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On uniform large-scale volume growth for the Carnot–Carathéodory metric on unbounded model hypersurfaces in $\mathbb{C}^2$

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We consider the rate of volume growth of large Carnot–Carathéodory metric balls on a class of unbounded model hypersurfaces in $\mathbb{C}^2$. When the hypersurface has a uniform global structure, we show that a metric ball of radius $\delta \gg 1$ either has volume on the order of δ^3 or δ^4. We also give necessary and sufficient conditions on the hypersurface to display either behavior.

1. Introduction

The study of holomorphic functions on pseudoconvex domains $\Omega \subseteq \mathbb{C}^n$ $(n \geq 2)$ often reduces to studying the partial differential operator $\bar{\partial}$ on Ω given by $\bar{\partial}(f) = \sum f_{\bar{z}_j}\, \mathrm{d}\bar{z}^j$. We can study the boundary values of holomorphic functions (on $\mathrm{b}\Omega$) by studying the partial differential operator $\bar{\partial}_b$ induced on $\mathrm{b}\Omega$ by $\bar{\partial}$. We locally express $\bar{\partial}_b$ in terms of differentiation with respect to $(n-1)$-antiholomorphic vector fields (the so-called Cauchy–Riemann, or CR, vector fields on $\mathrm{b}\Omega$) that are tangent to $\mathrm{b}\Omega$. Under mild nondegeneracy conditions on $\mathrm{b}\Omega$ we can access a family of metrics on $\mathrm{b}\Omega$ specifically adapted to the study of $\bar{\partial}$ and $\bar{\partial}_b$, in the sense that they capture important geometric aspects of $\mathrm{b}\Omega$. One of these, the Carnot–Carathéodory (CC) metric $d(\boldsymbol{p}, \boldsymbol{q})$, measures the infimal length of paths on $\mathrm{b}\Omega$ that not only connect the points $\boldsymbol{p}$ and $\boldsymbol{q}$, but are also almost-everywhere tangent to the real and imaginary parts of the CR vector fields; see [Street 2014] for an extensive history of this metric and its applications to the study of $\bar{\partial}$ and $\bar{\partial}_b$.

In this paper we consider the CC metric $d(\boldsymbol{p}, \boldsymbol{q})$ induced on the boundary of a model pseudoconvex domain $\Omega \subset \mathbb{C}^2$ by the real and imaginary parts of the CR vector field on $\mathrm{b}\Omega$. In particular, we seek to understand the volume growth of the metric balls $B_d(\boldsymbol{p}, \delta)$ when Ω is of the form

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im}(z_2) > P(z_1)\},$$

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where $P:\mathbb{C}\to\mathbb{R}$ is smooth, subharmonic, and nonharmonic. Under mild nondegeneracy conditions on ΔP is it known [Montanari and Morbidelli 2012; Nagel et al. 1985; 1988; 1989] that for $\delta \le 1$ the metric ball $B_d(\boldsymbol{p},\delta)$ is comparable to a "shorn" or "twisted" ellipsoid with radius δ in the directions spanned by the real and imaginary parts of the CR vector field and radius $\Lambda((z_1,z_2),\delta)$ in the $\operatorname{Re}(z_2)$-direction. If we equip $\mathrm{b}\Omega$ with the Lebesgue measure $dm(z,t)$ that it receives via its identification with $\mathbb{C}\times\mathbb{R}$ given by $(z_1,z_2)\mapsto(z,t)$, where $z=z_1=x+iy$ and $t=\operatorname{Re}(z_2)$, then this small CC metric ball has volume comparable to that of the twisted ellipsoid:

$$\operatorname{Vol}(B_d(\boldsymbol{p},\delta))\approx\delta^2\Lambda(\boldsymbol{p},\delta). \tag{1-1}$$

We build on the earlier work of the second author [Peterson 2014] which sought to understand the possible rate of growth of $\operatorname{Vol}(B_d(\boldsymbol{p},\delta))$ for model domains Ω such that when δ is large, the Euclidean radius

$$\Lambda((z_1,z_2),\delta)=\sup\bigl\{|\operatorname{Re}(z_2'-z_2)| : d((z_1,z_2),(z_1,z_2'))<\delta\bigr\}$$

of $B_d((z_1,z_2),\delta)$ in the $\operatorname{Re}(z_2)$-direction is essentially independent of (z_1,z_2). The quantity $\Lambda(\boldsymbol{p},\delta)$ is called the *global structure* of $\mathrm{b}\Omega$, and we make precise the (z_1,z_2)-independence condition described above with the following definition.

Definition 1.1. If there exists $\delta_0>0$, a function $f:[\delta_0,+\infty)\to[0,+\infty)$, and positive constants $0<c<C<+\infty$ such that $cf(\delta)\le\Lambda(\boldsymbol{p},\delta)\le Cf(\delta)$ for all $\delta\ge\delta_0$ and $\boldsymbol{p}\in\mathrm{b}\Omega$, then we say that $(f(\delta),\delta_0)$ is a *uniform global structure* or UGS for $\mathrm{b}\Omega$.

For such domains Ω we also have (1-1) when δ is large (see Remark 3.3), and therefore the volume growth of CC metric balls of any size is completely understood once we understand $\Lambda(\boldsymbol{p},\delta)$ for large δ.

Example 1.2. In [Nagel et al. 1988], it is shown that when $P(z_1)$ is a subharmonic, nonharmonic polynomial (and where ΔP has degree $m-2$),

$$\Lambda((z_1,z_2),\delta)\approx\sum_{k=0}^{m-2}\biggl(\sum_{\alpha=0}^{k}\biggl|\frac{\partial^k\Delta P}{\partial z_1^\alpha\partial\bar z_1^{k-\alpha}}(z_1)\biggr|\biggr)\delta^{k+2}.$$

In particular, when $P(z_1)=|z_1|^2$ (so that $\Delta P(z_1)\equiv 4$) we have $\Lambda((z_1,z_2),\delta)\approx 4\delta^2$, and therefore $(\delta^2,1)$ is a uniform global structure for $\mathrm{b}\Omega$.

On the other hand, if $P(z_1)=|z_1|^4$, then $\Lambda((z_1,z_2),\delta)\approx|z_1|^2\delta^2+|z_1|\delta^3+\delta^4\approx(|z_1|+\delta)^2\delta^2$, and therefore is not uniform in $z_1\in\mathbb{C}$. This shows that $\mathrm{b}\Omega$ has no uniform global structure. More generally, if P is a subharmonic, nonharmonic polynomial, then $\mathrm{b}\Omega$ does not have a uniform global structure when ΔP is not constant.

The following result from [Peterson 2014] controls the growth of uniform global structures.

