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In this paper, the dynamic behavior of a finite crack in functionally graded materials subjected to harmonic stress waves is investigated by means of nonlocal theory. The traditional concepts of nonlocal theory are extended to solve the dynamic fracture problem of functionally graded materials. To overcome mathematical difficulties, a one-dimensional nonlocal kernel is used instead of a two-dimensional one for the dynamic problem to obtain the stress fields near the crack tips. To make the analysis tractable, it is assumed that the shear modulus and the material density vary exponentially and vertically with respect to the crack. Using the Fourier transform and defining the jumps of the displacements across the crack surfaces as the unknown functions, two pairs of dual integral equations are derived. To solve the dual integral equations, the jumps of the displacements across the crack surfaces are expanded in a series of Jacobi polynomials. Unlike classical elasticity solutions, it is found that no stress singularities are present near crack tips. Numerical examples are provided to show the effects of the crack length, the parameter describing the functionally graded materials, the frequency of the incident waves, the lattice parameter of the materials and the material constants upon the dynamic stress fields near crack tips.

1. Introduction

A new class of engineered materials, functionally gradient materials (FGMs), has been developed primarily for use in high temperature applications [Koizumi 1993]. The composition of these FGMs, prepared using techniques like power metallurgy, chemical vapor deposition, centrifugal casting, etc., is graded along the thickness. The spatial variation of the material composition results in a medium with varying elastic and physical properties and calls for investigation into the fracture of FGMs under different loading conditions. In particular, the use of the graded material as interlayers in bonded media is one of the highly effective and promising applications to eliminate various shortcomings resulting from stepwise property

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mismatch inherent in piecewise homogeneous composite media [Lee and Erdogan 1994; Suresh and Mortensen 1977; Choi 2001].

From the viewpoint of fracture mechanics, the presence of a graded interlayer would play an important role in determining crack driving forces and fracture resistance parameters. In an attempt to address the issues pertaining to fracture analysis of bonded media with such transitional interfacial properties, a series of solutions to certain crack problems was obtained by Erdogan and his associates [Delae and Erdogan 1988; Ozturk and Erdogan 1996].

The dynamic crack problem for non-homogeneous composite materials was considered in [Wang et al. 2000] but they considered the FGM layer as a multi-layered homogeneous medium. The crack problem in FGM layers under thermal stresses was studied by Erdogan and Wu [1996]. They considered an unconstrained elastic layer under statically self-equilibrating thermal or residual stresses. More recently, the scattering of harmonic stress waves by a Mode-I crack in functionally graded materials was investigated by use of the Schmidt method in [Zhou et al. 2004]. However, it is found that all the solutions in [Koizumi 1993; Lee and Erdogan 1994; Suresh and Mortensen 1977; Choi 2001; Delae and Erdogan 1988; Ozturk and Erdogan 1996; Wang et al. 2000; Erdogan and Wu 1996; Zhou et al. 2004] contain stress singularities at the crack tips, which is not reasonable according to physical nature. As a result of this, beginning with Griffith, all fracture criteria in use today are based on other considerations, for example, energy, the J-integral [Rice 1968], and strain gradient theory [Xia and Hutchinson 1996].

To overcome the stress singularity in classical elastic fracture theory, Eringen [1977; 1978; 1979] used nonlocal theory to study the stress near tips of a sharp line crack in an isotropic elastic plate subject to uniform tension, shear, and antiplane shear, and the resulting solutions did not contain any stress singularities. This allows us to use maximum stress as a fracture criterion. Modern nonlocal continuum mechanics has originated and developed in the last four decades as an alternative to these local approaches of zero-range internal interactions. Edelen [1976] contributed some mathematical formalism while Green and Rivlin [1965] simply enunciated some postulates for nonlocal theory. On the other hand, Eringen [1976] contributed not only the complete physics and mathematics of nonlocal theory but also shaped the theory into concrete form, making it viable for practical applications to boundary value problems.

According to nonlocal theory, the stress at a point X in a body depends not only on the strain at point X but also on that at all other points of the body. This is contrary to the classical theory that the stress at a point X in a body depends only on the strain at point X. In [Pan and Takeda 1998], the basic theory of nonlocal elasticity was stated with emphasis on the difference between nonlocal theory and classical continuum mechanics. The basic idea of nonlocal elasticity is to build a relationship between macroscopic mechanical quantities and microscopic physical quantities within the framework of continuum mechanics.

The constitutive theory of nonlocal elasticity has been developed in [Edelen 1976], in which the elastic modulus is influenced by the microstructure of the material. Other results have been given by the application of nonlocal elasticity to the fields such as a dislocation near a crack [Pan 1992;1994] and fracture mechanics problems [Pan 1995; Pan and Fang 1993]. The literature on the fundamental aspects of nonlocal continuum mechanics is extensive. The results of those concrete problems that have been solved display a remarkable agreement with experimental evidence. This can be used to predict cohesive stress for various materials and the results are close to those obtained in atomic lattice dynamics [Eringen and Kim 1974;1977].

Likewise, a nonlocal study of the secondary flow of viscous fluid in a pipe furnishes a streamlined pattern similar to that obtained experimentally by Eringen [Eringen 1977]. Other examples of the effectiveness of the nonlocal approach are: (i) prediction of the dispersive character of elastic waves demonstrated experimentally (and lacking in classical theory) [Eringen 1972] and (ii) calculation of the velocity of short Love waves whose nonlocal estimates agree better with seismological observations than the local ones [Nowinski 1984b].

Several nonlocal theories have been formulated to address strain-gradient and size effects [Nowinski 1984b]. Recently, some fracture problems [Zhou et al. 1999b; 2002; Zhou and Wang 2002a] in an isotropic elastic material and piezoelectric material have been studied by use of nonlocal theory with a somewhat different method. The traditional concepts of nonlocal theory are extended to solve the fracture problem of piezoelectric materials [Zhou et al. 1999b; 2002; 2002a]. More recently, the traditional concepts of nonlocal theory have also been extended to solve the anti-plane shear fracture problem of functionally graded materials [Zhou and Wang 2006], and the results of the solution in [Zhou and Wang 2006] did not contain any stress singularity. However, to our knowledge, the effect of the lattice parameter on the dynamic stress field near the Mode-I crack tips has not been studied by use of nonlocal theory in functionally graded materials, in which the shear modulus and material density vary exponentially and vertically with respect to the crack. The present work is an attempt to fill this gap in research. Here, we attempt to give a theoretical solution for this problem.

In the present paper, the effect of the lattice parameter of functionally graded materials on dynamic stress fields near Mode-I crack tips is investigated using nonlocal theory in functionally graded materials with the Schmidt method [Morse and Feshbach 1958; Yau 1967]. When the lattice parameter of materials tends to zero, the present problem will revert to the same problem as in [Zhou et al. 2004]. To make the analysis tractable, it is assumed that the shear modulus and the material density vary exponentially and vertically with respect to the crack. To overcome the mathematical difficulties, a one-dimensional nonlocal kernel is used instead of a two-dimensional one for the dynamic problem of obtaining the stress fields near the crack tips.

The traditional concepts of nonlocal theory are extended to solve the dynamic fracture problem of functionally graded materials. The Fourier transform is applied and a mixed boundary value problem is reduced to two pairs of dual integral equations. To solve the dual integral equations, the jumps of displacements across crack surfaces are expanded in a series of Jacobi polynomials. Numerical solutions are obtained for the stress fields near the crack tips. Contrary to previous results, it is found that the solution does not contain any stress singularities at the crack tips.

