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THE ESHELBY TENSOR IN NONLOCAL ELASTICITY AND IN NONLOCAL MICROPOLAR ELASTICITY

MARKUS LAZAR AND HELMUT O. K. KIRCHNER

The Eshelby tensor is formulated for anisotropic linear nonlocal elasticity and nonlocal micropolar elasticity in a nonhomogeneous medium. The divergence of this tensor gives the configurational forces on geometric and physical defects in such a medium. Some examples of the Peach–Koehler force and the Mathison–Papapetrou force between dislocations and/or disclinations are given.

1. Introduction

We consider anisotropic nonlocal elasticity and anisotropic nonlocal micropolar elasticity for a medium of arbitrary inhomogeneity. Such nonlocal theories can predict dispersion relations in the entire Brillouin zone; they suppress nonphysical singularities: crack tip singularities do not occur, and the stresses of dislocations are finite [Eringen 2002]. These results are features of linear nonlocal theories which cannot be obtained in linear elasticity and linear micropolar elasticity. They agree very well with those predicted by atomistic theories and experiments.

The aim of this paper is to derive the Eshelby tensor [Eshelby 1951;1975] in the theories of nonlocal elasticity and nonlocal micropolar elasticity. This represents a vast generalization of this tensor written for local, linear elasticity by [Morse and Feshbach 1953]. The Eshelby tensor, which is the static energy-momentum tensor, is of fundamental importance in any field theory, and in particular in the field theory of generalized elasticity. The divergence of the Eshelby tensor gives the configurational forces on the sources of the field. Few results are known about the Peach–Koehler force and conservation laws in nonlocal elasticity [Kovács and Vörös 1979; Vukobrat and Kuzmanović 1992; Lazar 2005]. This is one motivation for the investigations in the present paper. We will derive all configurational forces felt by topological defects (dislocations and disclinations), physical sources (body force, body moment) and all others due to inhomogeneities in nonlocal elasticity and nonlocal micropolar elasticity. We calculate the J-integral for these nonlocal theories, relevant in fracture mechanics of nonlocal materials. In addition, we will present some examples of interaction forces between dislocations as well as disclinations in nonlocal theories.

Keywords: Eshelby tensor, J-integral, nonlocal elasticity.

2. Nonlocal elasticity

The goal of this section is the construction of the Eshelby tensor and the related configurational forces for nonlocal elasticity. In nonlocal elasticity the elastic energy is given by [Kröner and Datta 1966]

$$W = \frac{1}{2} \iint C_{ijkl}(\boldsymbol{x}, \boldsymbol{x}')\beta_{ij}(\boldsymbol{x})\beta_{kl}(\boldsymbol{x}')\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{x}', \tag{2.1}$$

where $C_{ijkl}(\boldsymbol{x}, \boldsymbol{x}')$ is the tensor of nonlocal elastic constants and $\beta_{ij}(\boldsymbol{x})$ denotes the elastic distortion. For simplicity, we assume a linear relationship. The nonlocal constitutive law for full anisotropy reads:

$$t_{ij}(\boldsymbol{x}) = \int C_{ijkl}(\boldsymbol{x}, \boldsymbol{x}')\beta_{kl}(\boldsymbol{x}')\,\mathrm{d}^3\boldsymbol{x}'. \tag{2.2}$$

The tensor of nonlocal elastic constants possesses the symmetry

$$C_{ijkl}(\boldsymbol{x}, \boldsymbol{x}') = C_{klij}(\boldsymbol{x}', \boldsymbol{x}). \tag{2.3}$$

The equilibrium condition is given by

$$\partial_j t_{ij}(\boldsymbol{x}) + f_i(\boldsymbol{x}) = 0, \tag{2.4}$$

where $f(\boldsymbol{x})$ denotes the body force in nonlocal elasticity. The incompatibility condition reads

$$\epsilon_{jkl}\partial_k\beta_{il}(\boldsymbol{x}) = \alpha_{ij}(\boldsymbol{x}). \tag{2.5}$$

Here α_{ij} is the dislocation density tensor, divergence free in the second index. The field (2.4) and the incompatibility condition (2.5) have the same form as in local elasticity; the generalization to nonlocal elasticity occurs through Hooke's law (2.2). By multiplying Equation (2.5) with ϵ_{mnj} one finds for the elastic distortion

$$\partial_m\beta_{in}(\boldsymbol{x}) - \partial_n\beta_{im}(\boldsymbol{x}) = \epsilon_{mnj}\alpha_{ij}(\boldsymbol{x}). \tag{2.6}$$

If no dislocations are present, the elastic distortion is just the gradient of a displacement $u_i(\boldsymbol{x})$: $\beta_{ij}(\boldsymbol{x}) = \partial_j u_i(\boldsymbol{x})$.