Theorem 1.3 [Peterson 2014, Theorem 1.2]. *If* $\mathrm{b}\Omega$ *has a UGS* $(f(\delta), \delta_0)$, *then there are positive constants* $0 < c < C < +\infty$ *such that* $c\delta \le f(\delta) \le C\delta^2$ *for all* $\delta \ge \delta_0$.

So when $\mathrm{b}\Omega$ has a UGS and $\delta \gg 1$, the global structure at any point grows at least linearly and at most quadratically in δ. Examples are given in [Peterson 2014] where $\mathrm{b}\Omega$ has a UGS linear in δ and quadratic in δ. Our question is whether there exist examples where the UGS grows somewhere "between" linear and quadratic. For instance, are there examples for $\mathrm{b}\Omega$ with UGS $(\delta^{3/2}, \delta_0)$ or $(\delta \log \delta, \delta_0)$?

Example 1.4. To see that this question is not trivial, fix $\alpha \in \left(0, \frac{2}{3}\right)$ and choose a subharmonic function $P : \mathbb{C} \to \mathbb{R}$ such that $\Delta P(z) = (1 + |z|^2)^{-\alpha/2}$. Using our techniques and those of [Peterson 2014] one can show that there exist constants $0 < c < C < +\infty$ such that for all $\delta > 0$,

$$c\delta^{2-\alpha} \le \Lambda((0,0), \delta) \le C\delta^{2-\alpha} \quad \text{and} \quad \Lambda((\delta^{3/2}, 0), \delta) \le C\delta^{2-3\alpha/2}.$$

Thus $\Lambda((0,0), \delta)$ grows at a rate comparable to $\delta^{2-\alpha}$, but $\Lambda((\delta^{3/2}, 0), \delta)$ grows no faster than $\delta^{2-3\alpha/2}$. This illustrates that it is possible for the global structure to grow (in δ) at nonpolynomial rates, but $\left(\text{since } \alpha < \frac{3}{2}\alpha\right)$ not necessarily uniformly in the base point (z_1, z_2).

Our first main theorem (proven in Section 4) answers our question negatively.

Theorem 1.5. *If* $\mathrm{b}\Omega$ *has UGS* $(f(\delta), \delta_0)$, *then either* (δ^2, δ^*) *or* (δ, δ^*) *is a UGS for* $\mathrm{b}\Omega$ *for some* $\delta^* > 0$.

We subsequently give necessary and sufficient conditions on $\mathrm{b}\Omega$ for both linear (Theorem 5.1) and quadratic (Theorem 5.2) growth of the UGS, thereby completely describing the conditions under which any particular model domain has a uniform global structure.

The volume growth of CC metric balls in model domains Ω as above for large δ is only explicitly understood when P is a subharmonic, nonharmonic polynomial [Nagel et al. 1988] or in the limited examples considered in [Peterson 2014] mentioned above. In some situations one can obtain upper bounds for the rate of volume growth (see [Chang and Chang 2014]), but one cannot hope for precise control of $\mathrm{Vol}(B_d(\boldsymbol{p}, \delta))$ for general P. On the other hand, applications of volume growth estimates are many and varied; for example, one can use these estimates to identify spaces of homogeneous type [Coifman and Weiss 1977], study singular integral operators [Stein 1993], and even to decide whether or not the boundaries of two model domains are quasiconformally equivalent [Fässler et al. 2015; Heinonen and Koskela 1998].

Our paper is structured as follows: Section 2 gives relevant definitions and notation that will be used extensively throughout the paper and recalls past results. In Section 3 we gain some intuition about how a UGS behaves and prove a key and explicit alternative characterization of the UGS. In Section 4 we prove Theorem 1.5, followed in Section 5 by necessary and sufficient conditions for a given model domain to possess a uniform global structure. Section 6 concludes the paper and offers future directions of study.

2. Preliminaries

With Ω as in the Introduction, the space of tangential CR vector fields on $\mathrm{b}\Omega$ is spanned by

$$\bar{Z} = 2\frac{\partial}{\partial \bar{z}_1} - 4i P_{\bar{z}_1}(z_1)\frac{\partial}{\partial \bar{z}_2}.$$

We identify $\mathrm{b}\Omega$ with $\mathbb{C}\times\mathbb{R}$ via the diffeomorphism $(z_1, z_2) \mapsto (z, t) \in \mathbb{C}\times\mathbb{R}$, where $z = z_1 = x + iy$ and $t = \mathrm{Re}(z_2)$. Under this transformation, $\bar{Z}$ becomes

$$\bar{Z} = 2\frac{\partial}{\partial \bar{z}} - 2i P_{\bar{z}}(z)\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial x} + P_y(x, y)\frac{\partial}{\partial t}\right) - i\left(-\frac{\partial}{\partial y} + P_x(x, y)\frac{\partial}{\partial t}\right) \overset{\text{def}}{=} X - iY.$$

As stated in Introduction, we give $\mathrm{b}\Omega$ the Lebesgue measure $dm(z, t)$ that it receives upon identification with $\mathbb{C}\times\mathbb{R}$. For the rest of the paper, we work on $\mathbb{C}\times\mathbb{R}$ instead of $\mathrm{b}\Omega$ to simplify notation.

We define the CC metric $d : (\mathbb{C}\times\mathbb{R}) \times (\mathbb{C}\times\mathbb{R}) \to [0, +\infty)$ by

$$\begin{aligned} d(\boldsymbol{p}, \boldsymbol{q}) = \inf\bigl\{\delta > 0 : \exists\gamma : [0, 1] \to \mathbb{C}\times\mathbb{R},\ \gamma(0) = \boldsymbol{p},\ \gamma(1) = \boldsymbol{q},\\ \gamma'(s) = \delta\alpha(s)X(\gamma(s)) + \delta\beta(s)Y(\gamma(s)) \text{ a.e.},\\ \alpha, \beta \in \mathrm{FPWS}[0, 1],\ |\alpha(s)|^2 + |\beta(s)|^2 < 1 \text{ a.e.}\bigr\}. \end{aligned} \tag{2-1}$$

Here FPWS[0, 1] (read “finite piecewise smooth”) denotes the set of functions $f : [0, 1] \to \mathbb{R}$ which are smooth except at a finite number of points and whose derivatives extend continuously to those points from each side separately.