2. Formulation of the problem

We assume that there is a crack of length 2ℓ along the x-axis in a functionally graded material plane, as shown in Figure 1. In this paper, the harmonic elastic stress wave is vertically incident. Let ω be the circular frequency of the incident wave, τ_0 a magnitude of the incident wave, and

$$u_0^{(j)}(x, y, t) \quad \text{and} \quad v_0^{(j)}(x, y, t)$$

are components of the displacement vectors. $\tau_{ik0}^{(j)}(x, y, t)$, $(i, k = x, y)$ are components of stress fields. Note that the superscript $j = 1, 2$ corresponds to the half-planes $y \le 0$ and $y \ge 0$ throughout this paper and as shown in Figure 1. Because the incident wave is an harmonic stress wave, all field quantities of

$$u_0^{(j)}(x, y, t), \quad v_0^{(j)}(x, y, t) \quad \text{and} \quad \tau_{ik0}^{(j)}(x, y, t)$$

can be assumed to be of the following forms:

$$[u_0^{(j)}(x, y, t), v_0^{(j)}(x, y, t), \tau_{ik0}^{(j)}(x, y, t)] \\ = [u^{(j)}(x, y), v^{(j)}(x, y), \tau_{ik}^{(j)}(x, y)]e^{-i\omega t}. \quad (1)$$

In what follows, the time dependence of $e^{-i\omega t}$ will be suppressed but understood. Here, the standard superposition technique was used. As discussed in [Eringen et al. 1977] and [Srivastava et al. 1983], the boundary conditions can be written as

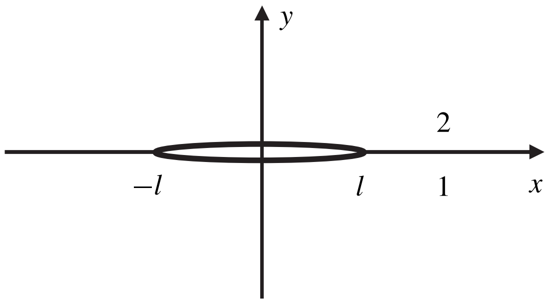

Figure 1. Geometry of a finite crack in the functionally graded materials.

follows. (In this paper, we consider just the perturbation fields.)

$$\tau_{yy}^{(1)}(x,0)=\tau_{yy}^{(2)}(x,0)=-\tau_0,\ \tau_{xy}^{(1)}(x,0)=\tau_{xy}^{(2)}(x,0)=0,\ |x|\le\ell \tag{2}$$

$$\tau_{yy}^{(1)}(x,0)=\tau_{yy}^{(2)}(x,0),\ \tau_{xy}^{(1)}(x,0)=\tau_{xy}^{(2)}(x,0),\ |x|>\ell \tag{3}$$

$$u^{(1)}(x,0)=u^{(2)}(x,0),\ v^{(1)}(x,0)=v^{(2)}(x,0),\ |x|>\ell \tag{4}$$

$$u^{(j)}(x,y)=0,\ v^{(j)}(x,y)=0,\ (j=1,2)\text{ for }\sqrt{x^2+y^2}\to\infty \tag{5}$$

3. Basic equations of nonlocal functionally graded materials

The basic equations of a plane of linear, non-homogeneous, isotropic, nonlocal functionally graded materials with variable shear modulus, variable material density and vanishing body force are given by Equations (6) and (7) [Suresh and Mortensen 1977; Nowinski 1984b]. (We assume here that the shear modulus and density function vary exponentially and vertically with respect to the crack.)

$$\frac{\partial\tau_{xx}^{(j)}(x,y)}{\partial x}+\frac{\partial\tau_{xy}^{(j)}(x,y)}{\partial y}=-\rho(y)\omega^2u^{(j)}(x,y),\quad (j=1,2) \tag{6}$$

$$\frac{\partial\tau_{xy}^{(j)}(x,y)}{\partial x}+\frac{\partial\tau_{yy}^{(j)}(x,y)}{\partial y}=-\rho(y)\omega^2v^{(j)}(x,y).\quad (j=1,2) \tag{7}$$

The following relationships were used in Equations (6)–(7)

$$-\rho(y)\omega^2 u^{(j)}(x,y)e^{-i\omega t} = \rho(y)\frac{\partial^2 u_0^{(j)}(x,y,t)}{\partial t^2} = \rho(y)\frac{\partial^2 (u^{(j)}(x,y)e^{-i\omega t})}{\partial t^2}, \tag{8}$$

$$-\rho(y)\omega^2 v^{(j)}(x,y)e^{-i\omega t} = \rho(y)\frac{\partial^2 v_0^{(j)}(x,y,t)}{\partial t^2} = \rho(y)\frac{\partial^2 (v^{(j)}(x,y)e^{-i\omega t})}{\partial t^2}, \tag{9}$$

$$\begin{cases} \tau_{xx}^{(j)}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu^*(|x'-x|,|y'-y|) \\ \qquad \left[\frac{1+k}{k-1}\,\frac{\partial u^{(j)}(x',y')}{\partial x'} + \frac{3-k}{k-1}\,\frac{\partial v^{(j)}(x',y')}{\partial y'}\right] dx'dy', \\ \tau_{yy}^{(j)}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu^*(|x'-x|,|y'-y|) \\ \qquad \left[\frac{1+k}{k-1}\,\frac{\partial v^{(j)}(x',y')}{\partial y'} + \frac{3-k}{k-1}\,\frac{\partial u^{(j)}(x',y')}{\partial x'}\right] dx'dy', \\ \tau_{xy}^{(j)}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu^*(|x'-x|,|y'-y|) \\ \qquad \left[\frac{\partial v^{(j)}(x',y')}{\partial x'} + \frac{\partial u^{(j)}(x',y')}{\partial y'}\right] dx'dy', \end{cases} \quad (j=1,2), \tag{10}$$

where $k=3-4\eta$ for the plane strain state and $k=(3-\eta)/(1+\eta)$ for the generalized plane stress state. $\mu^*(|x'-x|,|y'-y|)$ is the shear modulus, $\rho(y)$ is the material density. In this paper, we consider only the plane strain problem. η is the Poisson's ratio, and is taken to be a constant, owing to the fact that its variation within a practical range has a rather insignificant influence on the stress fields near the crack tips.

In the constitutive Equations (10), the only difference from classical elastic theory is that the stress

$$\tau_{ik}^{(j)}(x,y)(i,k=x,y)$$

at a point (x,y) depends on

$$u_{,k}^{(j)}(x,y) \quad \text{and} \quad v_k^{(j)}(x,y)$$

at all points of the body. As discussed in [Eringen and Kim 1974; 1977; Eringen 1977], it can be assumed in the form for which the dispersion curves of plane elastic waves coincide with those known in lattice dynamics. Among several possible curves the following has been found to be very useful.

$$\mu^*(|x'-x|,|y'-y|) = \mu(y')\alpha(|x'-x|,|y'-y|), \tag{11}$$

where $\alpha(|x'-x|,|y'-y|)$ is known as the influence function.

Crack problems in functionally graded materials do not appear to be analytically tractable for arbitrary variations of material properties. Usually one tries to generate forms of non-homogeneity for which the problem becomes tractable. As with the treatment of the crack problem for isotropic non-homogeneous materials in [Koizumi 1993; Lee and Erdogan 1994; Suresh and Mortensen 1977; Choi 2001; Delae and Erdogan 1988; Ozturk and Erdogan 1996], we assume that the shear modulus and the material density are described by

$$\mu(y) = \mu_0 \, e^{\gamma y}, \qquad \rho(y) = \rho_0 \, e^{\gamma y}, \tag{12}$$

where γ is a constant that describes the functionally graded materials; μ_0 and ρ_0 are the shear modulus and the material density along $y = 0$, respectively; and $\gamma \neq 0$ is the case for the functionally graded materials. When $\gamma = 0$, it returns to the homogeneous material case.