Following the procedure of [Kirchner 1999], we construct the Eshelby (or static energy-momentum) tensor for nonhomogeneous nonlocal elasticity. Let us take an arbitrary infinitesimal functional derivative δW of the elastic energy density. From Equation (2.1) we get

$$\delta W = \frac{1}{2} \iint \big\{C_{ijkl}(\boldsymbol{x}, \boldsymbol{x}')[\delta\beta_{ij}(\boldsymbol{x})]\beta_{kl}(\boldsymbol{x}') + C_{ijkl}(\boldsymbol{x}, \boldsymbol{x}')\beta_{ij}(\boldsymbol{x})[\delta\beta_{kl}(\boldsymbol{x}')] + [\delta C_{ijkl}(\boldsymbol{x}, \boldsymbol{x}')]\beta_{ij}(\boldsymbol{x})\beta_{kl}(\boldsymbol{x}')\big\}\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{x}'. \tag{2.7}$$

Using the symmetry (2.3) and Hooke's law (2.2) for nonlocality, there remains

$$\delta W = \int t_{ij}(\boldsymbol{x})[\delta\beta_{ij}(\boldsymbol{x})]\,\mathrm{d}^3\boldsymbol{x} + \frac{1}{2}\iint \beta_{ij}(\boldsymbol{x})[\delta C_{ijkl}(\boldsymbol{x},\boldsymbol{x}')]\beta_{kl}(\boldsymbol{x}')\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{x}'. \quad (2.8)$$

Since we want to obtain configurational forces, we specify the functional derivative to be translational:

$$\delta = (\delta x_k)\partial_k. \quad (2.9)$$

On the left hand side of Equation (2.7) we write

$$\begin{aligned}\delta W &= \int \delta w(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x}\\ &= \int [\partial_k w(\boldsymbol{x})](\delta x_k)\,\mathrm{d}^3\boldsymbol{x}\\ &= \int \partial_i[w(\boldsymbol{x})\delta_{ik}](\delta x_k)\,\mathrm{d}^3\boldsymbol{x}, \end{aligned} \quad (2.10)$$

with the energy density

$$w(\boldsymbol{x}) = \tfrac{1}{2}\, t_{ij}(\boldsymbol{x})\beta_{ij}(\boldsymbol{x}). \quad (2.11)$$

On the right hand side of Equation (2.7) we obtain with (2.3)

$$\begin{aligned}\delta W = \int &\left\{t_{ij}(\boldsymbol{x})[\partial_k\beta_{ij}(\boldsymbol{x}) - \partial_j\beta_{ik}(\boldsymbol{x})] + t_{ij}(\boldsymbol{x})[\partial_j\beta_{ik}(\boldsymbol{x})]\right\}(\delta x_k)\,\mathrm{d}^3\boldsymbol{x}\\ &+ \frac{1}{2}\iint \beta_{ij}(\boldsymbol{x})[\partial_k C_{ijmn}(\boldsymbol{x},\boldsymbol{x}')]\beta_{mn}(\boldsymbol{x}')(\delta x_k)\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{x}', \end{aligned} \quad (2.12)$$

where the second and third terms have been subtracted and added. The purpose is to obtain the square bracket with the meaning of Equation (2.6). The third term may be written with (2.4) as

$$\begin{aligned} t_{ij}(\boldsymbol{x})[\partial_j\beta_{ik}(\boldsymbol{x})] &= \partial_j[t_{ij}(\boldsymbol{x})\beta_{ik}(\boldsymbol{x})] - [\partial_j t_{ij}(\boldsymbol{x})]\beta_{ik}(\boldsymbol{x})\\ &= \partial_j[t_{ij}(\boldsymbol{x})\beta_{ik}(\boldsymbol{x})] + f_i(\boldsymbol{x})\beta_{ik}(\boldsymbol{x}). \end{aligned} \quad (2.13)$$

By equating (2.10) and (2.12), using Equations (2.11) and (2.13) we obtain the expression

$$\begin{aligned}\int \partial_i\big(w(\boldsymbol{x})\delta_{ik} - t_{li}(\boldsymbol{x})\beta_{lk}(\boldsymbol{x})\big)\mathrm{d}^3\boldsymbol{x} &= \int\bigg(\epsilon_{kjl}t_{ij}(\boldsymbol{x})\alpha_{il}(\boldsymbol{x}) + f_i(\boldsymbol{x})\beta_{ik}(\boldsymbol{x})\\ &\qquad + \frac{1}{2}\int \beta_{ij}(\boldsymbol{x})[\partial_k C_{ijmn}(\boldsymbol{x},\boldsymbol{x}')]\beta_{mn}(\boldsymbol{x}')\mathrm{d}^3\boldsymbol{x}'\bigg)\mathrm{d}^3\boldsymbol{x}\\ &= J_k. \end{aligned} \quad (2.14)$$

