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Forbidden subgraphs of coloring graphs

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Given a graph G, its k-coloring graph has vertex set given by the proper k-colorings of the vertices of G with two k-colorings adjacent if and only if they differ at exactly one vertex. Beier et al. (*Discrete Math.* **339**:8 (2016), 2100–2112) give various characterizations of coloring graphs, including finding graphs which never arise as induced subgraphs of coloring graphs. These are called forbidden subgraphs, and if no proper subgraph of a forbidden subgraph is forbidden, it is called minimal forbidden. In this paper, we construct a finite collection of minimal forbidden subgraphs that come from modifying theta graphs. We also construct an infinite family of minimal forbidden subgraphs similar to the infinite family found by Beier et al.

1. Introduction

A *graph* $G = (V, E)$ consists of a set $V = V[G] = \{v_1, \dots, v_n\}$ of *vertices* and a set $E = E[G] \subseteq \{vv' : v, v' \in V\}$ of *edges*, where vv' represents an unordered pair of vertices. In this paper, we assume G has finite order (i.e., $|V|$ is finite), $v \neq v'$ whenever $vv' \in E$, and G has at most one edge between a single pair of vertices. A graph H is an *induced subgraph of* G if $V[H] \subseteq V[G]$ and $vv' \in E[H]$ if and only if $vv' \in E[G]$.

Given $k \in \mathbb{N}$, a *proper* k*-coloring* of a graph G is a function $\alpha : V[G] \to \{1, 2, \dots, k\}$ such that $\alpha(v) \neq \alpha(v')$ whenever $vv' \in E[G]$. The k*-coloring graph of* G, denoted by $\mathcal{C}_k(G)$, is the graph with vertex set consisting of all proper k-colorings of G. Edges between colorings exist if and only if the colorings differ at precisely one vertex of G. Figure 1 shows an example. When discussing properties of $\mathcal{C}_k(G)$, we refer to G as the base graph for $\mathcal{C}_k(G)$.

Interest in coloring graphs stems from applications in theoretical physics. Coloring graphs model the Glauber dynamics of the antiferromagnetic Potts model at zero temperature [Dyer et al. 2006; Jerrum 1995; Molloy 2004; Vigoda 2000]. Beier et al. [2016] approach coloring graphs from an inverse perspective, asking "Given a

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graph G', does there exist a graph G and natural number k such that $\mathcal{C}_k(G) = G'$?" We build on their work on permissible and forbidden subgraphs of coloring graphs.

A graph H' is called *permissible* if it is an induced subgraph of some coloring graph. If H' is not an induced subgraph of any coloring graph, we say H' is *forbidden*. The graph H' is called *minimal forbidden* if H' is forbidden and each proper induced subgraph of H' is permissible. Beier et al. define an infinite two-parameter family of graphs $M_{n,p}$ and show an infinite number of them are minimal forbidden. They define another infinite collection of graphs called theta graphs and completely classify them into permissible, minimal forbidden, and forbidden but not minimal.

The goal of this paper is to formalize and enhance the tools and techniques for studying the forbidden and permissible subgraphs of coloring graphs introduced in [Beier et al. 2016] and to provide new examples. This will aid others investigating coloring graphs and, perhaps more interestingly, other types of transition graphs, like those found in [Cohen and Teicher 2014; Zhang et al. 1988; Haas 2012; Mohar 2007].

Section 2 expands on coloring edge labeling and edge labeling partitions, which were first introduced in [Beier et al. 2016]. We also recall necessary results from that paper involving permissible subgraphs. As an application of Section 2, we give two new collections of minimal forbidden subgraphs in Section 3. One collection comes from modifying theta graphs, and the other is an infinite subset of the two-parameter family of graphs $L_{n,p}$, which we define in that section. Finally, Section 4 provides several future directions for this work.

Our notation and conventions follow [Beier et al. 2016; Diestel 1997]. If we are unsure whether a graph is a coloring graph, we sometimes refer to it as a *candidate coloring graph*. Base graphs will be denoted by G, and candidate coloring graphs will be denoted by G'. Subgraphs (usually induced) of G and G' will be denoted by H and H' respectively. Vertices in the coloring graph will be identified by Greek letters $(\alpha, \beta, \gamma, \dots)$, and vertices in the base graph will be denoted by lowercase letters $(u, v, w, \dots)$.

We denote by I_n the graph consisting of n vertices and no edges. Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we denote their disjoint union by $G_1 \sqcup G_2$. The Cartesian product of G_1 and G_2, denoted by $G_1 \square G_2$, has vertex set $V_1 \times V_2$ with an edge between (v_1, v_2) and (w_1, w_2) exactly when $v_1 = w_1$ and $v_2 w_2 \in E_2$ or $v_2 = w_2$ and $v_1 w_1 \in E_1$.