The global structure $\Lambda((z, t), \delta)$, the radius in the t-direction of the CC ball, is then defined as

$$\Lambda((z, t), \delta) \overset{\text{def}}{=} \sup\bigl\{|t' - t| : d((z, t), (z, t')) < \delta\bigr\}. \tag{2-2}$$

Note that the quantity (2-2) is actually *independent* of the t-coordinate because the solutions to the differential equation in (2-1) are translation invariant in t. To simplify notation, we will therefore write $\Lambda(z, \delta)$ instead of $\Lambda((z, t), \delta)$ for the remainder of the paper, treating Λ as a function from $\mathbb{C}\times(0, +\infty) \mapsto [0, +\infty)$. The first observation of [Peterson 2014] is that definition (2-2) is in fact equivalent to the following statement in terms of curves in $\mathbb{C}$, independent of t:

$$\begin{aligned} \Lambda(z, \delta) = \sup\bigl\{\textstyle\oint_\gamma P_y\,\mathrm{d}x - P_x\,\mathrm{d}y : \gamma : [0, 1] \to \mathbb{C},\\ \gamma(0) = \gamma(1) = z,\ |\gamma'(s)| \le \delta \text{ a.e.},\\ \gamma'(s) = \alpha(s) + i\beta(s),\ \alpha, \beta \in \mathrm{FPWS}[0, 1]\bigr\}. \end{aligned} \tag{2-3}$$

We write $L(\gamma) = \int_a^b |\gamma'(s)|\,\mathrm{d}s$ for the usual Euclidean length of a piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$. The following geometric definition from [Peterson 2014] will be essential to our understanding of global structures.

Definition 2.1. We say $A \subset \mathbb{C}$ is a *pen* if A is open, connected, simply connected, and $\mathrm{b}A$ can be parametrized by a continuous piecewise smooth curve $\gamma : [0, 1] \to \mathbb{C}$ with $\gamma'(s) = \alpha(s) + i\beta(s)$, where $\alpha, \beta \in \mathrm{FPWS}[0, 1]$. We call $\mathrm{L}(\mathrm{b}A) = \mathrm{L}(\gamma)$ the amount of *fencing* used to enclose A. For a fixed $z \in \mathbb{C}$ and $\delta > 0$, we say that a finite collection of pens $R = (R_1, \ldots, R_N)$ is a (z, δ)-*stockyard* if

$$z \in \bigcup_{i=1}^{N} \mathrm{b}R_i, \quad \sum_{i=1}^{N} \mathrm{L}(\mathrm{b}R_i) \leq \delta, \quad \text{and} \quad \bigcup_{i=1}^{N} \mathrm{b}R_i \text{ is connected.}$$

Remark 2.2. We will often use the fact that given a pen A, we have $A \subseteq B(z, \mathrm{L}(\mathrm{b}A))$ for any point $z \in A$, where $B(z, \rho)$ denotes the open Euclidean disc in $\mathbb{C}$ of radius ρ centered at z.

Thinking of global structures in terms of (2-3), [Peterson 2014] provides the following theorem.

Theorem 2.3 [Peterson 2014, Theorem 1.1].

$$\Lambda(z, \delta) = \sup_{(z,\delta)-\text{stockyards } R} \sum_{R_i \in R} \int_{R_i} \Delta P(w)\, dm(w).$$

Here $dm(\cdot)$ denotes the Lebesgue measure on $\mathbb{C}$. The problem of calculating the global structure, an inherently three-dimensional problem, is therefore reduced to a question in two dimensions. Furthermore, notice that because P was assumed to be subharmonic and nonharmonic, we can think of ΔP as a density function in the plane. In this context, integration over a pen measures the "mass" of the region covered by the pen, and integration over a stockyard is then the sum of the mass collected by the individual pens. The global structure $\Lambda(z, \delta)$ is then just the most mass one can collect with a stockyard touching z constructed with at most δ amount of fencing.

We introduce the following simpler notation for use in our estimates. For two nonnegative quantities A and B, we write $A \lesssim B$ (read "A is controlled above by B") if there exists some constant $c > 0$, independent of all relevant quantities, such that $A \leq cB$. We say $A \gtrsim B$ (read "A is controlled below by B") if $B \lesssim A$, and $A \approx B$ (read "A is comparable to B") if both $A \lesssim B$ and $B \lesssim A$.

3. Alternate description of uniform global structures

When $\mathrm{b}\Omega$ has a UGS $(f(\delta), \delta_0)$ and when $\delta \geq \delta_0$, we expect that for every point z in the plane we can find a high density region whose distance from the point is no more than δ. We should then be able to construct a $(z, N\delta)$-stockyard for an appropriately fixed natural number N which covers this region with one or more pens. Otherwise $\Lambda(z, \delta)$ would be uncontrollably small at certain points. We also expect that no point should be within δ of a region of exceedingly high density. Otherwise $\Lambda(z, \delta)$ would be uncontrollably large at certain points. Before we make this notion precise in Proposition 3.4 of this section, we need two lemmas.

A simple observation about one formula for a UGS is the following.

Lemma 3.1. *If* bΩ *has UGS* $(f(\delta), \delta_0)$, *then* $(\sup_{z\in\mathbb{C}} \Lambda(z, \delta), \delta_0)$ *is also a UGS for* bΩ.

Proof. Fix some $z \in \mathbb{C}$. By the definition of UGS, there exist constants $c, C > 0$ independent of z and δ such that

$$cf(\delta) \le \Lambda(z, \delta) \le Cf(\delta).$$

So $Cf(\delta)$ is an upper bound for $\{\Lambda(z, \delta) : z \in \mathbb{C}\}$, which gives $\sup_{z\in\mathbb{C}} \Lambda(z, \delta) \le Cf(\delta)$ since the supremum is the least upper bound. Also $\sup_{z\in\mathbb{C}} \Lambda(z, \delta) \ge \Lambda(z, \delta) \ge cf(\delta)$. So then

$$\Lambda(z, \delta) \le Cf(\delta) \le \frac{C}{c} \sup_{z\in\mathbb{C}} \Lambda(z, \delta) \quad \text{and} \quad \Lambda(z, \delta) \ge cf(\delta) \ge \frac{c}{C} \sup_{z\in\mathbb{C}} \Lambda(z, \delta)$$

for all $\delta \ge \delta_0$. Therefore $(\sup_{z\in\mathbb{C}} \Lambda(z, \delta), \delta_0)$ is a UGS for bΩ. □

Lemma 3.1 makes it clear that we can take $f(\delta)$ to be a monotonically increasing function of δ. We next show that $f(\delta)$ does not increase too quickly in the sense that if we double the amount of fencing available to construct stockyards, then the amount of mass one can collect should not grow exceedingly fast.

Lemma 3.2. *If* bΩ *has UGS* $(f(\delta), \delta_0)$ *then* $f(\delta) \approx f(2\delta)$ *for all* $\delta \ge \delta_0$, *with constants independent of* δ.

Proof. By Lemma 3.1 we can without loss of generality take $f(\delta) = \sup_{z\in\mathbb{C}} \Lambda(z, \delta)$. For if $(g(\delta), \delta_0)$ is any other UGS for bΩ and we can prove the lemma for $f(\delta)$, then $g(\delta) \approx f(\delta) \approx f(2\delta) \approx g(2\delta)$. We prove first that $f(2\delta) \approx f(3\delta)$ for large δ and will show at the end of the proof that this is sufficient to establish the lemma.