The second integral contains the sources of the elastic fields: the dislocation density, the body force and the inhomogeneity of the material. The integrand of the first integral in Equation (2.14) is the divergence of the Eshelby tensor of nonlocal elasticity

$$P_{ki}(\boldsymbol{x}) = \big[w(\boldsymbol{x})\delta_{ik} - t_{li}(\boldsymbol{x})\beta_{lk}(\boldsymbol{x})\big]. \tag{2.15}$$

It may be transformed into a surface integral

$$J_k = \int P_{ki}(\boldsymbol{x}) n_i \,\mathrm{d}^2\boldsymbol{x}. \tag{2.16}$$

Equation (2.16) is the J-integral in nonlocal elasticity. Notice that in terms of energy, stresses and distortions, it is of the same form as in local elasticity. This is because the field (2.4) and the incompatibility condition (2.5) have the same form in local and nonlocal elasticity. The configurational force density is the divergence of the Eshelby tensor

$$\partial_i P_{ki} = F_k \tag{2.17}$$

with

$$F_k = \epsilon_{kjl} t_{ij}(\boldsymbol{x})\alpha_{il}(\boldsymbol{x}) + f_i(\boldsymbol{x})\beta_{ik}(\boldsymbol{x}) + \frac{1}{2}\int \beta_{ij}(\boldsymbol{x})[\partial_k C_{ijmn}(\boldsymbol{x},\boldsymbol{x}')]\beta_{mn}(\boldsymbol{x}')\,\mathrm{d}^3\boldsymbol{x}'. \tag{2.18}$$

The first term is the configurational force on a dislocation density like the Peach–Koehler force in local elasticity [Peach and Koehler 1950]. We have obtained the Peach–Koehler force generalized to nonlocal elasticity. The second term is the configurational force on a body force $f_i(\boldsymbol{x})$ in presence of an elastic distortion $\beta_{ik}(\boldsymbol{x})$—it is the nonlocal generalization of the Cherepanov force [Cherepanov 1981]. The third term is the material force on the inhomogeneity $\partial_k C_{ijmn}(\boldsymbol{x},\boldsymbol{x}')$ in nonlocal elasticity—the nonlocal generalization of the Eshelby force [Eshelby 1951].

For a homogeneous defect-free and source-free material the Eshelby tensor (2.15) reduces to

$$P_{ki}(\boldsymbol{x}) = \big[w(\boldsymbol{x})\delta_{ik} - t_{li}(\boldsymbol{x})\partial_k u_l(\boldsymbol{x})\big], \tag{2.19}$$

which is divergenceless. Then the J-integral (2.16) is zero.

On the other hand, if we use the dislocation density of a single straight dislocation

$$\alpha_{ij} = b_i' n_j\,\delta(x - x')\delta(y - y'), \tag{2.20}$$

we obtain the expression for the Peach–Koehler force in nonlocal elasticity as follows:

$$F_k^{\mathrm{PK}} = \epsilon_{kjl} b_i' n_l t_{ij}. \tag{2.21}$$

Here b_i is the Burgers vector and n_j the tangent line element of the dislocation, in agreement with the formula given by [Kovács and Vörös 1979]. These authors did not use the concept of the Eshelby tensor and configurational force in their calculation. They gave just a formal derivation of Equation (2.21).

From invariance arguments it follows that in an isotropic nonlocal medium the tensor of nonlocal elastic moduli must be of the form

$$C_{ijkl}(\boldsymbol{x}, \boldsymbol{x}') = \big\{\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\big\} G(|\boldsymbol{x} - \boldsymbol{x}'|), \tag{2.22}$$

where $G(|\boldsymbol{x} - \boldsymbol{x}'|)$ is called the nonlocal kernel [Eringen 2002].

In the following, we choose the two-dimensional nonlocal kernel (see, for example, [Eringen 2002])

$$G(|\boldsymbol{x} - \boldsymbol{x}'|) = \frac{1}{2\pi\varepsilon^2} K_0\left(\frac{\sqrt{(x-x')^2 + (y-y')^2}}{\varepsilon}\right), \quad \varepsilon \geq 0, \tag{2.23}$$

which is the Green function of the two-dimensional Helmholtz-equation and ε is the parameter of nonlocality. Here K_n denotes the modified Bessel function of the second kind and n is the order of this function.

With Equation (2.22) we obtain for the Peach–Koehler force of two parallel screw dislocations ($n_z = 1$)

$$F_r^{\mathrm{PK}} = b_z' t_{z\varphi}, \tag{2.24}$$

where

$$t_{z\varphi} = \frac{\mu b_z}{2\pi} \frac{1}{r} \{1 - \frac{r}{\varepsilon} K_1(r/\varepsilon)\}, \tag{2.25}$$

with $r = \sqrt{x^2 + y^2}$. Due to the nonlocal theory the $1/r$-singularity has disappeared. This force is zero at $r = 0$. It has an extremum value of $0.399\mu b_z' b_z/[2\pi\varepsilon]$ at $r \simeq 1.114\varepsilon$ (see Figure 1).

