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A ONE-DIMENSIONAL VARIATIONAL FORMULATION FOR QUASIBRITTLE FRACTURE

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Besides efficient techniques allowing for the finite-element modeling of propagating displacement discontinuities, the numerical simulation of fracture processes in quasibrittle materials requires the definition of criteria for crack initiation and propagation. Among several alternatives proposed in the literature, the possibility to characterize energetically the discontinuous solution has recently attracted special interest. In this work, the initiation and propagation of cohesive cracks in an inhomogeneous elastic bar, subject to an axial body force is considered. The incremental finite-step problem for the evolving discontinuity is formulated accounting for progressive damage in the cohesive interface. For assigned loading conditions, it is shown that the equilibrium of the system and the position where the crack actually forms can be obtained from the minimality conditions of an energy functional including the bulk elastic energy and the crack surface energy. The subsequent step-by-step propagation of the cohesive crack is also obtained from the minimality conditions of an energy functional defined for each step. The issue of the algorithmic selection of the energetically more convenient solution is briefly discussed.

1. Introduction

Computational finite element approaches to the simulation of crack inception and propagation in brittle and quasibrittle solids can be subdivided into the broad categories of smeared and discrete crack descriptions.

Smeared crack approaches, based on the simulation of damage growth in the bulk material, are particularly suited for the description of the initial phase of strain localization and consequent material degradation, but lack of physical foundation in the late stage of material separation and, in general, require a finer discretization to accurately resolve the localization band.

Conversely, discrete crack approaches, by their very nature, do not incorporate information on the initial stage of formation of microdamage in the bulk, but are particularly suited for the description of the propagation of displacement discontinuities in the material.

In the past, the simulation of a propagating discontinuity in a finite element mesh was one of the main problems involved with the discrete crack approach. Nowadays, computationally effective techniques are available, for example, adaptive remeshing [Askes and Sluys 2000; Rodríguez-Ferran and Huerta 2000; Pandolfi and Ortiz 2002], the strong discontinuity approach (SDA) [Simo et al. 1993; Oliver et al. 2002; Oliver et al. 2003], the extended finite element method (X-FEM) [Moës et al. 1999; Wells and Sluys

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2001; Moës and Belytschko 2002; Mariani and Perego 2003], which, in conjunction with the adoption of cohesive crack models, have greatly improved the accuracy and efficiency of the simulation.

If attention is restricted to the case of a cohesive crack propagating in an elastic medium, one of the main issues still open in the finite element implementation is the definition of a criterion for initiation and propagation. In a portion of the mesh which is not yet crossed by a crack, the displacement discontinuity does not exist until it is introduced according to a pre-defined criterion which is either local (that is, based on some measure of the local stress or strain state) or global (for example, based on some measure of the energy or other averaged quantities).

In the case of propagation, another problem is the definition of the correct shape and length of the incremental discontinuity corresponding to the assigned load increment. This is usually solved in an iterative way by assigning a tentative length of propagation, solving the problem with the augmented crack extension, and then verifying a posteriori whether at the new tip the propagation condition is still satisfied.

Variational approaches, in which the shape of the crack increment is chosen so as to minimize an energy functional, have attracted considerable attention in recent times in view of their strong mechanical foundation. Both numerical [Bourdin et al. 2000; Negri 2003; Angelillo et al. 2003] and theoretical results [Francfort and Marigo 1998; Dal Maso and Toader 2002; Chambolle 2003] have been presented for perfectly brittle materials, employing a potential given by the sum of elastic and fracture energies. In this framework, the propagation history is obtained as a quasistatic evolution, defined by means of a minimizing sequence: the time interval is discretized with a finite increment Δt and the configuration at the end of each time step is given by a minimizer, with suitable irreversibility constraints. In this kind of model the main source of mathematical difficulties is related to the convergence of the fracture sets as $\Delta t \to 0$. In the case of brittle materials this technical problem has been solved in different ways in [Dal Maso and Toader 2002; Chambolle 2003] and in [Francfort and Larsen 2003]. For cohesive energies some results have been obtained in [Dal Maso and Zanini 2007], assuming a priori the path of propagation, while the general case is still an open problem. A numerical application has been presented in [Dumstorff and Meschke 2005], without addressing, however, the convergence issue. In the one-dimensional framework, these kinds of difficulties are not encountered and more general forms of the fracture energy can be considered (see for instance [Braides et al. 1999; Del Piero and Truskinovsky 2001]). This is the setting adopted in the present work.