2. Background

In this section, we recall and formalize definitions, theorems, and techniques from [Beier et al. 2016] needed to analyze forbidden and permissible subgraphs. We begin with a discussion of edge labeling of coloring graphs, which is a key tool used to prove graphs are minimal forbidden.

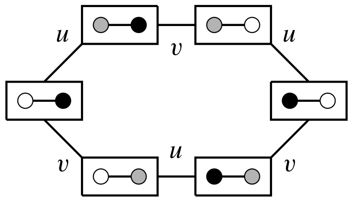


Figure 1. Coloring edge labeling of $\mathcal{C}_3(P_1)$.

An *edge labeling* of a graph is a function with domain the edge set of the graph. Given a graph G and $k \in \mathbb{N}$, the *coloring edge labeling* of the coloring graph $\mathcal{C}_k(G)$ is the map from $E[\mathcal{C}_k(G)]$ to $V[G]$ that labels each edge $\alpha\beta \in E[\mathcal{C}_k(G)]$ with the unique vertex of $V[G]$ at which the colorings α and β differ. This labeling technique was first introduced in [Beier et al. 2016, p. 2102], where it is referred to as edge labeling. Figure 1 shows the coloring edge labeling for coloring graph $\mathcal{C}_3(P_1)$, where $V[P_1] = \{u, v\}$.

For a graph H', we call an edge labeling a *proper edge labeling* if there exists a graph G and a $k \in \mathbb{N}$ such that H' is an induced subgraph of $\mathcal{C}_k(G)$ and the edge labeling of H' coincides with the coloring edge labeling of $\mathcal{C}_k(G)$. An *improper edge labeling* is an edge labeling that is not proper.

It follows from these definitions that a graph H' is permissible if and only if it has a proper edge labeling. In [Beier et al. 2016, Corollary 12], it is shown that all cycles except C_5 are permissible subgraphs, so a cycle C_n of size $n \neq 5$ must have at least one proper edge labeling. We use properties of proper edge labelings of cycles to analyze proper edge labelings of more complicated graphs. The following lemma summarizes properties of coloring edge labelings of cycles used in [Beier et al. 2016].

Lemma 1. *A proper edge labeling of a cycle C_n must satisfy the following conditions*:

(1) *Each label must occur at least twice.*

(2) *Adjacent edges have the same label if and only if $n = 3$.*

(3) *If a cycle has three edges consecutively labeled u, v, u with $u \neq v$ then either $n = 4$ or u occurs as a label at least three times.*

While the conditions outlined in Lemma 1 are necessary for a proper edge labeling of a cycle, they are not sufficient. One can show that the edge labeling in Figure 2 is not proper though it meets all conditions in Lemma 1. Also, we emphasize that the conditions in Lemma 1 are necessary only for cycles.

Also introduced in [Beier et al. 2016, p. 2102] is the concept of edge label partitioning. The *edge label partition* corresponding to a proper edge labeling of a cycle C_n is the partition of n consisting of the number of occurrences of each label.

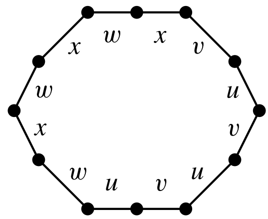


Figure 2. An improper edge labeling of C_{12}.

C_3	C_4	C_6	C_7	C_8	C_9
$3 \vdash 3$	$4 \vdash 2, 2$	$6 \vdash 3, 3$ $6 \vdash 2, 2, 2$	$7 \vdash 2, 2, 3$	$8 \vdash 4, 4$ $8 \vdash 2, 2, 4$ $8 \vdash 2, 3, 3$ $8 \vdash 2, 2, 2, 2$	$9 \vdash 2, 3, 4$ $9 \vdash 3, 3, 3$ $9 \vdash 2, 2, 2, 3$

Table 1. Edge label partition types for small cycles.

For example, the edge label partition corresponding to the proper edge labeling of C_6 shown in Figure 1 is $6 \vdash 3, 3$ since u and v each occur three times. Note that each proper edge labeling corresponds to a unique edge label partition. However, a partition does not necessarily uniquely determine a proper edge labeling.

Moreover, not every partition of n corresponds to a proper edge labeling. In fact, conditions on edge labelings stated in Lemma 1 give restrictions on which partitions can be edge label partitions. Each part of an edge label partition must be greater than 1 according to the first condition in Lemma 1. Also by the first condition, no part of an edge label partition can be greater than half of n. The following is a complete list of possible edge label partition types for C_n with $3 \leq n \leq 9$. (Recall C_5 is forbidden, so there are no edge label partitions of 5.)