The Peach–Koehler force between two parallel edge dislocations has been found by [Lazar 2005] in the framework of nonlocal elasticity. Unlike in classical elasticity, both for screw and edge dislocations the Peach–Koehler forces are finite and nonsingular in nonlocal elasticity.

3. Nonlocal micropolar elasticity

The aim of this section is to derive the Eshelby tensor and the corresponding configurational forces for nonlocal micropolar elasticity, another generalization of

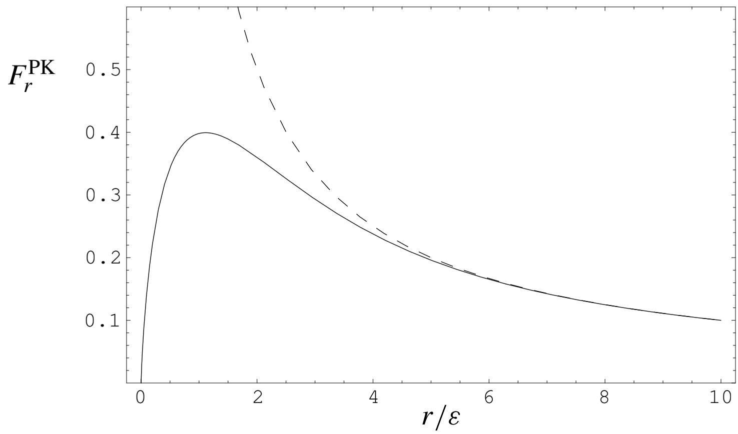


Figure 1. Peach–Koehler force $F_r^{\rm PK}$ between two screw dislocations. $F_r^{\rm PK}$ is given in units of $\mu b'_z b_z/[2\pi\varepsilon]$. The dashed curve represents the classical result.

classical elasticity. For linear anisotropic nonlocal micropolar elasticity, the strain energy density is given as follows [Eringen 2002]

$$W = \frac{1}{2}\iint \big\{\gamma_{ij}(\boldsymbol{x})\, C_{ijkl}(\boldsymbol{x},\boldsymbol{x}')\gamma_{kl}(\boldsymbol{x}') + \kappa_{ij}(\boldsymbol{x})\, A_{ijkl}(\boldsymbol{x},\boldsymbol{x}')\kappa_{kl}(\boldsymbol{x}') + 2\gamma_{ij}(\boldsymbol{x})\, B_{ijkl}(\boldsymbol{x},\boldsymbol{x}')\kappa_{kl}(\boldsymbol{x}')\big\}\, \mathrm{d}^3\boldsymbol{x}\, \mathrm{d}^3\boldsymbol{x}', \quad (3.1)$$

where $\gamma_{ij}(\boldsymbol{x})$ and $\kappa_{ij}(\boldsymbol{x})$ denote the relative distortion tensor and the wryness tensor, respectively. The nonlocal constitutive moduli possess the symmetries

$$A_{ijkl}(\boldsymbol{x},\boldsymbol{x}') = A_{klij}(\boldsymbol{x}',\boldsymbol{x}), \qquad C_{ijkl}(\boldsymbol{x},\boldsymbol{x}') = C_{klij}(\boldsymbol{x}',\boldsymbol{x}). \quad (3.2)$$

In nonlocal micropolar elasticity, the force stress tensor $t_{ij}(\boldsymbol{x})$ and the couple stress tensor $m_{ij}(\boldsymbol{x})$ are given in integral form by the nonlocal constitutive relations:

$$t_{ij}(\boldsymbol{x}) = \int \big\{C_{ijkl}(\boldsymbol{x},\boldsymbol{x}')\gamma_{kl}(\boldsymbol{x}') + B_{ijkl}(\boldsymbol{x},\boldsymbol{x}')\kappa_{kl}(\boldsymbol{x}')\big\}\, \mathrm{d}^3\boldsymbol{x}', \quad (3.3)$$

$$m_{ij}(\boldsymbol{x}) = \int \big\{B_{klij}(\boldsymbol{x},\boldsymbol{x}')\gamma_{kl}(\boldsymbol{x}') + A_{ijkl}(\boldsymbol{x},\boldsymbol{x}')\kappa_{kl}(\boldsymbol{x}')\big\}\, \mathrm{d}^3\boldsymbol{x}'. \quad (3.4)$$