Because $f(\delta)$ is a nondecreasing function, we trivially have $f(2\delta) \le f(3\delta)$. We need only show then that $f(3\delta) \lesssim f(2\delta)$. To this end, fix $z_0 \in \mathbb{C}$ and $\delta \ge \frac{2}{3}\delta_0$, and let R be any arbitrary $(z_0, 3\delta)$-stockyard. There is a FPWS curve $\gamma : [0, 1] \to \mathbb{C}$ with $\gamma(0) = \gamma(1) = z_0$, $\mathrm{L}(\gamma) \le 3\delta$, and

$$\sum_{R_i\in R} \int_{R_i} \Delta P(w)\, dm(w) = \oint_\gamma P_y\, \mathrm{d}x - P_x\, \mathrm{d}y.$$

We now produce seven continuous, piecewise smooth curves $\gamma_k : [0, 1] \to \mathbb{C}$, $k = 1, \ldots, 7$, with $L(\gamma_k) \le 2\delta$ and $\gamma_k'(s) = \alpha_k(s) + i\beta_k(s)$ with $\alpha_k, \beta_k \in \mathrm{FPWS}[0, 1]$

such that

$$\oint_{\gamma} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y = \sum_{k=1}^{7} \oint_{\gamma_k} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y.$$

Without loss of generality, suppose that γ has constant speed so that

$$\int_0^{1/3} |\gamma'(s)|\,\mathrm{d}s = \int_{1/3}^{2/3} |\gamma'(s)|\,\mathrm{d}s = \int_{2/3}^{1} |\gamma'(s)|\,\mathrm{d}s \le \delta. \tag{3-1}$$

For convenience, we define $z_1 = \gamma\left(\frac{1}{3}\right)$, $z_2 = \gamma\left(\frac{2}{3}\right)$, and $z_3 = \gamma(1) = z_0$. We also denote by $\overrightarrow{z, w}$ the directed line segment from z to w.

Now we have

$$\begin{aligned}
&\oint_{\gamma} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y \\
&= \int_{\gamma[0,1/3]} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y + \int_{\gamma[1/3,2/3]} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y + \int_{\gamma[2/3,1]} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y \\
&\qquad + \int_{\overrightarrow{z_0,z_1}} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y + \int_{\overrightarrow{z_1,z_2}} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y + \int_{\overrightarrow{z_2,z_3}} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y \\
&\qquad + \int_{\overrightarrow{z_1,z_0}} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y + \int_{\overrightarrow{z_2,z_1}} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y + \int_{\overrightarrow{z_3,z_2}} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y \\
&= \oint_{\gamma[0,1/3]+\overrightarrow{z_1,z_0}} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y + \oint_{\gamma[1/3,2/3]+\overrightarrow{z_2,z_1}} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y \\
&\qquad + \oint_{\gamma[2/3,1]+\overrightarrow{z_3,z_2}} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y + \oint_{\overrightarrow{z_0,z_1}+\overrightarrow{z_1,z_2}+\overrightarrow{z_2,z_3}} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y.
\end{aligned} \tag{3-2}$$

We consider the contours of integration in each integral.

We define $\gamma_i = \gamma\left[\frac{1}{3}(i-1), \frac{1}{3}i\right] + \overrightarrow{z_i, z_{i-1}}$ for $i = 1, 2, 3$. By (3-1), the length of each contour $\gamma\left[\frac{1}{3}(i-1), \frac{i}{3}\right]$ is no more than δ. And as the straight line between the endpoints of these contours, each directed line segment $\overrightarrow{z_i, z_{i-1}}$ also has length no more than δ. In other words, each γ_i for $i = 1, 2, 3$ is a closed curve of length no more than 2δ.

The last integral in (3-2) is taken over a closed contour composed of three line segments, each of length no more than δ. For each $j = 0, 1, 2$ define $b_j = \frac{1}{2}(z_j + z_{j+1})$ to be the bisector of segment $\overrightarrow{z_j, z_{j+1}}$, and for convenience define $b_{-1} = b_2$. We then define $\gamma_{j+4} = \overrightarrow{z_j, b_j} + \overrightarrow{b_j, b_{j-1}} + \overrightarrow{b_{j-1}, z_j}$ and define $\gamma_7 = \overrightarrow{b_0, b_1} + \overrightarrow{b_1, b_2} + \overrightarrow{b_2, b_0}$. Then we have

$$\oint_{\overrightarrow{z_0,z_1}+\overrightarrow{z_1,z_2}+\overrightarrow{z_2,z_3}} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y = \sum_{k=4}^{7} \oint_{\gamma_k} P_y\,\mathrm{d}x - P_x\,\mathrm{d}y.$$

But by similar triangles,

$$\mathrm{L}(\gamma_k) = \tfrac{1}{2}\mathrm{L}(\overrightarrow{z_0, z_1} + \overrightarrow{z_1, z_2} + \overrightarrow{z_2, z_3}) \le \tfrac{3}{2}\delta$$

for each $k = 4, 5, 6, 7$. Combining these observations with (3-2) and (2-3), we have

$$\begin{aligned}\sum_{R_i \in R} \int_{R_i} \Delta P(w)\, dm(w) &= \sum_{k=1}^{7} \oint_{\gamma_k} P_y \,\mathrm{d}x - P_x \,\mathrm{d}y \\ &\le \sum_{k=1}^{7} \Lambda(\gamma_k(0), \mathrm{L}(\gamma_k)) \le 3f(2\delta) + 4f\big(\tfrac{3}{2}\delta\big) \le 7f(2\delta)\end{aligned}$$

for all $(z_0, 3\delta)$-stockyards R. Therefore by Theorem 2.3 we see $\Lambda(z, 3\delta) \le 7f(2\delta)$ for all $z \in \mathbb{C}$; hence

$$f(3\delta) = \sup_{z \in \mathbb{C}} \Lambda(z, 3\delta) \le 7f(2\delta).$$

In summary, for all $\delta \ge \frac{2}{3}\delta_0$ we have

$$f(2\delta) \le f(3\delta) \le 7f(2\delta). \tag{3-3}$$

Now fix $\delta \ge \delta_0$. Because $f(\delta)$ is a nondecreasing function, we also trivially have $f(\delta) \le f(2\delta)$. But by monotonicity and (3-3) we see

$$f(2\delta) \le f\big(\tfrac{9}{4}\delta\big) \le 49 f(\delta).$$

Therefore, $f(\delta) \approx f(2\delta)$ for all $\delta \ge \delta_0$. □

Remark 3.3. Lemma 3.2 was used implicitly in [Peterson 2014] without proof or statement. The arguments of [Peterson 2014] show that for any fixed $z \in \mathbb{C}$,