A linear elastic bar, constrained at both ends, subjected to a uniformly distributed axial load and to an imposed displacement at one end is considered. The material has a limit strength with a cohesive fracture energy depending on the position. The axial force transmitted through the cohesive crack decreases with the crack opening and the loading-unloading behavior of the interface is governed by a nondecreasing, damage-like internal variable. The solution of the associated incremental problem is shown to be a local minimizer of the potential energy. Following [Braides et al. 1999], it is shown that a single crack is energetically more convenient than multiple cracks at the first crack initiation. This is also shown to be true for subsequent time steps, provided that the algorithmic criterion proposed for the selection of feasible solutions is adopted. The position of the first fracture and the amplitude of its jump are again determined by enforcing minimality. The resulting evolution is quasistatic and satisfies the loading-unloading conditions in Kuhn–Tucker form.

2. Problem definition

A bar of unit cross section, constrained at both ends and subject to a body force $b(x)$ directed along its axis, is considered. The reference configuration of the bar is represented by the interval $I = (0, L)$. In order to account for fractures, the admissible configurations are assumed to belong to a space of (possibly) discontinuous displacement fields u satisfying the boundary conditions $u(0) = 0$ and $u(L) = \eta$. The set of points where u is discontinuous, denoted by S_u, is not prescribed a priori and may contain the endpoints of the bar. Interpenetration is ruled out by the constraint $[u] \geq 0$, $[u]$ denoting the jump of u. A cohesive softening model is assumed for the opening crack. This means that the fracture energy, necessary to create the crack, is progressively released as the jump $[u]$ grows, until the critical value $[u]_{\text{crit}}$ is reached, beyond which no tractions can be transmitted across the discontinuity.

In the bulk, that is, at points belonging to $I \setminus S_u$, the current state of the bar is governed by the following equations for compatibility, equilibrium with nonzero body forces $b(x)$, and elastic (bulk) behavior, respectively,

$$\varepsilon = \frac{du}{dx}, \qquad \frac{d\sigma}{dx} + b = 0, \qquad \sigma = E\varepsilon,$$

where ε is the longitudinal strain, σ the axial stress and E the Young's modulus. On the other hand, for every crack point $z \in S_u$ we have conditions for compatibility at the interface and equilibrium across the crack

$$[u](z) = u^+(z) - u^-(z) \geq 0, \qquad \sigma^+(z) = \sigma^-(z).$$

The bar behaves elastically as long as the axial stress $\sigma(x)$ is below a threshold $\bar{p}(x)$, which is assumed to vary along the bar. The cohesive crack model is assumed to obey a nonreversible damaging law (see Figure 1). The accumulated damage is taken into account by a nondecreasing kinematic internal variable ξ, depending on the material point x. The traction p which can be transmitted across the crack is governed by a softening function $g(\xi)$, with $\frac{dg}{d\xi} \leq 0$, and the maximum $\bar{p}$ of p is such that $\bar{p}(x) = a(x)g(0)$, where $a(x)$ accounts for the variation of the resistance along the bar.

The inelastic potential $G(\xi)$ for $\xi \geq 0$ is defined as

$$G(\xi) = \int_0^{\xi} g(\mu)d\mu.$$

In this way, the softening function $g(\xi)$ is interpreted as the static internal variable conjugate to the damage ξ through the state equation

$$g(\xi) = \frac{dG(\xi)}{d\xi},$$

where, for $\xi = 0$, $\frac{dG(\xi)}{d\xi}$ should be intended as the right derivative of $G(\xi)$. The quantity $G(\xi = [u]_{\text{crit}})$ is commonly referred to as fracture energy and is a material property.

For increasing opening displacement $[u]$, following the softening branch $a(z)g(\xi)$ the traction decreases until the limit opening $[u]_{\text{crit}}$ is reached. For displacement jumps $[u] > [u]_{\text{crit}}$ no traction is transmitted across the crack (Figure 1). For decreasing opening displacement, a linear unloading path is followed, with slope $a(z)g(\xi)/\xi$ depending on the current value of damage. Upon reloading, the same linear path is followed until the current traction threshold $\bar{p}(z) = a(z)g(\xi)$ is reached, then the softening