The force and the moment equilibrium conditions read

$$\partial_j t_{ij}(\boldsymbol{x}) + f_i(\boldsymbol{x}) = 0, \quad (3.5)$$

$$\partial_j m_{ij}(\boldsymbol{x}) - \epsilon_{ijk} t_{jk}(\boldsymbol{x}) + l_i(\boldsymbol{x}) = 0, \quad (3.6)$$

where $f_i(\boldsymbol{x})$ and $l_i(\boldsymbol{x})$ are the body force and the body couple, respectively. The incompatibility conditions in micropolar elasticity [Eringen 1999] are the definitions for the dislocation density tensor $\alpha_{ij}(\boldsymbol{x})$ and the disclination density tensor $\Theta_{ij}(\boldsymbol{x})$:

$$\epsilon_{jkl}\,[\partial_k \gamma_{il}(\boldsymbol{x}) + \epsilon_{ikm}\kappa_{ml}(\boldsymbol{x})] = \alpha_{ij}(\boldsymbol{x}), \tag{3.7}$$

$$\epsilon_{jkl}\partial_k \kappa_{il}(\boldsymbol{x}) = \Theta_{ij}(\boldsymbol{x}). \tag{3.8}$$

Again, the form of Equations (3.5)–(3.8) is the same as in local micropolar elasticity. If no dislocations and disclinations are present, the micropolar strain quantities are of the form: $\gamma_{ij}(\boldsymbol{x}) = \partial_j u_i(\boldsymbol{x}) + \epsilon_{ijk}\varphi_k(\boldsymbol{x})$ and $\kappa_{ij}(\boldsymbol{x}) = \partial_j \varphi_i(\boldsymbol{x})$. Here $\varphi_k(\boldsymbol{x})$ denotes the micro-rotation.

Using the same procedure for the calculation of the Eshelby tensor in nonlocal micropolar elasticity as in Section 2 for the Eshelby tensor in nonlocal elasticity, we obtain

$$P_{ki}(\boldsymbol{x}) = \big[w(\boldsymbol{x})\delta_{ik} - t_{li}(\boldsymbol{x})\,\bar{\gamma}_{lk}(\boldsymbol{x}) - m_{li}(\boldsymbol{x})\kappa_{lk}(\boldsymbol{x})\big], \tag{3.9}$$

where $\bar{\gamma}_{lk} = \gamma_{lk} - \epsilon_{lkm}\varphi_m$ and

$$w(\boldsymbol{x}) = \tfrac{1}{2}\,t_{ij}(\boldsymbol{x})\gamma_{ij}(\boldsymbol{x}) + \tfrac{1}{2}\,m_{ij}(\boldsymbol{x})\kappa_{ij}(\boldsymbol{x}). \tag{3.10}$$

Equation (3.9) is the Eshelby tensor for nonlocal micropolar elasticity. Using the Noether theorem, in fact the translational invariance, it is the generalization of the Eshelby tensor for micropolar elasticity given by [Kluge 1969] to nonlocality and, on the other hand, it is the generalization of the Eshelby tensor for nonlocal elasticity derived in Section 2 to micropolarity. With Equation (3.9) we obtain a surface integral

$$J_k = \int P_{ki}(\boldsymbol{x}) n_i \,\mathrm{d}^2\boldsymbol{x}. \tag{3.11}$$

Equation (2.16) is the J-integral in nonlocal micropolar elasticity. The divergence of the Eshelby tensor (3.9) gives the configurational force density:

$$F_k(\boldsymbol{x}) = \partial_i P_{ki}(\boldsymbol{x}), \tag{3.12}$$

with

$$\begin{aligned} F_k(\boldsymbol{x}) = {} & \epsilon_{kjl} t_{ij}(\boldsymbol{x})\alpha_{il}(\boldsymbol{x}) + \epsilon_{kjl} m_{ij}(\boldsymbol{x})\Theta_{il}(\boldsymbol{x}) \\ & - \epsilon_{kjl} t_{ji}(\boldsymbol{x})\kappa^{\mathrm{P}}_{li}(\boldsymbol{x}) + f_i(\boldsymbol{x})\,\bar{\gamma}_{ik}(\boldsymbol{x}) + l_i(\boldsymbol{x})\kappa_{ik}(\boldsymbol{x}) \\ & + \tfrac{1}{2}\int \big\{\gamma_{ij}(\boldsymbol{x})[\partial_k C_{ijmn}(\boldsymbol{x},\boldsymbol{x}')]\gamma_{mn}(\boldsymbol{x}') + \kappa_{ij}(\boldsymbol{x})[\partial_k A_{ijmn}(\boldsymbol{x},\boldsymbol{x}')]\kappa_{mn}(\boldsymbol{x}') \\ & \qquad + 2\gamma_{ij}(\boldsymbol{x})[\partial_k B_{ijmn}(\boldsymbol{x},\boldsymbol{x}')]\kappa_{mn}(\boldsymbol{x}')\big\}\,\mathrm{d}^3\boldsymbol{x}'. \end{aligned} \tag{3.13}$$

It can be seen that Equation (3.13) is a sum of several configurational force densities in nonlocal micropolar elasticity:

(i) the Peach–Koehler force density on a dislocation density $\alpha_{il}(\boldsymbol{x})$ in the presence of the force stress $t_{ij}(\boldsymbol{x})$ [Kluge 1969];

(ii) the force density on a disclination density $\Theta_{il}(\boldsymbol{x})$ in the presence of the couple stress $m_{ij}(\boldsymbol{x})$, which is called a generalized Mathisson–Papapetrou type force density [Gairola 1981; Maugin 1993; Hehl et al. 1995];

(iii) a Cherepanov force density on a body force $f_i(\boldsymbol{x})$ in the presence of a distortion $\bar{\gamma}_{ik}(\boldsymbol{x})$;

(iv) a force density on a body couple $l_i(\boldsymbol{x})$ in presence of the elastic wryness $\kappa_{ik}(\boldsymbol{x})$;

(v) a force density on the force stress $t_{ji}(\boldsymbol{x})$ in presence of the plastic wryness $\kappa^{\mathrm{P}}_{li}(\boldsymbol{x})$;

(vi) three force densities on inhomogeneities: $\partial_k C_{ijmn}(\boldsymbol{x}, \boldsymbol{x}')$, $\partial_k A_{ijmn}(\boldsymbol{x}, \boldsymbol{x}')$ and $\partial_k B_{ijmn}(\boldsymbol{x}, \boldsymbol{x}')$.

For a homogeneous defect-free and source-free micropolar material, the Eshelby tensor (3.9) simplifies to

$$P_{ki}(\boldsymbol{x}) = \big[w(\boldsymbol{x})\delta_{ik} - t_{li}(\boldsymbol{x})\partial_k u_l(\boldsymbol{x}) - m_{li}(\boldsymbol{x})\partial_k \varphi_l(\boldsymbol{x})\big], \tag{3.14}$$

which is divergenceless. Then the J-integral (3.11) is zero. The formula (3.14) is the nonlocal generalization of the Eshelby tensor for micropolar elasticity given by [Lubarda and Markenscoff 2003]. The corresponding Eshelby tensor for finite local polar elasticity has been given by [Maugin 1998].

If we use the dislocation density tensor of a straight dislocation and the disclination density tensor of a straight disclination

$$\alpha_{ij} = b'_i n_j\, \delta(x - x')\delta(y - y'), \tag{3.15}$$

$$\Theta_{ij} = \Omega'_i n_j\, \delta(x - x')\delta(y - y'), \tag{3.16}$$

we obtain for the Peach–Koehler force and the Mathisson–Papapetrou force, respectively,

$$F^{\mathrm{PK}}_k = \epsilon_{kjl} b'_i n_l t_{ij}, \tag{3.17}$$

$$F^{\mathrm{MP}}_k = \epsilon_{kjl} \Omega'_i n_l m_{ij}. \tag{3.18}$$

Here Ω_i denotes the Frank vector (the topological charge of a disclination).

For isotropic nonlocal micropolar elasticity the nonlocal elastic moduli must be of the form

$$C_{ijkl}(\boldsymbol{x},\boldsymbol{x}')=\big\{\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\mu_c(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})\big\}G(|\boldsymbol{x}-\boldsymbol{x}'|), \quad (3.19)$$

$$A_{ijkl}(\boldsymbol{x},\boldsymbol{x}')=\big\{\alpha\delta_{ij}\delta_{kl}+\beta(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\gamma(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})\big\}G(|\boldsymbol{x}-\boldsymbol{x}'|), \quad (3.20)$$

$$B_{ijkl}(\boldsymbol{x},\boldsymbol{x}')=0, \quad (3.21)$$

in terms of six material constants λ, μ, μ_c, α, β, γ, characteristic for the medium under consideration. Again, $G(|\boldsymbol{x}-\boldsymbol{x}'|)$ is the nonlocal kernel. If we use the six material constants of micropolar elasticity, two characteristic lengths l and h can be defined by [Nowacki 1986]

$$l^2=\frac{(\mu+\mu_c)(\beta+\gamma)}{4\mu\,\mu_c}, \qquad h^2=\frac{\alpha+2\beta}{4\mu_c}. \quad (3.22)$$

In the following, we use the two-dimensional nonlocal kernel (2.23). Then the Peach–Koehler force for two parallel screw dislocations in a micropolar medium is

$$F_r^{\mathrm{PK}}=b_z' t_{z\varphi}, \quad (3.23)$$

with [Lazar et al. 2005]

$$t_{z\varphi}=\frac{b_z}{2\pi}\,\frac{1}{r}\Big\{\mu\Big[1-\frac{r}{\varepsilon}\,K_1(r/\varepsilon)\Big]+\frac{\mu_c h^2}{h^2-\varepsilon^2}\Big[\frac{r}{h}\,K_1(r/h)-\frac{r}{\varepsilon}\,K_1(r/\varepsilon)\Big]\Big\}. \quad (3.24)$$

The force (3.23) is nonsingular. It is zero at $r=0$ and has an extremum value which depends on the coefficients ε and h (see Figure 2). In addition, it can be seen that the Peach–Koehler force between two screw dislocations in nonlocal micropolar elasticity is slightly different from the force in nonlocal elasticity (2.24).

