{d-2}[\{x_1, y_2, \ldots, y_n\}]$$

is a tree. This implies that y_1 has exactly one fewer neighbor among $y_2, \ldots, y_n$ than x_1. To see this, note that $Q_{d-2}[\widehat{T}_{2_{d-2}} \setminus \{y_1\}]$ is composed of $1 + \deg(y_1)$ trees. Since

$$Q_{d-2}[T_{2_{d-2}}] = Q_{d-2}[(\widehat{T}_{2_{d-2}} \setminus \{y_1\}) \cup \{x_1\}]$$

is a tree, we must have $\deg(x_1) = 1 + \deg(y_1)$. Therefore, $01y_1$ must have one less neighbor than $01x_1$ among $01y_2, \ldots, 01y_n$. Hence, $01y_1$ and $01x_1$ have the same number of neighbors in A, and thus $01y_1$ has one more neighbor than $01x_1$ in T_{1_d}. We will now show that this last statement implies that $Q_d[T_{1_d}]$ is connected.

Since the graphs induced by $T_{1_{d-1}}$ and $T_{2_{d-1}}$ are trees, then

$$Q_d[T_{1_d}] = Q_d[T_{1_{d-1,0}} \cup T_{2_{d-1,1}}]$$

is a forest consisting of two disjoint trees. Hence, $Q_d[\widehat{T}_{1_d} \setminus \{01x_1\}]$ is a forest consisting of $1 + \deg(01x_1)$ trees. Since $\deg(01y_1) = 1 + \deg(01x_1)$, and since $Q_d[T_{1_d}]$ has no cycles, we have that each of the edges of $01y_1$ must be connected to a different component of the forest. Therefore, $Q_d[T_{1_d}]$ is a tree. An analogous argument shows that $Q_d[T_{2_d}]$ is a tree. Thus, $\{Q_d[T_{1_d}], Q_d[T_{2_d}]\}$ is a tree cover of size two of Q_d.

We now show that the covers and tree covers constructed for Q_{d-1} and Q_d satisfy the induction hypotheses. Note that since $Q_{d-2}[T_{1_{d-2}}]$ and $Q_{d-2}[T_{2_{d-2}}]$ are two disjoint trees, it follows from construction that $Q_d[\widehat{T}_{1_{d-1}}]$ is a forest consisting of two disjoint trees. Similarly, $Q_d[\widehat{T}_{2_{d-1}}]$ is a forest consisting of two disjoint trees, satisfying condition (1). For clarity, we relabel the vertices of $\widehat{T}_{1_{d-1}}$ and $\widehat{T}_{2_{d-1}}$ such that $\widehat{T}_{1_{d-1}} = \{w_1, \dots, w_m\}$ and $\widehat{T}_{2_{d-1}} = \{z_1, \dots, z_m\}$ where $w_1 = 0x_2$, $w_2 = 1x_1$, $z_1 = 0y_2$, and $z_2 = 1y_1$. Then by condition (3), swapping w_1 and z_1 results in sets $T_{1_{d-1}} = \{z_1, w_2, \dots, w_m\}$ and $T_{2_{d-1}} = \{w_1, z_2, \dots, z_m\}$ that induce a tree cover of Q_{d-1} of size two, which shows that condition (2) is satisfied. Note that with this relabeling, the sets $\widehat{T}_{1_d}$ and $\widehat{T}_{2_d}$ become

$$\widehat{T}_{1_d} = T_{1_{d-1,0}} \cup T_{2_{d-1,1}} = \{0z_1, 0w_2, 0w_3, \dots, 0w_m, 1w_1, 1z, \dots, 1z_m\}$$
$$\widehat{T}_{2_d} = T_{2_{d-1,0}} \cup T_{1_{d-1,1}} = \{0w_1, 0z_2, 0y_3, \dots, 0z_m, 1z_1, 1w_2, \dots, 1w_m\},$$

and we have shown above that swapping $0w_2 = 01x_1$ and $0z_2 = 01y_1$ results in the sets T_{1_d} and T_{2_d} which induce a tree cover of size two for Q_d, satisfying condition (3). Furthermore, since $w_1 = 0x_2 \in T_{1_{d-2,0}}$ and $w_2 = 1x_1 \in T_{2_{d-2,1}}$, we have that w_1 and w_2 are not in the same induced tree in $Q_{d-1}[\widehat{T}_{1_{d-1}}]$. Similarly, $z_1 = 0y_2 \in T_{2_{d-2,0}}$ and $z_2 = 1y_1 \in T_{1_{d-2,1}}$, so z_1 and z_2 are not in the same induced tree in $Q_{d-1}[\widehat{T}_{2_{d-1}}]$, showing that conditions (4) and (5) are satisfied.

Since the hypotheses still hold with the constructed covers and tree covers of Q_{d-1} and Q_d, then it follows, by inductively applying the above argument, that $\mathrm{T}(Q_d) = 2$ for all d. □

One may wonder why the base case of the proof starts with Q_6 and Q_7. We would like to note that starting as early as $d = 2$, we were able to use a tree cover of Q_d to produce a cover for Q_{d+1} such that there exists two vertices that could be swapped in order to produce a tree cover for Q_{d+1}. In fact, this is how we constructed the tree covers for Q_3, Q_4, Q_5 given at the start of the proof. However, there is a choice to be made when switching vertices, and the point at which the above constructive pattern holds is dependent upon the initial choice of vertices that are swapped. For example, we experimented with using a different initial swap and found that the pattern did not hold until $d = 11$ or later. It may also be the case that there is an initial swap that allows the pattern to begin sooner than $d = 8$. This is a very interesting phenomenon that is worth further exploration.

We also investigated the idea of generalizing the above proof to all Hamming graphs. For $H(2,3)$, we found that $T(H(2,3)) = 3$, and evidence suggests that $T(H(d,q)) = q$.

Conjecture 11. *$T(H(d,q)) = q$, for $H(d,q)$ the Hamming graph of dimension d.*

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