$$\big\{(w, s) \in \mathbb{C} \times \mathbb{R} : |w - z| < \tfrac{1}{4}\delta,\ |s - t - T(z, w)| < \Lambda\big(z, \tfrac{1}{4}\delta\big)\big\} \subseteq B_d((z, t), \delta)$$

and

$$B_d((z, t), \delta) \subseteq \big\{(w, s) \in \mathbb{C} \times \mathbb{R} : |w - z| < 3\delta,\ |s - t - T(z, w)| < \Lambda(z, 3\delta)\big\},$$

where

$$T(z, w) = -2\mathrm{Im}\bigg(\int_0^1 (w - z) P_z(r(w - z) + z)\,\mathrm{d}r\bigg)$$

is the "twist" of the CC ball. Lemma 3.2 then yields the formula

$$\mathrm{Vol}(B_d((z, t), \delta)) \approx \delta^2 \Lambda(z, \delta) \quad \text{for} \quad \delta \ge \delta_0$$

when $\mathrm{b}\Omega$ has UGS $(f(\delta), \delta_0)$. This shows that we can think of $B_d((z, t), \delta)$ as a "twisted" ellipsoid in the case of large δ, not just small δ as in (1-1).

We are now ready to make precise the intuition laid out in the beginning of this section.

Proposition 3.4. *If* bΩ *has UGS* $(f(\delta), \delta_0)$, *then*

$$\Lambda(z, \delta) \approx \sup_{\hat{z} \in B(z,\delta)} \sup_{0<\hat{\delta}\leq\delta} \frac{\delta}{\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w)$$

uniformly for $z \in \mathbb{C}$ *and* $\delta \geq \delta_0$.

Proof. As in the proof of Lemma 3.2, we assume without loss of generality that $f(\delta)$ is a nondecreasing function. For any choice of $\hat{z} \in B(z, \delta)$ and $0 < \hat{\delta} \leq \delta$, define a $(z, 4\pi\delta)$-stockyard $R = (R_0, R_1, \ldots, R_N)$ composed of one pen R_0 which is a circle touching z and some point on b$B(\hat{z}, \hat{\delta})$ and $N = \lfloor \delta/\hat{\delta} \rfloor$ copies of $B(\hat{z}, \hat{\delta})$. Using the fact that $\lfloor \delta/\hat{\delta} \rfloor \geq \delta/(2\hat{\delta})$ because $\delta \geq \hat{\delta} > 0$, we have

$$\begin{aligned}
\Lambda(z, \delta) \approx f(\delta) \approx f(16\delta) \geq f(4\pi\delta) \gtrsim \Lambda(z, 4\pi\delta)& \\
\geq \sum_{R_i \in R} \int_{R_i} \Delta P(w)\, dm(w) &\geq \left\lfloor \frac{\delta}{\hat{\delta}} \right\rfloor \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \\
\geq \frac{\delta}{2\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w).&
\end{aligned}$$

Therefore,

$$\Lambda(z, \delta) \gtrsim \sup_{\hat{z} \in B(z,\delta)} \sup_{0<\hat{\delta}\leq\delta} \frac{\delta}{\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w).$$

Now let $R = (R_1, \ldots, R_M)$ be an arbitrary (z, δ)-stockyard. For $i = 1, \ldots, M$, fix some point $z_i \in R_i$. Then, recalling Remark 2.2, we have

$$\begin{aligned}
\sum_{R_i \in R} \int_{R_i} \Delta P(w)\, dm(w) &\leq \sum_{i=1}^{M} \int_{B(z_i, \mathrm{L}(\mathrm{b}R_i))} \Delta P(w)\, dm(w) \\
&= \sum_{i=1}^{M} \frac{\mathrm{L}(\mathrm{b}R_i)}{\delta} \frac{\delta}{\mathrm{L}(\mathrm{b}R_i)} \int_{B(z_i, \mathrm{L}(\mathrm{b}R_i))} \Delta P(w)\, dm(w) \\
&\leq \sum_{i=1}^{M} \left(\frac{\mathrm{L}(\mathrm{b}R_i)}{\delta} \right) \sup_{\hat{z} \in B(z,\delta)} \sup_{0<\hat{\delta}\leq\delta} \frac{\delta}{\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \\
&\leq \frac{\delta}{\delta} \sup_{\hat{z} \in B(z,\delta)} \sup_{0<\hat{\delta}\leq\delta} \frac{\delta}{\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \\
&= \sup_{\hat{z} \in B(z,\delta)} \sup_{0<\hat{\delta}\leq\delta} \frac{\delta}{\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w).
\end{aligned}$$

Therefore

$$\Lambda(z, \delta) \leq \sup_{\hat{z} \in B(z,\delta)} \sup_{0<\hat{\delta}\leq\delta} \frac{\delta}{\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w).$$

□

4. Proof of Theorem 1.5

Proposition 3.4 reveals very strong information about the density in the space around a point when there is a UGS. Armed with this knowledge, we are almost ready to prove Theorem 1.5. We begin by recalling and proving two lemmas, the first of which is a technical result from [Peterson 2014].

Lemma 4.1 [Peterson 2014, Lemma 4.1]. *If* $b\Omega$ *has a UGS, then there are constants* $C_1, C_2 > 0$, *depending only on* ΔP *and* δ_0, *such that*

(a) $\displaystyle\inf_{z\in\mathbb{C}} \sup_{\hat{z}\in B(z,\delta)} \sup_{0<\hat{\delta}\le\delta} (\hat{\delta}+\hat{\delta}^2)^{-1}\int_{B(\hat{z},\hat{\delta})} \Delta P(w)\,dm(w) \ge C_1$ *for all* $\delta \ge \delta_0$;

(b) $\displaystyle\sup_{z\in\mathbb{C}} \sup_{\delta>0}(\delta+\delta^2)^{-1}\int_{B(z,\delta)} \Delta P(w)\,dm(w) \le C_2.$

Remark 4.2. Note that increasing δ_0 can only possibly increase C_1 and will not affect the constant C_2.

We also need a short geometric lemma.

Lemma 4.3. *Let* $0 < a \le b$. *Then within any disc of radius* b *in* $\mathbb{C}$, *one can pack at least* $b^2/(16a^2)$ *disjoint discs of radius* a.