Another interesting situation is the interaction of two parallel wedge disclinations. The Mathisson–Papapetrou force for two parallel wedge disclinations in a micropolar medium is

$$F_r^{\mathrm{MK}}=\Omega_z' m_{z\varphi}, \quad (3.25)$$

with [Lazar and Maugin 2004]

$$m_{z\varphi}=\frac{(\beta+\mu_c)\Omega_z}{2\pi}\,\frac{1}{r}\Big\{1-\frac{r}{\varepsilon}\,K_1(r/\varepsilon)\Big\}. \quad (3.26)$$

It is zero at $r=0$ and has an extremum value of $0.399(\beta+\mu_c)\Omega_z'\Omega_z/[2\pi\varepsilon]$ at $r\simeq 1.114\varepsilon$. It is similar in form to the Peach–Koehler force between two screw dislocations (2.24).

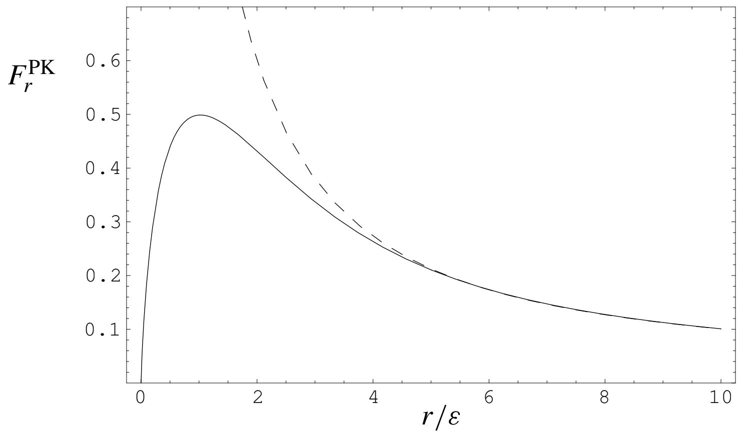


Figure 2. Peach–Koehler force F_r^{PK} between two screw dislocations in nonlocal micropolar elasticity. F_r is given in units of $\mu b_z/[2\pi\varepsilon]$ with $h = 2\varepsilon$ and $\mu = 3\eta$. The dashed curve represents the micropolar result.

The Peach–Koehler force between an edge dislocation and the force stress produced by a wedge disclination is given by

$$F_x^{\mathrm{PK}} = b'_x t_{xy}, \tag{3.27}$$

$$F_y^{\mathrm{PK}} = -b'_x t_{xx}, \tag{3.28}$$

where [Lazar and Maugin 2004]

$$t_{xx}=\frac{\mu\Omega_z}{2\pi(1-\nu)}\left\{\ln r+\frac{y^2}{r^2}+K_0(r/\varepsilon)+\frac{(x^2-y^2)\varepsilon^2}{r^4}\left(2-\frac{r^2}{\varepsilon^2}K_2(r/\varepsilon)\right)\right\}, \tag{3.29}$$

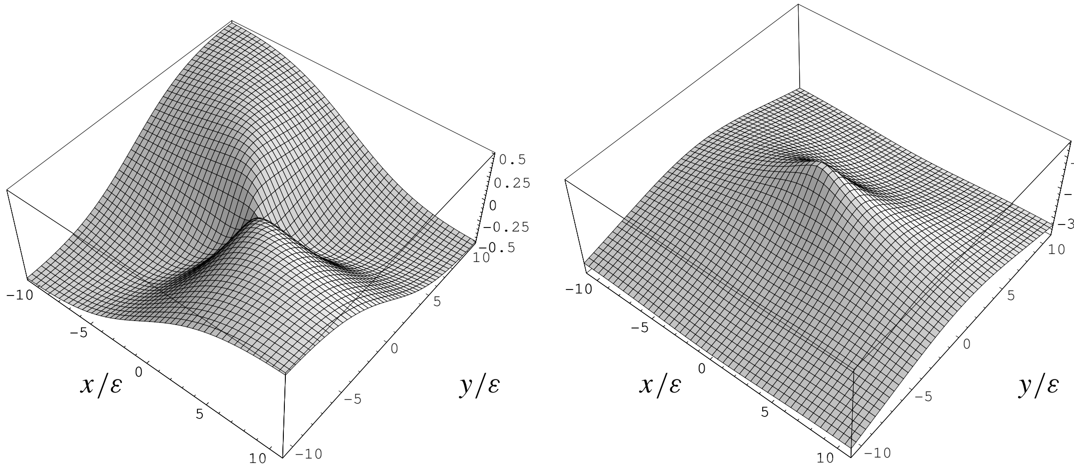
$$t_{xy}=-\frac{\mu\Omega_z}{2\pi(1-\nu)}\frac{xy}{r^2}\left\{1-\frac{2\varepsilon^2}{r^2}\left(2-\frac{r^2}{\varepsilon^2}K_2(r/\varepsilon^2)\right)\right\}. \tag{3.30}$$