Substituting Equations (10) for Equations (6)–(7) and using Equations (11)–(12) and the Green–Gauss theorem leads to

$$\begin{aligned}
&\mu_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha |x'-x|, |y'-y|) \, e^{\gamma y'} \\
&\left[(1+k)\frac{\partial^2 u^{(j)}}{\partial x'^2} + (k-1)\,\frac{\partial^2 u^{(j)}}{\partial y'^2} + 2\,\frac{\partial^2 v^{(j)}}{\partial x' \partial y'} + (k-1)\gamma\left(\frac{\partial u^{(j)}}{\partial y'} + \frac{\partial v^{(j)}}{\partial x'}\right)\right] dx' dy' \\
&\qquad - \int_{-\ell}^{\ell} \alpha |x'-x|, |0|) \boldsymbol{[}\sigma_{xy}^{(j)}(x', 0)\boldsymbol{]} \, dx' \;=\; -\rho(y)\omega^2 u^{(j)}(x, y)
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\mu_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha |x'-x|, |y'-y|) \, e^{\gamma y'} \\
&\left\{(1+k)\frac{\partial^2 v^{(j)}}{\partial y'^2} + (k-1)\,\frac{\partial^2 v^{(j)}}{\partial x'^2} + 2\frac{\partial^2 u^{(j)}}{\partial x' \partial y'} + \gamma\left[(1+k)\frac{\partial v^{(j)}}{\partial y'} + (3-k)\frac{\partial u^{(j)}}{\partial x'}\right]\right\} dx' dy' \\
&\qquad - \int_{-\ell}^{\ell} \alpha |x'-x|, |0|) \; \boldsymbol{[}\sigma_{yy}^{(j)}(x', 0)\boldsymbol{]} \, dx' = -\rho(y)\omega^2 v^{(j)}(x, y),
\end{aligned} \tag{14}$$

where

$$\sigma_{yy}^{(j)}(x, y) = \mu_0 e^{\gamma y} \left[\frac{1+k}{k-1}\;\frac{\partial v^{(j)}(x, y)}{\partial y} + \frac{3-k}{k-1}\;\frac{\partial u^{(j)}(x, y)}{\partial x}\right] \tag{15}$$

and

$$\sigma_{xy}^{(j)}(x, y) = \mu_0 e^{\gamma y} \left[\frac{\partial v^{(j)}(x, y)}{\partial x} + \frac{\partial u^{(j)}(x, y)}{\partial y}\right]. \tag{16}$$

The bold brackets in Equations (13)–(14) indicate a jump at the crack line, that is,

$$\boldsymbol{[}\sigma_{xy}^{(j)}(x, 0)\boldsymbol{]} = \; \sigma_{xy}^{(2)}(x, 0^+) - \sigma_{xy}^{(1)}(x, 0^-), \tag{17}$$

$$\boldsymbol{[}\sigma_{yy}^{(j)}(x, 0)\boldsymbol{]} = \; \sigma_{yy}^{(2)}(x, 0^+) - \sigma_{yy}^{(1)}(x, 0^-); \tag{18}$$

Expressions (15)–(16) are the classical constitutive equations. Here the surface integral may be dropped since the displacement field vanishes at infinity as shown in Equations (13)–(14).

4. The dual integral equations

As discussed in [Eringen et al. 1977], we see that

$$[\![\sigma_{xy}^{(j)}(x,0)]\!] = 0\,, \qquad [\![\sigma_{yy}^{(j)}(x,0)]\!] = 0. \tag{19}$$

What remains now is to solve the integrodifferential equations (13)–(14) for functions $u^{(j)}(x, y)$ and $v^{(j)}(x, y)$, $(j = 1, 2)$. It is impossible to obtain a rigorous solution at the present stage. It seems obvious that in the solution of such a problem we encounter serious, if not insurmountable, mathematical difficulties and must resort to an approximation procedure. In the given problem, as discussed in [Nowinski 1984b; 1984a; Zhou and Wang 2002b], we assume that the nonlocal interaction in the y direction is ignored. This is purely an assumption for mathematical tractability. In view of our assumptions, we can state that

$$\begin{cases}\alpha|x'-x|,\,|y'-y|) = \alpha_0(|x'-x|)\delta(y'-y),\\ \alpha_0(|x'-x|) = \frac{1}{\sqrt{\pi}}\,(\beta\alpha\exp[-(\beta/\alpha^2(x'-x)^2],\end{cases} \tag{20}$$

where β is a constant and can be determined by experiment, and where a is the characteristic length. The characteristic length may be selected according to the range and sensitivity of the physical phenomena. For instance, for a perfect crystal, a may be taken as the lattice parameter. For a granular material, a may be considered to be the average granular distance and, for a fiber composite, the fiber distance, etc. In the present paper, a is taken to be the lattice parameter. From Equations (13)–(14), we have

$$\int_{-\infty}^{\infty}\alpha_0(|x'-x|)e^{\gamma y}\left[(1+k)\frac{\partial^2u^{(j)}}{\partial x'^2}+(k-1)\frac{\partial^2u^{(j)}}{\partial y^2}+2\frac{\partial^2v^{(j)}}{\partial x'\partial y}\right.$$
$$\left.+(k-1)\gamma\left(\frac{\partial u^{(j)}}{\partial y}+\frac{\partial v^{(j)}}{\partial x'}\right)\right]dx' = -\,\frac{\rho_0}{\mu_0}\omega^2u^{(j)}, \tag{21}$$

$$\int_{-\infty}^{\infty}\alpha_0(|x'-x|)e^{\gamma y}\left\{(1+k)\frac{\partial^2v^{(j)}}{\partial y^2}+(k-1)\frac{\partial^2v^{(j)}}{\partial x'^2}+2\frac{\partial^2u^{(j)}}{\partial x'\partial y}\right.$$
$$\left.+\;\gamma\left[(1+k)\frac{\partial v^{(j)}}{\partial y}+(3-k)\frac{\partial u^{(j)}}{\partial x'}\right]\right\}dx' = -\,\frac{\rho_0}{\mu_0}\,\omega^2v^{(j)}. \tag{22}$$

To solve the problem, the Fourier transform of Equations (21)–(22) with x can be given as follows:

$$-s^2(1+k)\bar{u}^{(j)}+(k-1)\frac{\partial^2\bar{u}^{(j)}}{\partial y^2}-2s\frac{\partial\bar{v}^{(j)}}{\partial y} +(k-1)\gamma\left(\frac{\partial\bar{u}^{(j)}}{\partial y}-s\bar{v}^{(j)}\right)=-\frac{\rho_0}{\mu_0\bar{\alpha}_0}\,\omega^2\bar{u}^{(j)}, \quad (23)$$

$$(1+k)\frac{\partial^2\bar{v}^{(j)}}{\partial y^2}-s^2(k-1)\bar{v}+2s\frac{\partial\bar{u}^{(j)}}{\partial y} +\gamma\left[(1+k)\frac{\partial\bar{v}^{(j)}}{\partial y}+s(3-k)\bar{u}\right]=-\frac{\rho_0}{\mu_0\bar{\alpha}_0}\,\omega^2\bar{v}^{(j)}. \quad (24)$$

Throughout the paper a superposed bar indicates the Fourier transform.