Table 1 is very useful when attempting to find proper edge labelings of graphs. For instance, if H' contains an induced copy of C_7, then a proper edge labeling of H' must have exactly three distinct labels on that cycle and a corresponding edge label partition type of $7 \vdash 2, 2, 3$. We can then use Lemma 1 to further investigate how those labels could be arranged.

In addition to examining cycles, our analysis of forbidden subgraphs builds on the following results about permissible and forbidden subgraphs.

Theorem 2 [Beier et al. 2016, Theorem 9]. *If H_1' and H_2' are permissible, then $H_1' \sqcup H_2'$ is permissible. Alternately, if $H_1' \sqcup H_2'$ is forbidden, then either H_1' or H_2' is forbidden.*

Theorem 3 [Beier et al. 2016, Theorem 10]. *If H_1' and H_2' are permissible, then $H_1' \square H_2'$ is permissible. Alternately, if $H_1' \square H_2'$ is forbidden, then either H_1' or H_2' is forbidden.*

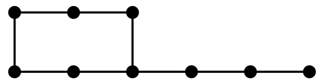


Figure 3. Attaching P_3 to C_6.

These two preceding theorems allow us to construct a number of permissible subgraphs. Since the path P_1 is permissible, given any permissible subgraph H', it follows that $P_1 \square H'$ is permissible. Note that H' with one new vertex and an edge from that new vertex to any other vertex is an induced subgraph of $P_1 \square H'$. We call this *attaching a copy of* P_1. More generally, we refer to the process of identifying an endpoint of a path with some vertex of a graph H' as *attaching a path to* H'. Figure 3 shows an example of attaching a path to a 6-cycle. By an inductive argument, we arrive at Corollary 4. By similar arguments, Corollaries 5 and 6 also follow from Theorems 2 and 3.

Corollary 4 [Beier et al. 2016, p. 2104]. *A permissible subgraph with any number of paths of any length attached is permissible.*

Corollary 5 [Beier et al. 2016, Corollary 11]. *All trees are permissible.*

Corollary 6 [Beier et al. 2016, Corollary 12]. *The graph C_n for $n \neq 5$ is permissible. The graph C_5 is forbidden.*

In addition to appending paths to build new permissible subgraphs, we can sometimes add additional vertices along induced paths of permissible subgraphs to get new permissible subgraphs. The next theorem explains the conditions under which this can be done. The result of replacing an edge of a graph with P_2 will be called *subdividing an edge*.

Theorem 7 [Beier et al. 2016]. *Let H' be a permissible subgraph containing a degree-2 vertex whose neighbors are not adjacent. The graph obtained by subdividing both edges incident to the vertex of degree 2 is also permissible.*

This subdivision theorem is useful when studying permissibility of so-called theta graphs. A (*generalized*) *theta graph*, denoted by $T(m_1, m_2, \dots, m_k)$ where $m_i \leq m_{i+1}$ for all i, consists of a collection of internally disjoint paths of lengths m_1, $m_2, \dots, m_k$ with a single common initial vertex u and terminal vertex v where $u \neq v$. Thus, u and v will have degree k, while all other vertices have degree 2. Note that theta graphs generalize cycles since $C_n = T(1, n-1)$. The collection of generalized theta graphs are completely categorized as permissible, minimal forbidden, or forbidden not minimal in [Beier et al. 2016]. Any theta graph not containing those listed in the following theorem is permissible.

Theorem 8 [Beier et al. 2016, Theorem 15]. *The complete list of minimal forbidden theta graphs is*

$$T(1,4), \quad T(1,2,2), \quad T(2,2,2), \quad T(3,3,3) \quad \textit{and} \quad T(2,2,4).$$

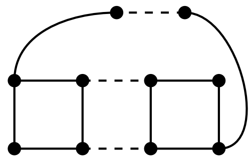


Figure 4. $M_{n,p}$, where $n, p \geq 1$.

An infinite set of minimal forbidden subgraphs is introduced in [Beier et al. 2016]. These are part of the set of graphs denoted by $M_{n,p}$, where $n, p \geq 1$. These graphs contain a chain of $n-1$ induced copies of C_4 with a path of length $p+1$ between two vertices, as seen in Figure 4. The following theorem, which is needed in our arguments, summarizes the results on $M_{n,p}$ graphs from [Beier et al. 2016].