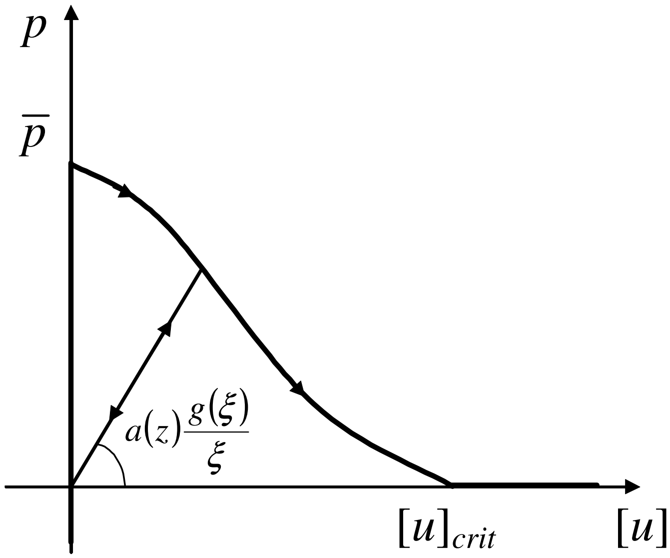

Figure 1. Traction-displacement discontinuity law for cohesive damaging crack.

branch is followed again. The traction-crack opening displacement relation is

$$p(z) \in [0, \bar{p}(z)] \qquad \text{for } \xi = 0,$$
$$p(z) = a(z)\frac{g(\xi)}{\xi}[u](z) \qquad \text{for } \xi > 0,$$

and the loading-unloading conditions are

$$p - a(z)g(\xi) \le 0, \quad (p - a(z)g(\xi))\dot{\xi} = 0 \qquad \text{for } \dot{\xi} \ge 0, \tag{1}$$

while the consistency condition $p - a(x)g(\xi) \le 0$ must be satisfied everywhere in the bulk.

These definitions imply that, for a crack at $z \in S_u$,

$$\xi(t, z) = \max_{\tau \le t}[u](\tau, z).$$

Note that the standard form of time-independent inelastic constitutive laws requires also the definition of an inelastic multiplier and of an evolution equation for the internal variable ξ. In the particularly simple damage law here adopted for the cohesive interface, the rate of the internal variable and the inelastic multiplier coincide. To simplify the notation, the rate of the internal variable has been directly introduced into the loading-unloading conditions (1) instead of the inelastic multiplier.

The "local" fracture energy, that is, the energy necessary to create the discontinuity in the point z, is defined as

$$G_f(z) = \lim_{[u]\to[u]_{\text{crit}}} a(z)G([u]) < \infty.$$

Note that cohesive models with $[u]_{\text{crit}} \to \infty$ are also feasible as long as $G_f(z)$ remains bounded.

Stable global response (that is, no snap-back) of the bar is assumed under the imposed end displacement η. It is easy to show that a snap-back response is ruled out if

$$L < -\frac{E}{a(z)\frac{dg}{d\xi}}, \qquad \text{for } 0 \le \xi \le [u]_{\text{crit}}.$$

3. Finite-step problem

In view of the nonreversible nature of the crack evolution, the analysis of the bar response to an assigned history $\eta(t)$, with $t \in [0, T]$, of the imposed displacement, requires the definition of a step-by-step time marching procedure. The structural response u_{n+1} at time $t_{n+1} = t_n + \Delta t$ satisfies in the bulk

$$\varepsilon_{n+1} = \frac{du_{n+1}}{dx}, \qquad \sigma_{n+1} = E\varepsilon_{n+1}, \qquad \frac{d\sigma_{n+1}}{dx} + b = 0, \tag{2}$$

and in the crack points

$$[u]_{n+1}(z) = u^+_{n+1}(z) - u^-_{n+1}(z) \ge 0, \qquad \sigma^+_{n+1}(z) = \sigma^-_{n+1}(z) = p_{n+1}(z).$$

Knowing the configuration at time t_n, the following stepwise-reversible behavior is assumed for the cohesive cracks

$$\xi_{n+1} = \xi_n + \Delta\xi, \qquad p_{n+1} = a(z)\frac{g(\xi_{n+1})}{\xi_{n+1}}[u]_{n+1}, \qquad \text{for } \xi_{n+1} > 0, \tag{3}$$

$$p_{n+1} - a(z)g(\xi_{n+1}) \le 0, \qquad (p_{n+1} - a(z)g(\xi_{n+1}))\Delta\xi = 0, \qquad \text{for } \Delta\xi \ge 0, \tag{4}$$

where the second equation in (3) is the traction-crack opening displacement relation, and the relations in (4) represent the loading-unloading conditions. Note that the above defined finite-step problem can be conceived as resulting from a backward-difference integration of the incremental problem, while the original (continuous) problem will be recovered for $\Delta t \to 0$.