Proof. Without loss of generality, assume the disc of radius b is centered at the origin. Since $B(0,a) \subset B(0,b)$, we can always pack at least one disc of radius a inside of $B(0,b)$. If $2a > \sqrt{2}b$, then we have at least one disc of radius a inside of $B(0,b)$, and

$$1 > \frac{\sqrt{2}b}{2a} > \frac{b^2}{2a^2} > \frac{b^2}{16a^2}.$$

Note now that for all $x \ge 1$, we have $x = \lfloor x\rfloor + \alpha$ for some $\alpha \in [0,1)$ so that

$$\lfloor x^2\rfloor = \left\lfloor(\lfloor x\rfloor+\alpha)^2\right\rfloor < \left\lfloor(\lfloor x\rfloor+\lfloor x\rfloor)^2\right\rfloor = \lfloor 4\lfloor x\rfloor^2\rfloor = 4\lfloor x\rfloor^2.$$

Assume that $2a \le \sqrt{2}b$. The disc $B(0,b)$ contains a square of side length

$$\left\lfloor\frac{\sqrt{2}b}{2a}\right\rfloor 2a \le \sqrt{2}b$$

centered at the origin. This square contains exactly $\lfloor\sqrt{2}b/(2a)\rfloor^2$ disjoint squares of side length $2a$, each of which contains a disc of radius a. So we again see that $B(0,b)$ contains at least

$$\left\lfloor\frac{\sqrt{2}b}{2a}\right\rfloor^2 > \frac{1}{4}\left\lfloor\frac{b^2}{2a^2}\right\rfloor \ge \frac{b^2}{16a^2}$$

discs of radius a. □

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Proposition 3.4 shows that there is some constant $c > 0$ such that

$$\sup_{\hat{z}\in B(z,\delta)} \sup_{0<\hat{\delta}\le\delta} \frac{\delta}{\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \ge cf(\delta)$$

for all $z \in \mathbb{C}$ and $\delta \ge \delta_0$. So for all $z \in \mathbb{C}$ and $\delta \ge \delta_0$, there exists $\hat{z} \in B(z, \delta)$ and $0 < \hat{\delta} \le \delta$ such that

$$\frac{1}{\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \ge \frac{c}{2} \frac{f(\delta)}{\delta}.$$

Now suppose $f(\delta) = \delta$ is not a UGS for $b\Omega$. That is, $\limsup_{\delta\to+\infty} f(\delta)/\delta = +\infty$. Then, taking $C_2 > 0$ as in Lemma 4.1, we can choose $\delta_1 > \max(1, \delta_0)$ such that $f(\delta_1)/\delta_1 > 4C_2/c$. Choose $\hat{\delta}$ associated to $\delta = \delta_1$ as above. If $\hat{\delta} \le 1$, then by Lemma 4.1 we have

$$2C_2 < \frac{c}{2}\frac{f(\delta_1)}{\delta_1} \le \frac{1}{\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \le \frac{2}{\hat{\delta}+\hat{\delta}^2} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \le 2C_2,$$

which is impossible. Therefore for all $z \in \mathbb{C}$, there exists $\hat{z} \in B(z, \delta_1)$ and $1 \le \hat{\delta} \le \delta_1$ such that

$$\int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \ge \frac{c}{2}\frac{f(\delta_1)}{\delta_1}\hat{\delta} \ge 2C_2 > 0.$$

It follows that for all $z \in \mathbb{C}$,

$$\int_{B(z,2\delta_1)} \Delta P(w)\, dm(w) \ge \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \ge 2C_2.$$

By Lemma 4.3, for all $\delta \ge \delta_1$, we can pack $N > \delta^2/(16\delta_1^2)$ disjoint discs of radius $2\delta_1$ within a disc of radius 2δ. So for all $z \in \mathbb{C}$,

$$\int_{B(z,2\delta)} \Delta P(w)\, dm(w) \ge N2C_2 > \frac{\delta^2}{16\delta_1^2} \cdot 2C_2 \approx (2\delta)^2.$$

Then for all $\delta \ge 2\delta_1$ and some $z_1 \in bB(z, \delta)$,

$$f(\delta) \approx f(2\pi\delta) \approx \Lambda(z_1, 2\pi\delta) \ge \int_{B(z,\delta)} \Delta P(w)\, dm(w) \gtrsim \delta^2.$$

But Theorem 1.3 implies $f(\delta) \lesssim \delta^2$ for all $\delta \ge 2\delta_1 \ge \delta_0$. Therefore setting $\delta^* = 2\delta_1$ we see that if $f(\delta) = \delta$ is not a UGS for $b\Omega$, then (δ^2, δ^*) is a UGS for $b\Omega$. □

So a UGS must grow in a linear or quadratic fashion. Linear growth means that for any point, the stockyards which pick up the most mass enclose a small, dense, nearby disc as many times as possible. Quadratic growth means a stockyard which picks up the most mass does so by taking a pen consisting of one large disc, collecting as much area as possible.

5. Identifying uniform global structures

So far, almost all of the results of this paper have taken as hypothesis that $b\Omega$ has a UGS and considered what that means for the global structure Λ. To look at an arbitrary model domain and determine if there is a UGS is a much more difficult question. But with Theorem 1.5, we see that we only need to provide conditions to identify uniform global structures where either $f(\delta)=\delta$ or $f(\delta)=\delta^2$. The following two theorems provide necessary and sufficient conditions for each case.

Theorem 5.1. *The hyperspace* $b\Omega$ *has UGS* (δ, δ_0) *if and only if*

(a) $\displaystyle\int_{B(z,\delta)} \Delta P(w)\, dm(w) \lesssim \delta$ *for all* $z \in \mathbb{C}$ *and* $\delta > 0$, *and*

(b) *there exist constants* $\delta^* > M > 0$ *such that*

$$\inf_{z\in\mathbb{C}} \sup_{\hat{z}\in B(z,\delta^*)} \sup_{0<\hat{\delta}\le M} \frac{1}{\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \gtrsim 1.$$

Proof. Suppose (δ, δ_0) is a UGS for $b\Omega$. For any $z \in \mathbb{C}$, fix some point z_1 with $|z_1 - z| = \delta$. If $2\pi\delta \ge \delta_0$ then

$$\int_{B(z,\delta)} \Delta P(w)\, dm(w) \le \Lambda(z_1, 2\pi\delta) \approx 2\pi\delta \approx \delta.$$

If $0 < 2\pi\delta < \delta_0$, then taking a stockyard consisting of $\lfloor \delta_0/(2\pi\delta) \rfloor$ copies of $B(z,\delta)$ gives

$$\frac{\delta_0}{4\pi\delta} \int_{B(z,\delta)} \Delta P(w)\, dm(w) \le \left\lfloor \frac{\delta_0}{2\pi\delta} \right\rfloor \int_{B(z,\delta)} \Delta P(w)\, dm(w) \le \Lambda(z_1, \delta_0) \approx 1.$$

Therefore (a) holds.