Theorem 9 [Beier et al. 2016, Lemma 16, Theorem 17]. *The family $M_{n,p}$ is forbidden but not minimal if and only if $n \geq 1$ and $p \leq 2$. The family $M_{n,3}$ is minimal forbidden if and only if $n \geq 2$.*

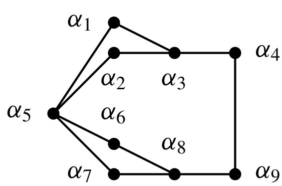
3. Two collections of minimal forbidden subgraphs

Figure 5 has 14 new examples of minimal forbidden subgraphs that come from modifying the structure of generalized theta graphs. We will prove graph (a) is minimal forbidden by applying the language and lemmas from the previous section. The proofs that the other graphs are minimal forbidden are very similar in style and are therefore left to the reader.

The proof that a graph is minimal forbidden breaks into two parts: showing it is forbidden and showing it is minimal. To prove a graph is forbidden, we focus on its induced cycles examining their interactions and showing they have no simultaneous proper edge labelings. To prove a forbidden subgraph is minimal, we show that each of its proper induced subgraphs is permissible. Since subgraphs of permissible subgraphs are permissible, it is sufficient to show that all induced subgraphs obtained by removing one vertex are permissible.

Theorem 10. *Graph (a) in Figure 5 is a minimal forbidden subgraph.*

Proof. Consider the following graph H', which is Figure 5(a) with a choice of vertex names:



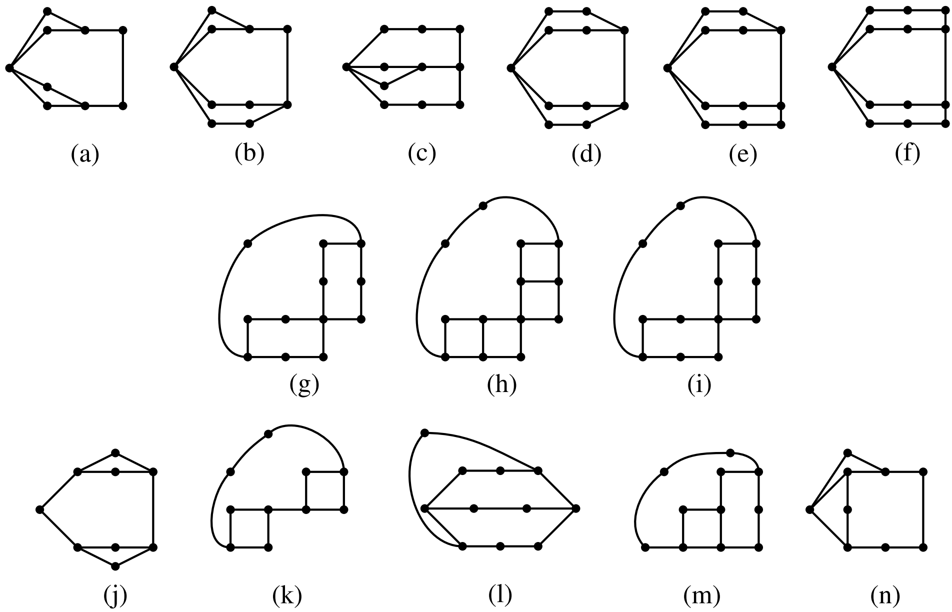


Figure 5. A finite collection of minimal forbidden subgraphs.

Since α_5 is not a vertex of an induced C_3, a proper edge labeling of H' must assign a different label to each of the four edges incident to α_5. Since the edges incident to α_5 are part of two edge-disjoint induced copies of C_4, these induced copies of C_4 will have repeated edge labels, say u, v, u, v and w, x, w, x. However, each induced copy of C_4 shares two consecutive edges with an induced copy of C_7. Thus this C_7 would have at least four distinct edge labels, which is not possible by Lemma 1. We conclude H' is forbidden.

Next, we demonstrate that H' is permissible by arguing that removing any vertex from H' yields a permissible subgraph. Removing α_1, α_2, α_6, or α_7 from H' results in a copy of theta graph $T(2, 2, 5)$, which is permissible by Theorem 8. Removing α_5 forms a tree, which is permissible by Corollary 5. Upon removing α_3 or α_8 from H' we obtain C_4 with three paths of length 1 or 2 attached. Such graphs are permissible by Theorem 3 and Corollary 5. Removing α_4 or α_9 results in a tree attached to two copies of C_4 that share a vertex. Two copies of C_4 glued at one vertex is an induced subgraph of $C_4 \square P_2$, which is permissible by Theorem 3. Appending a tree is then permissible by Corollary 5. It follows that all proper induced subgraphs of H' are permissible, and so H' is a minimal forbidden subgraph. □

We now construct a new infinite family of minimal forbidden subgraphs similar to the subset of $M_{n,p}$ graphs discussed in [Beier et al. 2016]. Figure 6 shows the graph $L_{n,p}$ with $n, p \geq 0$; the vertex names in the figure will be referenced in our arguments.