It is therefore possible to define an energy $\tilde{G}([u], \xi_n)$ associated to the reversible finite-step law

$$\tilde{G}([u], \xi_n) = \begin{cases} \frac{g(\xi_n)}{2\xi_n}[u]^2, & \text{for } [u] \le \xi_n, \\ G([u]) - G(\xi_n) + \frac{\xi_n g(\xi_n)}{2}, & \text{for } [u] \ge \xi_n. \end{cases} \tag{5}$$

For $\xi_n = 0$, one has $\tilde{G}([u], \xi_n) = G([u])$. The adopted cohesive finite-step law with the associated energy is shown in Figure 2a at initiation and in Figure 2b in correspondence of a generic time-step.

For $\eta = \eta_{n+1}$, the following functional is defined for the current time-step

$$U^\eta(u, \xi_n) = \tfrac{1}{2}\int_I E\Big(\frac{du}{dx}\Big)^2 dx + \sum_{z\in S_u} a(z)\tilde{G}([u](z), \xi_n(z)) - \int_I bu\, dx. \tag{6}$$

In the expression of the functional, ξ_n plays the role of an assigned parameter. At the end of each step, the internal variable is updated and the functional changes its expression. The updating procedure is schematically shown in Figure 3 where numbered dots denote the value of the functional at the end of the step, before updating, while dots with starred numbers denote the corresponding updated value.

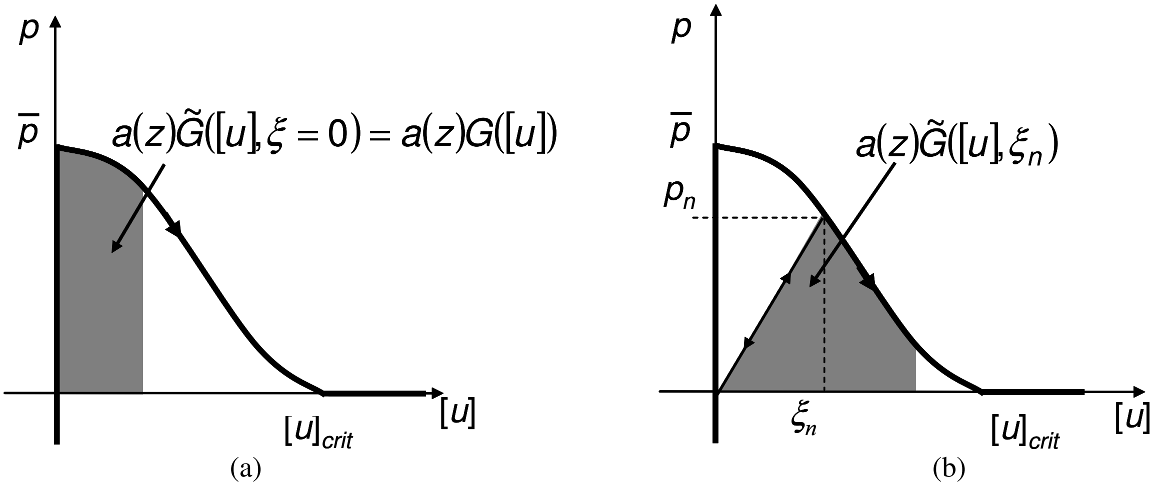

Figure 2. Finite-step cohesive law at initiation (a), at a time-step t_n (b).

Let u' denote the (distributional) derivative of u, that is,

$$u' = \frac{du}{dx}dx + \sum_{z \in S_u} [u](z)\delta_z, \tag{7}$$

where δ_z denotes the Dirac delta concentrated in z. We assume that u' is a bounded measure, that is,

$$\left|u'\right|(I) = \int_I \left|\frac{du}{dx}\right| dx + \sum_{z \in S_u} |[u]| < \infty.$$

Following [Braides et al. 1999] let $B(x)$ be a primitive of $b(x)$ vanishing in zero. Integrating by parts the body force work, the functional can be rewritten as

$$U^\eta(u, \xi_n) = \tfrac{1}{2} \int_I E\left(\frac{du}{dx}\right)^2 dx + \sum_{z \in S_u} a(z)\tilde{G}([u](z), \xi_n(z)) + \int_{[0,L]} Bu' \; - B(L)\eta.$$