Also, for any fixed $\delta^* \ge \delta_0 > 0$, Lemma 4.1 gives some constant $C_1 > 0$ such that

$$\begin{aligned}
&\inf_{z\in\mathbb{C}} \sup_{\hat{z}\in B(z,\delta^*)} \sup_{0<\hat{\delta}\le \delta_0} \frac{1}{\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \\
&\qquad \ge \inf_{z\in\mathbb{C}} \sup_{\hat{z}\in B(z,\delta_0)} \sup_{0<\hat{\delta}\le \delta_0} \frac{1}{\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \\
&\qquad \ge \inf_{z\in\mathbb{C}} \sup_{\hat{z}\in B(z,\delta_0)} \sup_{0<\hat{\delta}\le \delta_0} \frac{1}{\hat{\delta}+\hat{\delta}^2} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \ge C_1.
\end{aligned}$$

Therefore (b) holds (with $M = \delta_0$).

Now we suppose (a) and (b) hold. For any $\delta > 0$ and $z \in \mathbb{C}$, let $R = (R_1, \dots, R_N)$ be an arbitrary (z, δ)-stockyard. For each $i = 1, \dots, N$, fix some point $z_i \in R_i$.

Then recalling Remark 2.2, (a) gives

$$\sum_{R_i \in R} \int_{R_i} \Delta P(w)\, dm(w) \le \sum_{R_i \in R} \int_{B(z_i, \mathrm{L}(\mathrm{b}R_i))} \Delta P(w)\, dm(w) \lesssim \sum_{R_i \in R} \mathrm{L}(\mathrm{b}R_i) \le \delta.$$

Therefore $\Lambda(z, \delta) \lesssim \delta$ uniformly for $z \in \mathbb{C}$ and $\delta > 0$.

For any $z \in \mathbb{C}$, fix a $\hat{z} \in B(z, \delta^*)$ and $0 < \hat{\delta} \le M$ such that

$$\frac{1}{\hat{\delta}} \int_{B(\hat{z}, \hat{\delta})} \Delta P(w) dm(w) \gtrsim 1,$$

as given by (b). Then for all $\delta \ge 2\pi M \ge 2\pi \hat{\delta}$, there is a $(z, \pi\delta^* + \delta)$-stockyard R which consists of one circular pen touching z and some point on $\mathrm{b}B(\hat{z}, \hat{\delta})$ and $\lfloor \delta/(2\pi\hat{\delta}) \rfloor$ copies of $B(\hat{z}, \hat{\delta})$. Then

$$\begin{aligned} \Lambda(z, \pi\delta^* + \delta) &\ge \sum_{R_i \in R} \int_{R_i} \Delta P(w)\, dm(w) \\ &\ge \left\lfloor \frac{\delta}{2\pi\hat{\delta}} \right\rfloor \int_{B(\hat{z}, \hat{\delta})} \Delta P(w)\, dm(w) \ge \frac{\delta}{4\pi\hat{\delta}} \int_{B(\hat{z}, \hat{\delta})} \Delta P(w)\, dm(w) \\ &\gtrsim \delta = 2\pi M \frac{\delta}{2\pi M} \ge \frac{2\pi M}{2\pi M + \pi\delta^*} (\pi\delta^* + \delta), \end{aligned}$$

where here we have used the fact that if $c \ge 0$ and $a \ge b > 0$, then $a/b \ge (a+c)/(b+c)$. Therefore $\Lambda(z, \delta) \approx \delta$ for all $\delta \ge \delta_0$ with $\delta_0 = \pi\delta^* + 2\pi M$. □

Theorem 5.2. *The hypersurface* $\mathrm{b}\Omega$ *has UGS* (δ^2, δ_0) *if and only if there exists* $\delta^* > 0$ *such that*, *uniformly for* $z \in \mathbb{C}$,

(a) $\displaystyle\int_{B(z,\delta)} \Delta P(w)\, dm(w) \lesssim \delta$ *when* $\delta \le \delta^*$, *and*

(b) $\displaystyle\int_{B(z,\delta)} \Delta P(w)\, dm(w) \approx \delta^2$ *when* $\delta \ge \delta^*$.

Proof. Suppose (δ^2, δ_0) is a UGS for $\mathrm{b}\Omega$. Then for any $z \in \mathbb{C}$ and some point z_1 with $|z_1 - z| = \delta$ we have

$$\int_{B(z,\delta)} \Delta P(w)\, dm(w) \le \Lambda(z_1, 2\pi\delta) \approx (2\pi\delta)^2 \approx \delta^2$$

for all $\delta \ge \delta_0$.

Proposition 3.4 shows that there is some constant $c > 0$ such that

$$\sup_{\hat{z} \in B(z,\delta)} \sup_{0 < \hat{\delta} \le \delta} \frac{\delta}{\hat{\delta}} \int_{B(\hat{z}, \hat{\delta})} \Delta P(w)\, dm(w) \ge c\delta^2$$

for all $z \in \mathbb{C}$ and $\delta \geq \delta_0$. So for all $z \in \mathbb{C}$ and $\delta \geq \delta_0$, there exists $\hat{z} \in B(z, \delta)$ and $0 < \hat{\delta} \leq \delta$ such that

$$\frac{1}{\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \geq \tfrac{1}{2} c\delta.$$

Taking $C_2 > 0$ as in Lemma 4.1, choose some $\delta_1 > \max(1, \delta_0, 4C_2/c)$. Choose $\hat{\delta}$ associated to $\delta = \delta_1$ as above. If $\hat{\delta} \leq 1$, then by Lemma 4.1 we have

$$2C_2 < \frac{c}{2}\delta_1 \leq \frac{1}{\hat{\delta}} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \leq \frac{2}{\hat{\delta} + \hat{\delta}^2} \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \leq 2C_2,$$

which is impossible. Therefore for all $z \in \mathbb{C}$, there exists $\hat{z} \in B(z, \delta_1)$ and $1 \leq \hat{\delta} \leq \delta_1$ such that

$$\int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \geq \frac{c}{2}\delta_1 \hat{\delta} \geq 2C_2 > 0.$$

It follows that for all $z \in \mathbb{C}$,

$$\int_{B(z,2\delta_1)} \Delta P(w)\, dm(w) \geq \int_{B(\hat{z},\hat{\delta})} \Delta P(w)\, dm(w) \geq 2C_2.$$

By Lemma 4.3, for all $\delta \geq \delta_1$, we can pack $N > \delta^2/(16\delta_1^2)$ disjoint discs of radius $2\delta_1$ within a disc of radius 2δ. So for all $z \in \mathbb{C}$,

$$\int_{B(z,2\delta)} \Delta P(w)\, dm(w) \geq N \int_{B(z,2\delta_1)} \Delta P(w)\, dm(w) > \frac{\delta^2}{16\delta_1^2} \cdot 2C_2 \approx (2\delta)^2.$$

Therefore,

$$\int_{B(z,\delta)} \Delta P(w)\, dm(w) \approx \delta^2$$

for all $\delta \geq 2\delta_1 > \delta_0$. Setting $\delta^* = 2\delta_1$, we see (b) holds. Also, Lemma 4.1 yields

$$\int_{B(z,\delta)} \Delta P(w)\, dm(w) \leq C_2(\delta + \delta^2).$$

But if $\delta \leq \delta^*$ then

$$\int_{B(z,\delta)} \Delta P(w)\, dm(w) \leq C_2(\delta^* + 1)\delta \approx \delta,$$

so (b) holds.