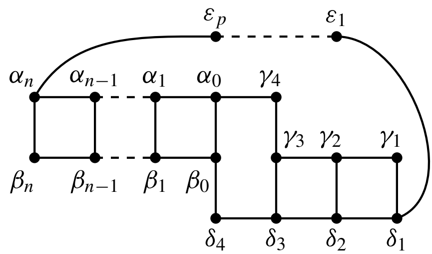


Figure 6. Vertex labels for $L_{n,p}$, where $n,\ p \geq 0$.

Lemma 11. *For all* $p \geq 1$ *and* $n \geq 0$ *with* $p+n \geq 3$, *the graph* $L_{n,p}$ *is permissible.*

Proof. We begin by arguing that $L_{2k,1}$ is an induced subgraph of $\mathcal{C}_{k+1}(I_4)$ and $L_{2k+1,1}$ is an induced subgraph of $\mathcal{C}_{k+2}(I_4)$ for $k \geq 1$. Note that we include 0 as a color. Consider I_4 with $V[I_4] = \{u, v, w, y\}$. We represent colorings of I_4 as sequences of four numbers. For instance, the sequence 1230 corresponds to the coloring α where $\alpha(u)=1$, $\alpha(v)=2$, $\alpha(w)=3$, and $\alpha(y)=0$. Figure 7 shows a set of colorings of I_4 using $k+1$ colors that span a copy of $L_{2k,1}$ in $\mathcal{C}_{k+1}(I_4)$.

For $n = 2k+1$, the construction is almost the same. Consider the colorings in Figure 7 with the following modifications. Add colorings $00k(k{+}1)$ and $10k(k{+}1)$ on the left. Change the top coloring from $01kk$ to $01k(k{+}1)$ and the rightmost colorings from $22kk$ and $21kk$ to $22k(k{+}1)$ and $21k(k{+}1)$. One can check that these colorings span a copy of $L_{2k+1,1}$ in $\mathcal{C}_{k+2}(I_4)$.

Consider $L_{n,p}$ with $p > 1$, $n \geq 0$, and $p+n \geq 3$. Then $n+p-1 \geq 2$, and hence it follows by the previous argument that $L_{n+p-1,1}$ is permissible. Removing all vertices β_i from $L_{n+p-1,1}$ with $n+1 \leq i \leq n+p-1$ yields an induced copy of $L_{n,p}$. Induced subgraphs of permissible subgraphs are permissible, so $L_{n,p}$ is permissible. □

For $n+p < 3$, the graphs $L_{n,p}$ are forbidden but not minimal. Indeed, the graph $L_{0,0}$ contains an induced copy of $T(2, 2, 4)$, the graph $L_{0,1}$ contains an induced

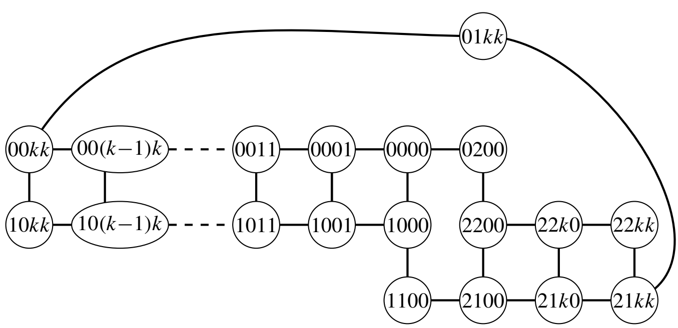


Figure 7. $L_{2k,1}$ is an induced subgraph of $\mathcal{C}_{k+1}(I_4)$.

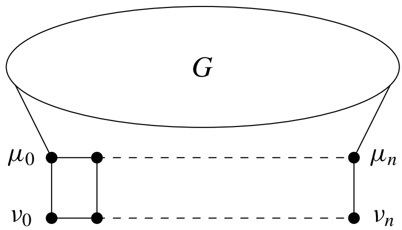


Figure 8. The graph G' described in Lemma 12.

copy of $M_{2,3}$, and the graph $L_{1,0}$ contains an induced copy of $M_{3,3}$. The graphs $L_{2,0}$, $L_{0,2}$, and $L_{1,1}$ each contain an induced copy of graph (d) in Figure 5. Our final goal is to show that $L_{n,0}$ is minimal forbidden for all $n \geq 3$, but first we need one additional lemma.

Lemma 12. *Let G be a permissible subgraph containing an induced path of length n on vertices $\mu_0, \ldots, \mu_n$. Then the graph G' with $V[G'] = V[G] \cup \{\nu_0, \ldots, \nu_n\}$ and*

$$E[G'] = E[G] \cup \{\nu_i \nu_{i+1} : 0 \leq i < n\} \cup \{\mu_i \nu_i : 0 \leq i \leq n\}$$

is also permissible. The graph G' is shown in Figure 8.