Because of the symmetry, it suffices to consider the problem for $x \geq 0$, $|y| < \infty$. The above systems governing Equations (23)–(24) are solved using the Fourier integral transform to obtain the general expressions for the displacement components as

$$\begin{cases} u^{(1)}(x,y) &= \frac{2}{\pi}\int_0^\infty \sum_{i=1}^2 A_i(s)\,e^{-\lambda_{i+2}y}\sin(sx)\,ds,\\ v^{(1)}(x,y) &= \frac{2}{\pi}\int_0^\infty \sum_{i=1}^2 m_{i+2}(s)A_i(s)\,e^{-\lambda_{i+2}y}\cos(sx)\,ds\ , \end{cases} \quad (y\geq 0) \quad (25)$$

$$\begin{cases} u^{(2)}(x,y) &= \frac{2}{\pi}\int_0^\infty \sum_{i=1}^2 B_i(s)\,e^{-\lambda_{i}y}\sin(sx)\,ds,\\ v^{(2)}(x,y) &= \frac{2}{\pi}\int_0^\infty \sum_{i=1}^2 m_{i}(s)B_i(s)\,e^{-\lambda_{i}y}\cos(sx)\,ds, \end{cases} \quad (y\geq 0) \quad (26)$$

and from Equations (15) and (16), the stress components are obtained as

$$\begin{cases} \sigma_{yy}^{(1)}(x,y)=\frac{2\mu_0e^{\gamma y}}{\pi(k-1)}\int_0^\infty \sum_{i=1}^2[-(k+1)m_{i+2}(s)\lambda_{i+2}],\\ \qquad\qquad +s(3-k)]A_i(s)\,e^{-\lambda_{i+2}y}\cos(sx)\,ds,\\ \sigma_{xy}^{(1)}(x,y)=\frac{2\mu_0e^{\gamma y}}{\pi}\int_0^\infty \sum_{i=1}^2[-\lambda_{i+2}-m_{i+2}(s)s]A_i(s)e^{-\lambda_{i+2}y}\sin(sx)\,ds, \end{cases} \quad (y\leq 0) \quad (27)$$

$$\begin{cases} \sigma_{yy}^{(2)}(x,y)=\frac{2\mu_0e^{\gamma y}}{\pi(k-1)}\int_0^\infty \sum_{i=1}^2[-(k+1)m_{i}(s)\lambda_{i},\\ \qquad\qquad +s(3-k)]B_i(s)\,e^{-\lambda_{i}y}\cos(sx)\,ds,\\ \sigma_{xy}^{(2)}(x,y)=\frac{2\mu_0e^{\gamma y}}{\pi}\int_0^\infty \sum_{i=1}^2[-\lambda_{i}-m_{i}(s)s]B_i(s)e^{-\lambda_{i}y}\sin(sx)\,ds, \end{cases} \quad (y\geq 0) \quad (28)$$

where s is the transform variable. A_1, A_2, B_1 and B_2 are arbitrary unknowns, and $\lambda_i(s)$ $(i = 1, 2, 3, 4)$ are the roots of the characteristic equation

$$\lambda^4 - 2\lambda^3\gamma + (\gamma^2 - 2s^2)\lambda^2 + 2\gamma s^2\lambda + s^4 + \frac{3-k}{k+1}\,\gamma^2 s^2 + \frac{2k\rho_0\omega^2}{(k+1)\mu_0\alpha_0(s)}(-s^2 + \lambda^2 - \gamma\lambda) + \frac{k-1}{k+1}\left(\frac{\rho_0\omega^2}{\mu_0\alpha_0(s)}\right)^2 = 0, \tag{29}$$

and $m_i(s)$ $(i = 1, 2, 3, 4)$ is expressed for each root $\lambda_i(s)$ as

$$m_i(s) = \frac{-(k+1)s^2 + (k-1)\lambda_i^2 - \gamma(k-1)\lambda_i}{-2s\lambda_i + s\gamma(k-1)}\,. \tag{30}$$

Equation (29) can be rewritten in the following form

$$(\lambda^2 - \lambda\gamma - s^2)^2 + \frac{3-k}{k+1}\gamma^2 s^2 + \frac{2kc_1^2(\lambda^2 - \gamma\lambda - s^2)}{k+1} + \frac{c_1^4(k-1)}{k+1} = 0, \tag{31}$$

where

$$c_1^2 = \frac{c^2}{\alpha_0(s)} \quad \text{and} \quad c^2 = \frac{\rho_0\omega^2}{\mu_0}.$$

The roots may be obtained as

$$\lambda_1 = \frac{1}{2}\left(\gamma + \sqrt{\gamma^2 - 4\left(\frac{kc_1^2}{k+1} - s^2 - \sqrt{\frac{c_1^4}{(k+1)^2} - \frac{s^2\gamma^2(3-k)}{k+1}}\right)}\right), \tag{32}$$

$$\lambda_2 = \frac{1}{2}\left(\gamma + \sqrt{\gamma^2 - 4\left(\frac{kc_1^2}{k+1} - s^2 + \sqrt{\frac{c_1^4}{(k+1)^2} - \frac{s^2\gamma^2(3-k)}{k+1}}\right)}\right), \tag{33}$$

$$\lambda_3 = \frac{1}{2}\left(\gamma - \sqrt{\gamma^2 - 4\left(\frac{kc_1^2}{k+1} - s^2 - \sqrt{\frac{c_1^4}{(k+1)^2} - \frac{s^2\gamma^2(3-k)}{k+1}}\right)}\right), \tag{34}$$

$$\lambda_4 = \frac{1}{2}\left(\gamma - \sqrt{\gamma^2 - 4\left(\frac{kc_1^2}{k+1} - s^2 + \sqrt{\frac{c_1^4}{(k+1)^2} - \frac{s^2\gamma^2(3-k)}{k+1}}\right)}\right). \tag{35}$$

From Equations (25)–(28), we can see that there are four unknown constants (in Fourier space they are functions of s), that is, A_1, A_2, B_1, and B_2, which can be obtained from the boundary conditions (2)–(4). To solve the present problem, the jumps of the displacements across the crack surfaces can be defined as follows:

$$f_1(x) - u^{(2)}(x, 0) - u^{(1)}(x, 0), \tag{36}$$

$$f_2(x) - v^{(2)}(x, 0) - v^{(1)}(x, 0), \tag{37}$$

where $f_1(x)$ is an odd function and $f_2(x)$ an even one.

Applying the Fourier transforms and the boundary conditions (2)–(4), we obtain

$$[X_1]\begin{bmatrix} B_1(s) \\ B_2(s) \end{bmatrix} - [X_2]\begin{bmatrix} A_1(s) \\ A_2(s) \end{bmatrix} = \begin{bmatrix} \bar{f}_1(s) \\ \bar{f}_2(s) \end{bmatrix}, \tag{38}$$

$$[X_3]\begin{bmatrix} B_1(s) \\ B_2(s) \end{bmatrix} = [X_4]\begin{bmatrix} A_1(s) \\ A_2(s) \end{bmatrix}, \tag{39}$$

where the matrices $[X_i]$ $(i = 1, 2, 3, 4)$ can be seen in the Appendix.