Now we suppose (a) and (b) hold so that for $\delta \leq \delta^*$ we have

$$\int_{B(z,\delta)} \Delta P(w) dm(w) \leq a\delta$$

and for $\delta \geq \delta^*$ we have

$$\int_{B(z,\delta)} \Delta P(w) dm(w) \leq b\delta^2$$

for some constants $a, b > 0$ independent of $z \in \mathbb{C}$. For $\delta \geq 1$, let $R = (R_1, \ldots, R_N)$ be an arbitrary (z, δ)-stockyard. Without loss of generality, we may relabel the pens so that $\mathrm{L}(\mathrm{b}R_i) \leq \delta^*$ for $i = 1, \ldots, L$ and $\mathrm{L}(\mathrm{b}R_i) \geq \delta^*$ for $i = L+1, \ldots, N$ for some integer $L \in \{0, \ldots, N\}$. For each $i = 1, \ldots, N$, fix some $z_i \in R_i$. Recalling Remark 2.2, we have

$$\begin{aligned}\sum_{R_i \in R} \int_{R_i} \Delta P(w)\, dm(w) &= \sum_{i=1}^{L} \int_{R_i} \Delta P(w)\, dm(w) + \sum_{i=L+1}^{N} \int_{R_i} \Delta P(w)\, dm(w) \\ &\leq \sum_{i=1}^{L} \int_{B(z_i, \mathrm{L}(\mathrm{b}R_i))} \Delta P(w)\, dm(w) + \sum_{i=L+1}^{N} \int_{B(z_i, \mathrm{L}(\mathrm{b}R_i))} \Delta P(w)\, dm(w) \\ &\leq a \sum_{i=1}^{L} \mathrm{L}(\mathrm{b}R_i) + b \sum_{i=L+1}^{N} \mathrm{L}(\mathrm{b}R_i)^2 \\ &\leq a \sum_{i=1}^{L} \mathrm{L}(\mathrm{b}R_i) + b \Big(\sum_{i=L+1}^{N} \mathrm{L}(\mathrm{b}R_i) \Big)^2 \leq a\delta + b\delta^2 \lesssim \delta^2.\end{aligned}$$

So $\Lambda(z, \delta) \lesssim \delta^2$ for all $\delta \geq 1$.

Using (b), we may take a stockyard consisting of one large circular pen with radius $\delta \geq \delta^*$ and center z_1 satisfying $|z_1 - z| = \delta$ to see that

$$\Lambda(z, 2\pi\delta) \geq \int_{B(z_1, \delta)} \Delta P(w)\, dm(w) \approx \delta^2 \approx (2\pi\delta)^2.$$

Therefore $\Lambda(z, \delta) \approx \delta^2$ for all $\delta \geq \delta_0$ with $\delta_0 = \max(1, \delta^*/(2\pi))$. □

6. Future directions

Although the results of this paper completely describe the nature of uniform global structures for the model domains we consider, several interesting avenues for further study present themselves when we weaken our hypotheses. One such direction would be to extend the results of this paper to higher dimensions. That is, is there an appropriate notion of stockyards in higher dimensions with which to analyze the global structure on the boundary of a model domain in $\mathbb{C}^n$? It is not clear how the Green's theorem argument used in [Peterson 2014] to prove Theorem 2.3 would generalize or even how (if at all) the notion of stockyards should generalize to higher dimensions.

One could also relax the conditions on P which determine the boundary $\mathrm{b}\Omega$. For example, do similar results hold assuming that P is only once differentiable and that ΔP as a distribution is nonnegative? One could also allow P to be a more general function for which Ω is pseudoconvex, that is, take $P = P(z_1, \mathrm{Re}(z_2))$. In such a situation, the volume of CC balls with such a choice of P would a priori depend on the $\mathrm{Re}(z_2)$-direction. Since the methods of this paper heavily exploited the $\mathrm{Re}(z_2)$-translation invariance of Ω, it is unclear if these methods can be easily extended to handle the more general situation.

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References

[Chang and Chang 2014] S.-C. Chang and T.-H. Chang, "On CR volume growth estimate in a complete pseudohermitian 3-manifold", *Int. J. Math.* **25**:4 (2014), art. id. 1450035, 22 pp. MR Zbl

[Coifman and Weiss 1977] R. R. Coifman and G. Weiss, "Extensions of Hardy spaces and their use in analysis", *Bull. Amer. Math. Soc.* **83**:4 (1977), 569–645. MR Zbl

[Fässler et al. 2015] K. Fässler, P. Koskela, and E. Le Donne, "Nonexistence of quasiconformal maps between certain metric measure spaces", *Int. Math. Res. Not.* **2015**:16 (2015), 6968–6987. MR Zbl

[Heinonen and Koskela 1998] J. Heinonen and P. Koskela, "Quasiconformal maps in metric spaces with controlled geometry", *Acta Math.* **181**:1 (1998), 1–61. MR Zbl

[Montanari and Morbidelli 2012] A. Montanari and D. Morbidelli, "Nonsmooth Hörmander vector fields and their control balls", *Trans. Amer. Math. Soc.* **364**:5 (2012), 2339–2375. MR Zbl

[Nagel et al. 1985] A. Nagel, E. M. Stein, and S. Wainger, "Balls and metrics defined by vector fields, I: Basic properties", *Acta Math.* **155**:1 (1985), 103–147. MR Zbl

[Nagel et al. 1988] A. Nagel, J.-P. Rosay, E. M. Stein, and S. Wainger, "Estimates for the Bergman and Szegő kernels in certain weakly pseudoconvex domains", *Bull. Amer. Math. Soc.* (*N.S.*) **18**:1 (1988), 55–59. MR Zbl

[Nagel et al. 1989] A. Nagel, J.-P. Rosay, E. M. Stein, and S. Wainger, "Estimates for the Bergman and Szegő kernels in $\mathbb{C}^2$", *Ann. of Math.* (2) **129**:1 (1989), 113–149. MR Zbl

[Peterson 2014] A. Peterson, "Carnot–Carathéodory metrics in unbounded subdomains of $\mathbb{C}^2$", *Arch. Math.* (*Basel*) **102**:5 (2014), 437–447. MR Zbl

[Stein 1993] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series **43**, Princeton University Press, 1993. MR Zbl

[Street 2014] B. Street, *Multi-parameter singular integrals*, Annals of Mathematics Studies **189**, Princeton Univ. Press, 2014. MR Zbl

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