Proof. Since G is permissible, the graph $G \square P_1$ is permissible. The graph G' is an induced subgraph of $G \square P_1$. □

Theorem 13. *For $n \geq 3$, the graph $L_{n,0}$ is minimal forbidden.*

Proof. Any proper edge labeling of $L_{n,0}$ must restrict to a proper edge labeling of the central copy of C_6 spanned by α_0, β_0, γ_3, γ_4, δ_3 and δ_4. By Lemma 1, the only possible proper edge labelings of C_6 are (a) u, v, u, v, u, v or (b) u, w, v, u, w, v, where u, v, and w are distinct vertices in a base graph. Without loss of generality, assume edge $\beta_0\alpha_0$ has label u. This is illustrated in Figure 9.

In case (a), edges $\alpha_0\gamma_4$, $\delta_4\beta_0$, and $\gamma_3\delta_3$ have label v, while edges $\gamma_3\gamma_4$ and $\delta_3\delta_4$ have label u. By Lemma 1, all edges $\alpha_i\beta_i$ for $1 \leq i \leq n$ must have label u, and edges $\delta_j\gamma_j$ for $1 \leq j \leq 3$ must have label v. Furthermore u must label at least one more edge in the induced $(n+6)$-cycle highlighted in Figure 9.

Invoking Lemma 1 once again, we see that u cannot label $\alpha_{i-1}\alpha_i$ for any $1 \leq i \leq n$ or edges $\alpha_n\delta_1$ and $\gamma_2\gamma_3$ since u labels an adjacent edge in each case. Thus u can only label edge $\gamma_1\gamma_2$. This contradicts the third statement in Lemma 1 since the cycle under consideration has size greater than 4. We conclude that a proper edge labeling of $L_{n,0}$ restricting to the labeling of C_6 in case (a) does not exist.

In case (b) by Lemma 1, the edges $\alpha_i\beta_i$ for $1 \leq i \leq n$ and $\gamma_j\delta_j$ for $1 \leq j \leq 3$ must have label u. With these forced edge labelings, the $(n+6)$-cycle shown in Figure 9 does not have a proper edge labeling satisfying the conditions of Lemma 1

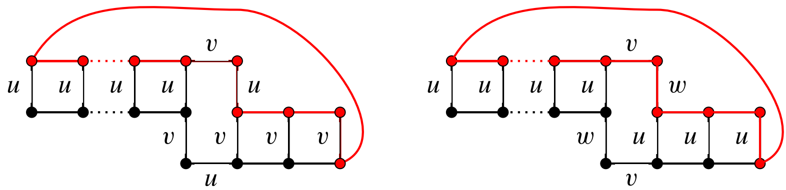


Figure 9. Possible edge labelings for $L_{n,0}$ from the proof of Theorem 13, case (a) on the left and case (b) on the right.

since u cannot label two edges. We conclude that a proper edge labeling of $L_{n,0}$ restricting to the labeling of C_6 in case (b) does not exist. Since no proper edge labeling of $L_{n,0}$ restricts to a proper edge labeling of the central copy of C_6, we conclude that $L_{n,0}$ has no proper edge labeling and is therefore forbidden.

We now check that each induced subgraph of $L_{n,0}$ spanned by all but one vertex is permissible. We refer to the vertex labels of $L_{n,0}$ shown in Figure 6. There are seven cases.

Case 1: Removing δ_1 or α_n yields a 6-cycle with two disjoint chains of 4-cycles, one of which has an attached copy of P_1. Recall that 6-cycles are permissible by Corollary 6 and attaching paths to permissible subgraphs yields new permissible subgraphs by Corollary 4. Finally, note that by inductively applying Lemma 12 to a path of length 1, one can append a chain of 4-cycles to a permissible subgraph to obtain another permissible subgraph.

Case 2: Removing β_0, α_0, δ_3, δ_4, γ_3, or γ_4 yields a proper induced subgraph of $M_{n+4,0}$ possibly with paths attached. The graph $M_{n+4,0}$ is permissible by Theorem 9, so every induced subgraph is also permissible. Once again, Corollary 4 allows us to attach a paths.

Case 3: Removing β_i for $1 \le i \le n$ yields a copy of P_1 attached to $L_{i-1,n-i+1}$ with a chain of 4-cycles attached along a path of length $n-i$, as in Lemma 12. Since $n \ge 3$ and $i \ge 1$, it follows that $(i-1)+(n-i+1) \ge 3$. Thus by Section 3 we see that $L_{i-1,n-i+1}$ is permissible.