From Equations (10), and using Equations (20), we have

$$\tau_{yy}^{(2)}(x, y) = \int_{-\infty}^{\infty} \alpha_0(|x' - x|)\sigma_{yy}^{(2)}(x', y)\, dx', \tag{40}$$

$$\tau_{xy}^{(2)}(x, y) = \int_{-\infty}^{\infty} \alpha_0(|x' - x|)\sigma_{xy}^{(2)}(x', y)\, dx'. \tag{41}$$

Using the relations as follows [Gradshteyn and Ryzhik 1980]

$$I_1 = \int_{-\infty}^{\infty} \exp(-px'^2) \begin{Bmatrix} \sin\xi(x'+x) \\ \cos\xi(x'+x) \end{Bmatrix} dx' = (\pi/p)^{\frac{1}{2}} \exp\left(\frac{-\xi^2}{4p}\right) \begin{Bmatrix} \sin(\xi x) \\ \cos(\xi x) \end{Bmatrix}, \tag{42}$$

we have

$$\tau_{yy}^{(2)}(x, y) = \frac{2\mu_0 e^{\gamma y}}{\pi(k-1)} \int^{\infty} 0 e^{-\frac{s^2}{4p}} \left[\sum_{i=1}^{2} g_i(s) B_i(s) + \sum_{i=1}^{2} g_{i+2}(s) A_i(s)\right] \cos(sx)\, ds, \tag{43}$$

$$\tau_{xy}^{(2)}(x, y) = \frac{2\mu_0 e^{\gamma y}}{\pi} \int_0^{\infty} e^{-\frac{s^2}{4p}} \left[\sum_{i=1}^{2} h_i(s) B_i(s) + \sum_{i=1}^{2} h_{i+2}(s) A_i(s)\right] \sin(sx)\, ds, \tag{44}$$

where

$$g_i(s) = -(k+1)m_i(s)\lambda_i + s(3-k) \quad \text{and} \quad h_i(s) = -\lambda_i - m_i(s)s,$$

with $(i = 1, 2, 3, 4)$, $p = \left(\frac{\beta}{a}\right)^2$.

By solving the four expressions in Equations (38)–(39) with four unknown functions A_1, A_2, B_1 and B_2, substituting those solutions for Equations (43)–(44), and applying the boundary conditions (2)–(4) to the results, we have

$$\begin{aligned}\tau_{yy}^{(1)}(x,0) &= \tau_{yy}^{(2)}(x,0)\\ &= \frac{2\mu_0}{\pi(k-1)}\int_0^\infty e^{-\frac{s^2}{4p}}[d_1(s)\bar{f}_1(s)+d_2(s)\bar{f}_2(s)]\cos(sx)\,ds\\ &= -\tau_0, \qquad 0\le x\le \ell,\end{aligned} \tag{45}$$

$$\begin{aligned}\tau_{xy}^{(1)}(x,0) &= \tau_{xy}^{(2)}(x,0)\\ &= \frac{2\mu_0}{\pi}\int_0^\infty e^{-\frac{s^2}{4p}}[d_3(s)\bar{f}_1(s)+d_4(s)\bar{f}_2(s)]\sin(sx)\,ds\\ &= 0, \qquad 0\le x\le \ell,\end{aligned} \tag{46}$$

$$\int_0^\infty \bar{f}_1(s)\sin(sx)ds = 0, \qquad x>\ell, \tag{47}$$

$$\int_0^\infty \bar{f}_2(s)\cos(sx)ds = 0, \qquad x>\ell, \tag{48}$$

where

$$\begin{aligned}d_1(s) &= g_1(s)e_{11}(s)+g_2(s)e_{21}(s)+g_3(s)c_{11}(s)+g_4(s)c_{21}(s),\\ d_2(s) &= g_1(s)e_{12}(s)+g_2(s)e_{22}(s)+g_3(s)c_{12}(s)+g_4(s)c_{22}(s),\\ d_3(s) &= h_1(s)e_{11}(s)+h_2(s)e_{21}(s)+h_3(s)c_{11}(s)+h_4(s)c_{21}(s),\\ d_4(s) &= h_1(s)e_{12}(s)+h_2(s)e_{22}(s)+h_3(s)c_{12}(s)+h_4(s)c_{22}(s),\end{aligned}$$

and where $e_{ij}(s)$ and $c_{ij}(s)$ $(i=1,2,\ j=1,2)$ are non-zero constants, as can be seen in the Appendix. To determine the unknown functions $\bar{f}_1(s)$ and $\bar{f}_2(s)$, the dual integral equations in (45)–(48) must be solved. For the lattice parameter $a\to 0$, then

$$d_1(s)e^{-\frac{s^2}{4p}}, \qquad (i=1,2,3,4)$$

is equal to a non-zero constant and Equations (45)–(48) reduce to two pairs of dual integral equations for the same problem in the classical functionally graded materials case.

5. Solution of the dual integral equations

The only difference between the classical and nonlocal equations is in the influence functions $d_i(s)$ $(i = 1, 2, 3, 4)$. It is logical to utilize the classical solution to convert the system of equations in (45)–(48) to two pairs of integral equations of the second kind, since the latter is generally better behaved. For the lattice parameter $a \to 0$, then

$$d_1(s)e^{-\frac{s^2}{4p}}, \qquad (i = 1, 2, 3, 4)$$

is equal to a non-zero constant and Equations (45)–(48) reduce to two pairs of dual integral equations for the same problem in classical elasticity. In the case of $(a \to 0)$, the present problem is the same as that discussed in [Zhou et al. 2004]. As we find in [Eringen et al. 1977], the dual integral equations (48)–(51) cannot be transformed into a Fredholm integral equation of the second kind, because

$$d_1(s)e^{-\frac{s^2}{4p}}/s, \qquad (i = 1, 2, 3, 4)$$

does not tend to a constant C $(C \neq 0)$ for $s \to \infty$. Of course, the dual equations (45)–(48) can be considered to be a single integral equation of the first kind with discontinuous kernel. It is well known in the literature that integral equations of the first kind are generally ill-posed in the sense of Hadamard, that is, small perturbations of the data can yield arbitrarily large changes in the solution. This makes the numerical solution of such equations quite difficult. To overcome this difficulty, the Schmidt method [Morse and Feshbach 1958; Yau 1967] is used to solve the dual integral equations (45)–(48).

From the nature of the displacement along the crack line, it can be shown that the jumps of the displacements across the crack surface are finite, differentiable, and continuum functions. Hence, the jumps of the displacements across the crack surface can be expanded by the following series:

$$f_1(x) = \sum_{n=0}^{\infty} a_n P_{2n+1}^{(\frac{1}{2},\frac{1}{2})}\left(\frac{x}{\ell}\right)\left(1 - \frac{x^2}{\ell^2}\right)^{\frac{1}{2}}, \qquad \text{for } 0 \le x \le \ell, \tag{49}$$

$$f_1(x) = 0, \qquad \text{for } x > \ell, \tag{50}$$

$$f_2(x) = \sum_{n=0}^{\infty} b_n P_{2n}^{(\frac{1}{2},\frac{1}{2})}\left(\frac{x}{\ell}\right)\left(1 - \frac{x^2}{\ell^2}\right)^{\frac{1}{2}}, \qquad \text{for } 0 \le x \le \ell, \tag{51}$$

$$f_2(x) = 0, \qquad \text{for } x > \ell, \tag{52}$$

where a_n and b_n are unknown coefficients and

$$P_n^{(\frac{1}{2},\frac{1}{2})}(x)$$

is a Jacobi polynomial [Gradshteyn and Ryzhik 1980].