F_x^{PK} is zero at $x = 0$ and $y = 0$ and has extremum values at $x = y$. On the other hand F_y^{PK} has a finite extremum at $r = 0$ (see Figure 3). F_x^{PK} is the glide force and F_y^{PK} is the climb force for the edge dislocation caused by the stress field of the wedge disclination.

The Mathisson–Papapetrou force between a wedge disclination and the couple stress produced by an edge dislocation reads

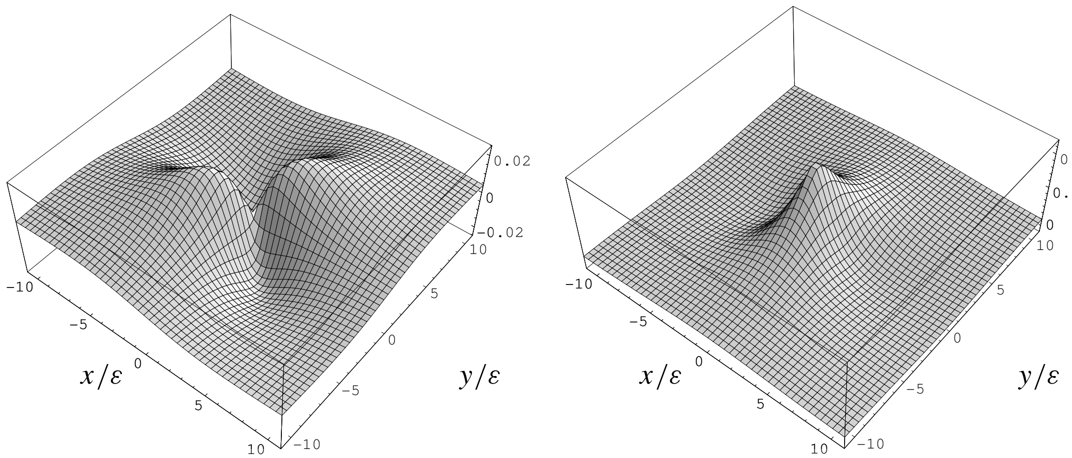
$$F_x^{\mathrm{MP}} = \Omega'_z m_{zy}, \tag{3.31}$$

$$F_y^{\mathrm{MP}} = -\Omega'_z m_{zx}, \tag{3.32}$$



(a) (b)

Figure 3. Peach–Koehler force between an edge dislocation and a wedge disclination: (a) F_x^{PK} and (b) F_y^{PK} are given in units of $\mu b'_x \Omega_z / [2\pi(1-\nu)]$.



(a) (b)

Figure 4. Mathisson–Papapetrou force between a wedge disclination and an edge dislocation: (a) F_x^{MP} and (b) F_y^{MP} are given in units of $(\beta+\gamma)\Omega_z b/[2\pi]$ with $l = 2\varepsilon$.

with the couple stress [Lazar and Maugin 2004]

$$m_{zx} = \frac{(\beta+\gamma)b_x}{2\pi}\left\{\frac{x^2-y^2}{r^4}\left(1-\frac{1}{l^2-\varepsilon^2}\Big[lr\,K_1(r/l)-\varepsilon r\,K_1(r/\varepsilon)\Big]\right) - \frac{x^2}{r^2}\,\frac{1}{l^2-\varepsilon^2}\Big[K_0(r/l)-K_0(r/\varepsilon)\Big]\right\}, \quad (3.33)$$

$$m_{zy} = \frac{(\beta+\gamma)b_x}{2\pi}\,\frac{xy}{r^4}\left\{2\Big(1 - \frac{1}{l^2-\varepsilon^2}\Big[lrK_1(r/l) - \varepsilon rK_1(r/\varepsilon)\Big]\Big) - \frac{r^2}{l^2-\varepsilon^2}\Big[K_0(r/l) - K_0(r/\varepsilon)\Big]\right\}. \quad (3.34)$$

F_x^{MP} is zero at $x = 0$ and $y = 0$ and has extremum values at $x = y$. F_y^{MP} has a finite extremum at $r = 0$ (see Figure 4).

The main feature in nonlocal micropolar elasticity is that the Peach–Koehler and the Mathisson–Papapetrou forces are nonsingular and they have finite extremum values unlike the results obtained in micropolar elasticity.

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