The remaining cases are proven with explicit constructions. In the figures, strings of lengths 3 and 4 represent colorings of I_3 and I_4. If n is even, we say $n = 2k$, and if n is odd, say $n = 2k+1$. Since $n \ge 3$, we have $k \ge 1$ for n even or odd.

Case 4: For $n \ge 3$, the graph spanned by all but vertex δ_2 of $L_{n,0}$ is an induced subgraph of $\mathcal{C}_{k+3}(I_3)$, as is shown in Figure 10 for n even and Figure 11 for n odd.

Case 5: If n is odd, the subgraph of $L_{n,0}$ spanned by all but vertex γ_1 is an induced subgraph of $\mathcal{C}_{k+4}(I_3)$, as is shown in Figure 12. For n even, the subgraph of $L_{n,0}$ spanned by all but vertex γ_1 is an induced subgraph of $\mathcal{C}_{k+2}(I_4)$, as is shown in Figure 13.

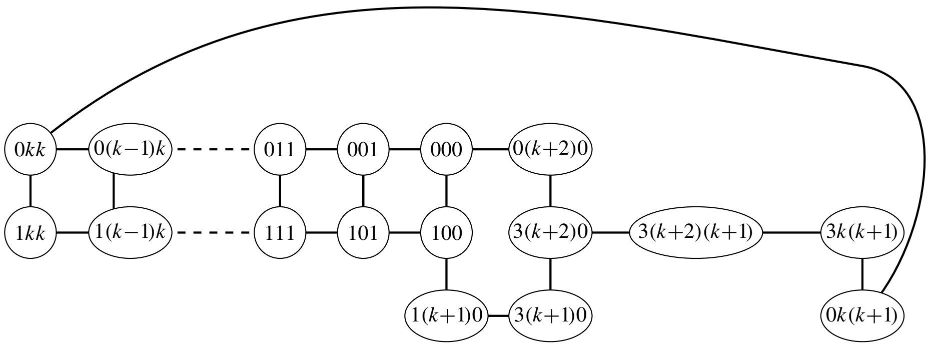


Figure 10. Case 4 for n even.

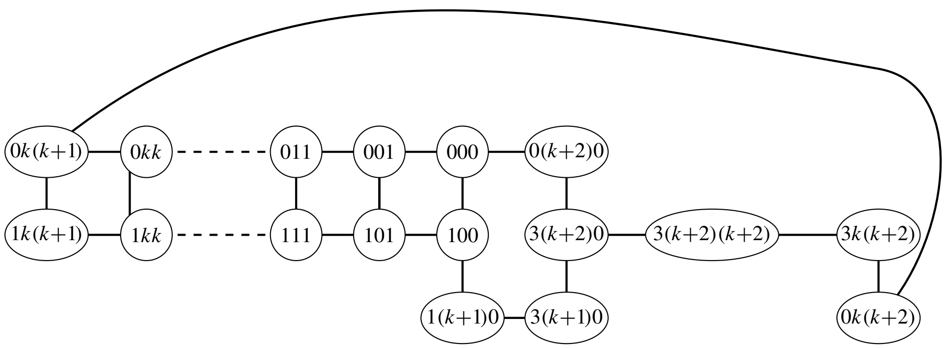


Figure 11. Case 4 for n odd.

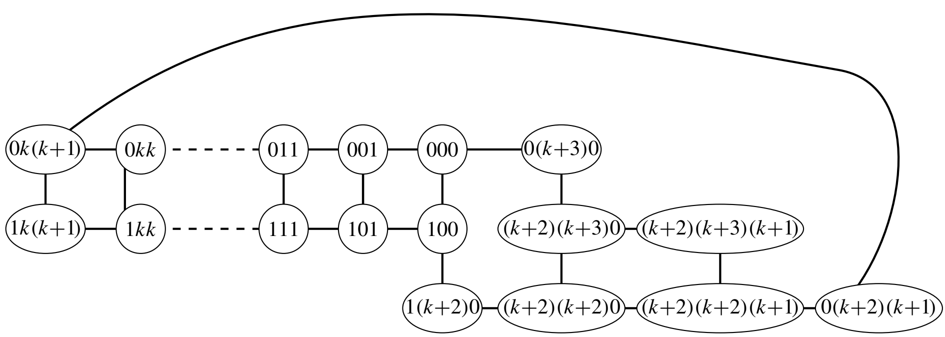


Figure 12. Case 5 for n odd.

Case 6: The graph spanned by all but vertex γ_1 of $L_{n,0}$ is the result of removing one vertex from one of the graphs in Figures 12 and 13 (depending on parity of n) and attaching a copy of P_1. By the previous case and Corollary 4, this is permissible.