The Fourier transforms of Equations (49)–(52) are [Erdelyi 1954]

$$\bar{f}_1(s) = \sum_{n=0}^{\infty} a_n G_n^{(1)} \frac{1}{s} J_{2n+2}(s\ell) , \quad G_n^{(1)} = \sqrt{\pi}\,(-1)^n \frac{\Gamma(2n+2+\frac{1}{2})}{(2n+1)!}, \tag{53}$$

$$\bar{f}_2(s) = \sum_{n=0}^{\infty} b_n G_n^{(2)} \frac{1}{s} J_{2n+1}(s\ell) , \quad G_n^{(2)} = \sqrt{\pi}\,(-1)^n \frac{\Gamma(2n+1+\frac{1}{2})}{(2n)!}, \tag{54}$$

where $\Gamma(x)$ and $J_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting Equations (53)–(54) for Equations (45)–(48), it can be shown that Equations (47)–(48) are automatically satisfied. Equations (45)–(46) reduce to

$$\frac{2\mu_0}{\pi(k-1)} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{1}{s} e^{-\frac{s^2}{4p}} [d_1(s) a_n G_n^{(1)} J_{2n+2}(sl) + d_2(s) b_n G_n^{(2)} J_{2n+2}(sl)] \cos(sx) ds = -\tau_0, \quad 0 \le x \le \ell, \tag{55}$$

$$\sum_{n=0}^{\infty} \int_0^{\infty} \frac{1}{s} e^{-\frac{s^2}{4p}} [d_3(s) a_n G_n^{(1)} J_{2n+2}(sl) + d_4(s) b_n G_n^{(2)} J_{2n+1}(sl)] \sin(sx) ds = 0 , \quad 0 \le x \le \ell. \tag{56}$$

The multi-valued functions λ_1, λ_2, λ_3 and λ_4 have branch points. We choose the branches such that $\Re(\lambda_1) \ge 0$, $\Re(\lambda_2) \ge 0$, $\Re(\lambda_3) \le 0$ and $\Re(\lambda_4) \le 0$ are on the path of integration. For large s, the integrands of Equations (55)–(56) almost all decrease exponentially. So the semi-infinite integral in Equations (55)–(56) can be evaluated numerically by Filon's method. Equations (55)–(56) can now be solved for the coefficients a_n and b_n by the Schmidt method [Morse and Feshbach 1958; Yau 1967]. Briefly, Equations (55)–(56) can be rewritten as

$$\sum_{n=0}^{\infty} a_n E_n^*(x) + \sum_{n=0}^{\infty} b_n F_n^*(x) = U_0(x) , \quad 0 \le x \le \ell, \tag{57}$$

$$\sum_{n=0}^{\infty} a_n G_n^*(x) + \sum_{n=0}^{\infty} b_n H_n^*(x) = 0 , \quad 0 \le x \le \ell, \tag{58}$$

where $E_n^*(x)$, $F_n^*(x)$, $G_n^*(x)$ and $H_n^*(x)$ and $U_0(x)$ are known functions, and a_n and b_n are unknown coefficients.

From Equation (58), we obtain

$$\sum_{n=0}^{\infty} b_n H_n^*(x) \;=\; -\sum_{n=0}^{\infty} a_n G_n^*(x) \,. \tag{59}$$

This can now be solved for coefficients b_n by the Schmidt method. Here, the form $-\sum_{n=0}^{\infty} a_n G_n^*(x)$ will be considered temporarily as a known function. A set of functions $P_n(x)$, which satisfies the orthogonality condition

$$\int_0^\ell P_m(x)P_n(x)dx = N_n\delta_{mn}\ , \quad N_n = \int_0^\ell P_n^2(x)\,dx, \tag{60}$$

can be constructed from the function, $H_n^*(x)$, such that

$$P_n(x) = \sum_{i=0}^{n} \frac{M_{in}}{M_{mn}},\ H_i^*(x), \tag{61}$$

where M_{ij} is the cofactor of the element d_{ij} of D_n, which is defined as

$$D_n = \begin{bmatrix} d_{00}, d_{01}, d_{02}, \ldots, d_{0n} \\ d_{10}, d_{11}, d_{12}, \ldots, d_{1n} \\ d_{20}, d_{21}, d_{22}, \ldots, d_{2n} \\ \ldots \\ \ldots \\ \ldots \\ d_{n0}, d_{n1}, d_{n2}, \ldots, d_{nn} \end{bmatrix}, \quad d_{ij} = \int_0^\ell H_i^*(x)H_j^*(x)\,dx. \tag{62}$$

Using Equations (59)–(62), we obtain

$$b_n = \sum_{j=n}^{\infty} \frac{M_{nj}}{M_{jj}} \quad \text{with} \ \ q_j = -\sum_{i=0}^{\infty} a_i \frac{1}{N_j} \int_0^\ell G_i^*(x)P_j(x)\,dx. \tag{63}$$

This can be rewritten as

$$b_n = \sum_{i=0}^{\infty} a_i K_{in}^*\ , \quad K_{in}^* = -\sum_{j=n}^{\infty} q_j \frac{M_{nj}}{N_j M_{jj}} \int_0^\ell G_i^*(x)P_j(x)dx\ . \tag{64}$$

Substituting Equation (64) for Equation (57), we obtain

$$\sum_{n=0}^{\infty} a_n Y_n^*(x) = U_0(x)\ , \quad Y_n^*(x) = E_n^*(x) + \sum_{i=0}^{\infty} K_{ni}^* F_i^*(x)\ . \tag{65}$$

This can now be solved for the coefficients a_n by the Schmidt method, again as mentioned above. With the aid of Equation (64), the coefficients b_n can be obtained.

6. Numerical calculations and discussion

The coefficients a_n and b_n are known, so that the entire stress field can be obtained. In the case of the present study, $\tau_{yy}^{(1)}(x, y)$ and $\tau_{xy}^{(1)}(x, y)$ along the crack line can

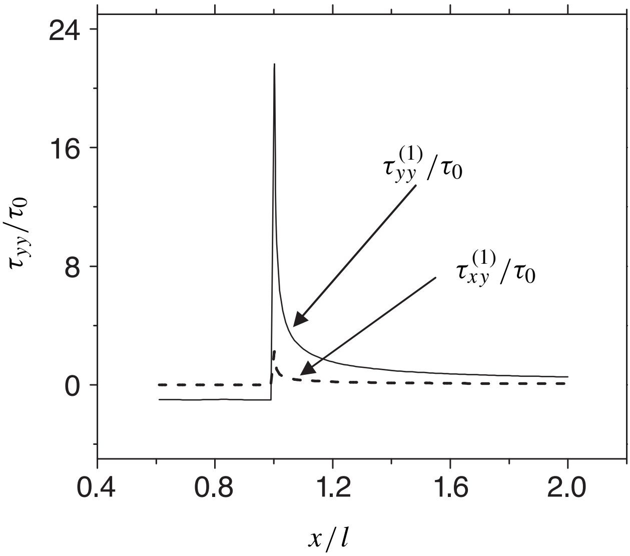

Figure 2. The stress along the crack line versus x/l for $l = 1.0$, $wl/c = 0.2$, $\gamma l = 0.4$, $\eta = 0.23$ and $a/\beta l = 0.003$.

be expressed as

$$\tau_{yy} = \tau_{yy}^{(1)}(x,0) = \frac{2\mu_0}{\pi(k-1)} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{1}{s} e^{-\frac{s^2}{4p}} \tag{66}$$

$$[d_1(s) a_n G_n^{(1)} J_{2n+2}(s\ell) + d_2(s) b_n G_n^{(2)} J_{2n+2}(s\ell)] \cos(sx)\, ds,$$

$$\tau_{xy} = \tau_{xy}^{(1)}(x,0) = \frac{2\mu_0}{\pi} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{1}{s} e^{-\frac{s^2}{4p}} \tag{67}$$

$$[d_3(s) a_n G_n^{(1)} J_{2n+2}(s\ell) + d_4(s) b_n G_n^{(2)} J_{2n+2}(s\ell)] \sin(sx)\, ds.$$

When the lattice parameter $a \neq 0$, the semi-infinite integration and the series in Equations (66)–(67) are convergent for any variable x, and they give finite stresses along $y = 0$, so there are no stress singularities at crack tips. For $-\ell < x < \ell$, $\tau_{yy}^{(1)}(x,0)/\tau_0$ is very close to negative unity. Hence, the solution of this paper can also be proved to satisfy the boundary conditions (2). For $x > \ell$, $\tau_{yy}^{(1)}(x,0)/\tau_0$ possesses finite values diminishing from a finite value at $x = \ell$ to zero at $x = \infty$. Since $a/\beta\ell > 1/100$ represents a crack length of less than 100 atomic distances [Eringen et al. 1977; Eringen 1978; 1979], and, for such submicroscopic sizes, other serious questions arise regarding the interatomic arrangements and force laws, we do not

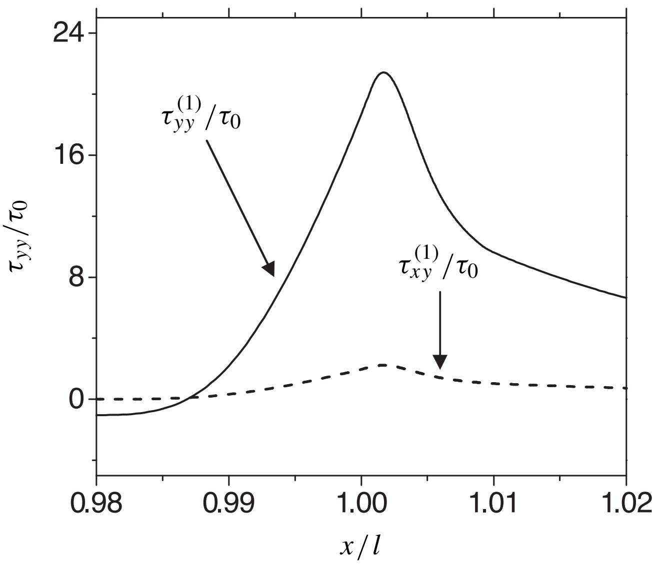

Figure 3. The locally enlarged graph of Figure 2 near the crack tip.

pursue valid solutions at such small crack sizes. The semi-infinite integrals that occur are easily evaluated because of the rapid diminution of the integrands. From [Itou 1978; Zhou et al. 1999a], it can be seen that the Schmidt method is performed satisfactorily if the first ten terms of the infinite series in Equations (57)–(58) are retained. The results of this paper are shown in Figures 2–8. From the results, the following observations are very significant:

(i) Nonlocal theory can be used to solve dynamic fracture problems in functionally graded materials subjected to harmonic stress waves. The traditional concepts of nonlocal theory can be extended to solve the fracture problem of functionally graded materials. When the lattice parameter, $a \to 0$ the present problem will revert to the same problem as discussed in [Zhou et al. 2004]. The dynamic stress fields can be directly obtained in the present paper. However, the dynamic stress fields cannot be directly obtained in [Zhou et al. 2004]; only the stress intensity factors are given there.

(ii For $a/\beta\ell \neq 0$, it can be proved that the semi-infinite integration in Equations (66)–(67) and the series in Equations (66)–(67) are convergent for any variable x. So the stresses give finite values all along the crack line, as shown in Figures 2 and 3. Contrary to the classical theory solution, we find that no stress singularities are present at the crack tips, and also that the present results converge to the classical ones when far away from the crack tips. The nonlocal

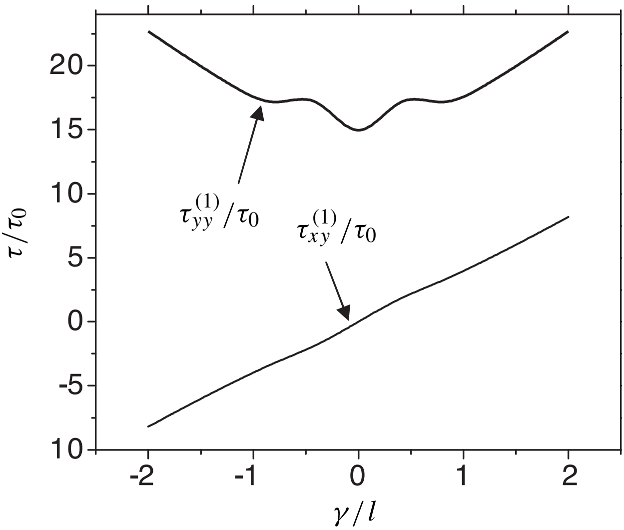

Figure 4. The stress at the crack tip versus γl for $a/\beta l = 0.003$, $wl/c = 0.2$, $\eta = 0.23$ and $l = 1.0$.

elastic solutions yield a finite hoop stress at the crack tips, thus allowing us to use the maximum stress as a fracture criterion. The maximum stress does not occur at the crack tips, but slightly away from it, as shown in Figure 3. This phenomenon has been thoroughly substantiated in [Eringen 1983]. The distance between the crack tip and the maximum stress point is very small, and it depends on the crack length, the lattice parameter, the parameter describing the functionally graded materials, and the frequency of the incident waves. As shown in Figures 2 and 3, it can be seen that the shear stress $\tau_{xy}^{(1)}$ is equal to zero for $|x| < \ell$. However, the shear stress $\tau_{xy}^{(1)}$ is not equal to zero for $x \geq \ell$. This inequality is caused by the shear modulus and mass density not being symmetric with respect to the cracked plane. The shear stress is smaller than the normal stress along the crack line.

(iii) Stresses at the crack tips become infinite as the lattice parameter $a \to 0$. This is the classical continuum limit of square root singularity. This can be shown from Equations (45)–(48). For $a \to 0$,

$$e^{-\frac{s^2}{4p}} = 1,$$

Equations (45)–(48) will reduce to the dual integral equations for the same problem in the classical functionally graded materials [Zhou et al. 2004].

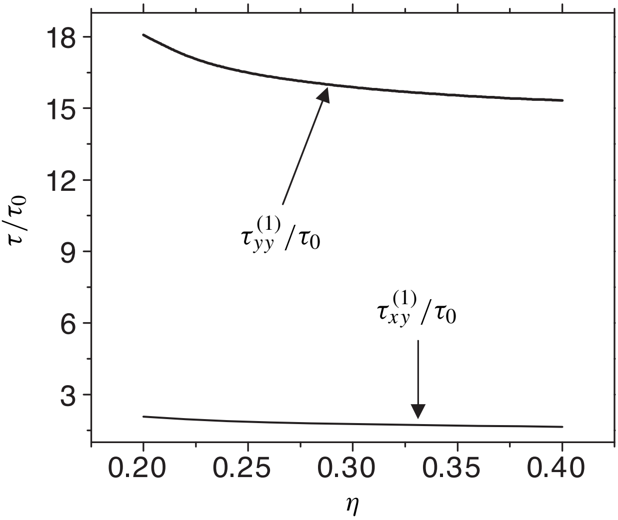

Figure 5. The stress at the crack tip versus η for $\gamma l = 0.4$, $a/\beta l = 0.003$, $wl/c = 0.3$, and $l = 1.0$.