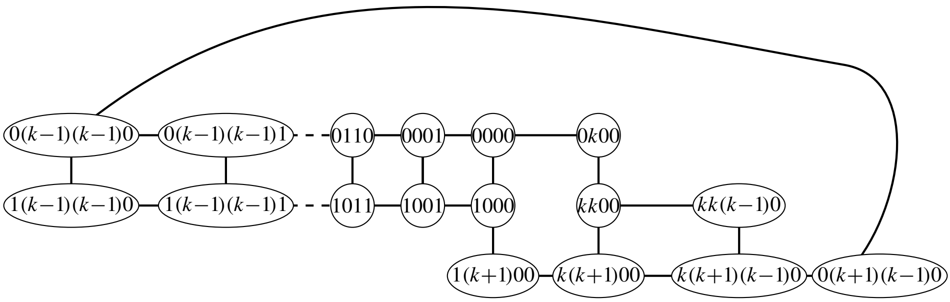


Figure 13. Case 5 for n even.

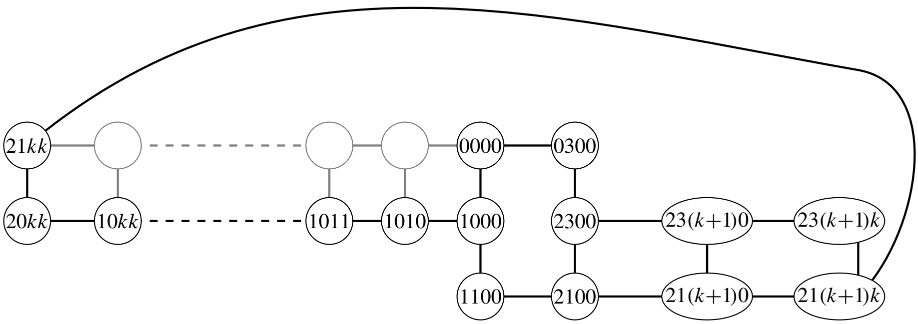


Figure 14. Case 7 for n odd.

Case 7: In Figure 14, the colors of the last two vertices are alternately incremented as we move to the left away from the central 6-cycle until the very last step where the color of the first vertex is changed. When a vertex α_i for $1 \le i \le n$ is removed from $L_{n,0}$, there will be one missing vertex among the empty vertices shown in Figure 14. Vertices in positions α_j for $j > i$ should be assigned colorings by changing the second vertex in the coloring below from color 0 to color 1. For the vertices in positions α_j with $j < i$, the colorings should differ from the ones below by changing the first vertex from color from 1 to 0. One can check that the colorings shown in Figure 14, together with the ones just described, span $L_{n,0}$ with vertex α_i removed inside of $\mathcal{C}_{k+2}(I_4)$. The same concept works for n even. The labels are shown in Figure 15. □

4. Future directions

Given time and creativity, it seems certain one could find many other examples of minimal forbidden subgraphs of coloring graphs, but there are several other interesting directions one could explore related to this research. This paper builds on

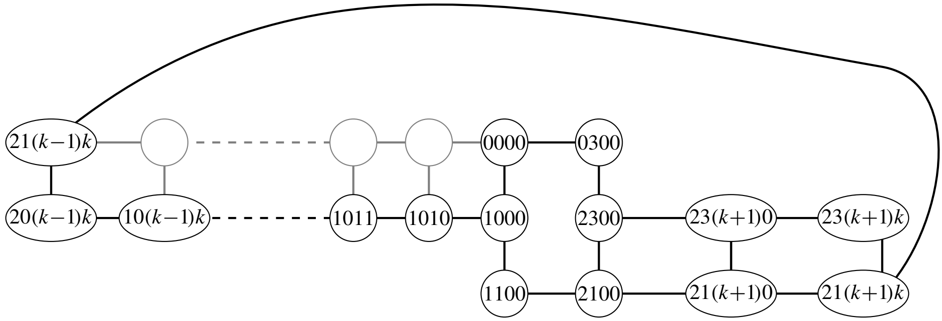


Figure 15. Case 7 for n even.

[Beier et al. 2016] to provide a template for showing graphs are minimal forbidden, but are there other less brute-force ways to show graphs are minimal forbidden?

Coloring edge labelings are still not completely understood. We provide some necessary conditions for edge labelings of cycles to be proper, but are there others? Are there sufficient conditions for an edge labeling of a cycle to be proper? Closely related to this, can we find a simple method for determining when a partition is an edge labeling partition? Finally, coloring graphs are a particular type of transition graph. To what extent will the methods presented here apply to other types of transition graphs? Other examples of transition graphs can be found in [Cohen and Teicher 2014; Zhang et al. 1988; Haas 2012; Mohar 2007].

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