These dual integral equations can be solved by using the singular integral equation for the same problem in the local functionally graded materials case. However, that stress singularities are present at the crack tips in the local functionally graded materials problem is well known.

(iv) The stress fields $\tau_{yy}^{(1)}$ at crack tips are symmetric about the line $\gamma\ell = 0$, as shown in Figure 4. The stress fields $\tau_{yy}^{(1)}$ at the crack tips decrease with an increase in the gradient parameter for $\gamma\ell < -1.0$, and increase with the gradient parameter reaching a peak near $\gamma\ell = -0.5$. They then decrease in magnitude for $\gamma\ell < 0$, as shown in Figure 4. In the case of $\gamma\ell > 0$, the stress fields $\tau_{yy}^{(1)}$ at the crack tips are symmetric, as in the case of $\gamma\ell < 0$. This means that by adjusting the gradient parameter of FGMs, dynamic stress fields near the crack tips can be reduced. However, the shear stress fields $\tau_{xy}^{(1)}$ at the crack tips increase almost linearly with an increase in the gradient parameter for all $\gamma\ell$. In this case, the shear stress $\tau_{xy}^{(1)}$ is smaller than the normal stress $\tau_{yy}^{(1)}$.

(v) The stress fields at the crack tips decrease with an increase in Poisson's ratio η, as shown in Figure 5. However, the changing ranges are small—that is, the variation of Poisson's ratio η within a practical range has a rather insignificant influence on the stress value near crack tips as discussed in [Delae and Erdogan 1988; Ozturk and Erdogan 1996].

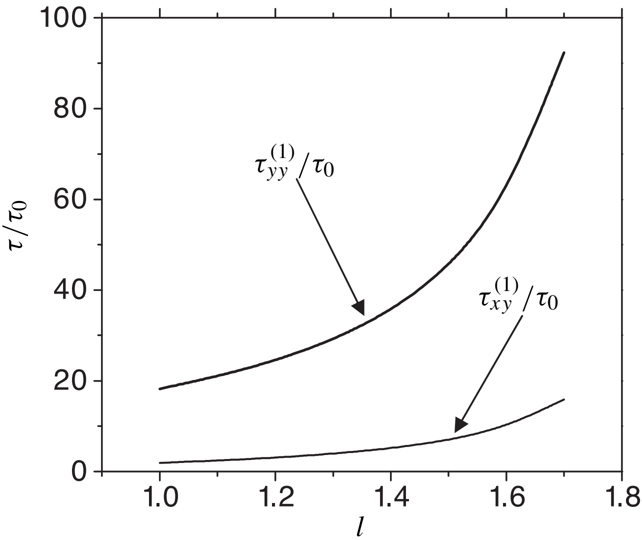

Figure 6. The stress at the crack tip versus l for $a/\beta = 0.003$, $\gamma = 0.4$, $\eta = 0.23$ and $w/c = 0.2$.

(vi) The stress fields at the crack tips increase non-linearly with an increase in crack length, as shown in Figure 6. This is similar to results of classical fracture theory. For classical fracture theory, the stress intensity factors increase with an increase in crack length.

(vii) The dynamic stresses of $\tau_{yy}^{(1)}$ and $\tau_{xy}^{(1)}$ at the crack tips in functionally graded materials tend to increase, with the frequency reaching a peak, and then decrease in magnitude, as shown in Figure 7. We can see that this conclusion is the same as that of the fracture problem in isotropic homogeneous materials.

(viii) The effect of the lattice parameter of functionally graded materials on the stress fields near the crack tips decreases with an increase in the lattice parameter, as shown in Figure 8. This phenomenon is discussed in [Eringen et al. 1977; 1978; 1979].

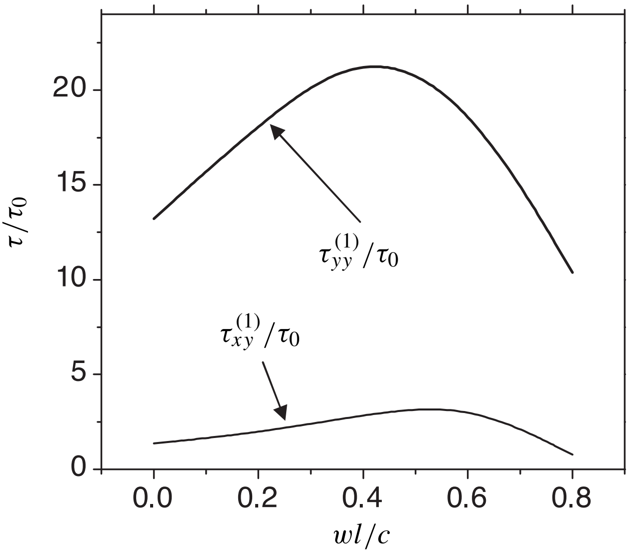

Figure 7. The stress at the crack tip versus wl/c for $a/\beta l = 0.0p03$, $\gamma l = -0.4$, $\eta = 0.23$ and $l = 1.0$.

Appendix

$$X_1 = \begin{bmatrix} 1 & 1 \\ m_1(s) & m_2(s) \end{bmatrix},$$

$$X_1 = \begin{bmatrix} 1 & 1 \\ m_3(s) & m_4(s) \end{bmatrix},$$

$$X_3 = \begin{bmatrix} g_1(s) & g_2(s) \\ h_1(s) & h_2(s) \end{bmatrix},$$

$$X_4 = \begin{bmatrix} g_3(s) & g_4(s) \\ h_3(s) & h_4(s) \end{bmatrix},$$

$$[X_5] = [X_1] - [X_2][X_4]^{-1}[X_3],$$

$$\begin{bmatrix} e_{11}(s) & e_{12}(s) \\ e_{21}(s) & e_{22}(s) \end{bmatrix} = [X_5]^{-1},$$

$$[X_6] = [X_4]^{-1}[X_3][X_5]^{-1} = \begin{bmatrix} c_{11}(s) & c_{12}(s) \\ c_{21}(s) & c_{22}(s) \end{bmatrix}.$$

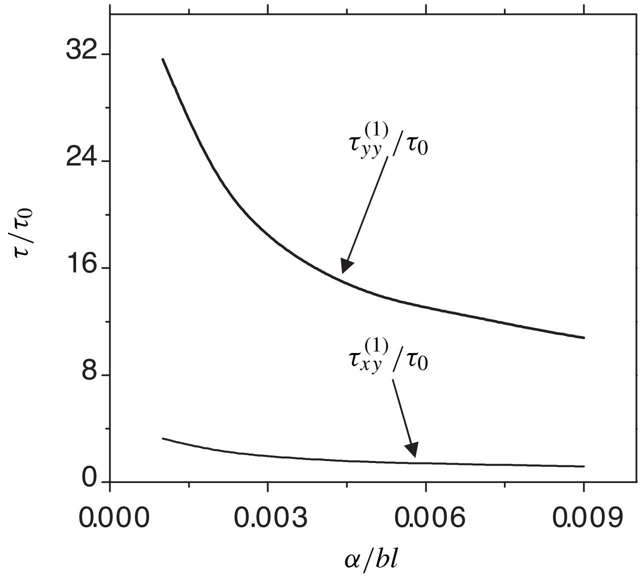

Figure 8. The stress at the crack tip versus $a/\beta l$ for $l = 1.0$, $\gamma l = 0.4$, $\eta = 0.23$ and $wl/c = 0.2$.

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