Taking into account the definition of the distributional derivative in Equation (7), one obtains

$$U^\eta(u, \xi_n) = \int_I \frac{du}{dx}\left(\tfrac{1}{2}E\frac{du}{dx} + B\right) dx + \sum_{z \in S_u} \big(a(z)\tilde{G}([u](z), \xi_n(z)) + B(z)[u](z)\big) - B(L)\eta. \tag{8}$$

Remark 1. Such functionals have been widely studied in the recent mathematical literature on free discontinuity problems (see, for instance, [Braides et al. 1999]). In this framework the potential energy would be defined in the space $SBV(0, L)$ of special functions with bounded variation (for a definition of SBV spaces, see for example, [Ambrosio et al. 2000]), that is, for the displacements u whose distributional derivative is a bounded measure which can be written as in Equation (7). Existence of minimizers (local and global) in SBV will be discussed in the next section.

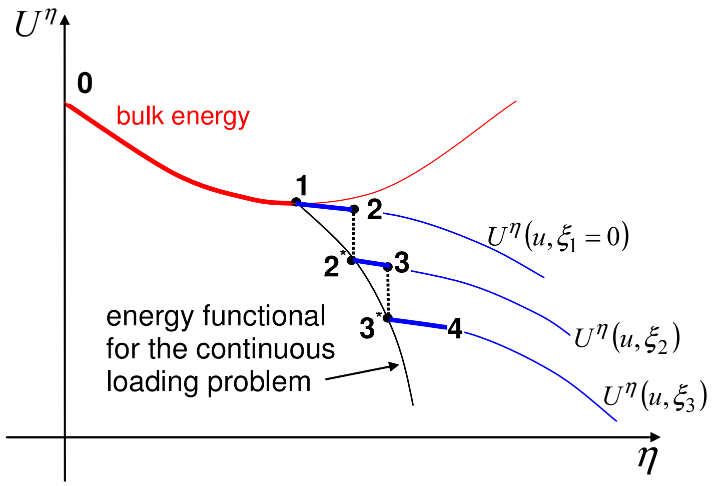

Figure 3. Updating procedure for energy functional. Numbers refer to end of step. Starred numbers refer to updated values of functional.

A displacement u_{n+1} at the end of the finite-step is considered a local minimizer of U^η defined in Equation (6) if $U^\eta(v,\xi) \geq U^\eta(u,\xi)$ for every $v \in SBV(0,L)$ such that

$$\int_I |v-u|\,dx \leq \alpha,$$

for some α sufficiently small. In the following sections it will be shown that the discrete evolution defined by a sequence of local minimizers of Equation (6) satisfies the governing equations (2)-(4). To proceed in the discussion, it is convenient to consider separately the crack initiation problem ($\xi_n = 0$) and the subsequent crack opening problem ($\xi_n \neq 0$).

4. Crack initiation problem

The following crack initiation problem is considered. The bar is subject to an assigned body force $b(x)$, whose intensity is such that the tensile strength $\bar{p}(x)$ is not exceeded in any point of the bar. Then a growing positive displacement η is imposed at $x = L$ until the threshold value $\bar{\eta}$ is reached for which, at a position $\bar{z}$, the stress σ reaches its limit value $\bar{p}(\bar{z})$. The value of $\bar{\eta}$ depends on the strength of the bar and on the body force.

Following the path of reasoning proposed by Braides, Dal Maso and Garroni [Braides et al. 1999], first we prove that the minimizers (both local and global) have a single fracture. Assume that two cracks are activated at z_1 and z_2, with jumps w_1 and w_2 respectively. Let $w = w_1 + w_2$ and let $a_i = a(z_i)$, $B_i = B(z_i)$. Let $H([u])$ be the energy associated to the discontinuity in Equation (8), that is,

$$H([u]) = \sum_{z\in S_u} \{a(z)\tilde{G}([u](z), \xi_n(z)) + B(z)[u](z)\}\,.$$

Noting that at crack initiation ($\xi_n = 0$) one has $\tilde{G} \equiv G$ and considering z_1, z_2 and w as fixed, we can write H and its second derivative as a function of w_1 alone as

$$H(w_1) = a_1 G(w_1) + B_1 w_1 + a_2 G(w - w_1) + B_2(w - w_1),$$

$$\frac{d^2 H(w_1)}{dw_1^2} = a_1 \frac{dg(w_1)}{dw_1} + a_2 \frac{dg(w - w_1)}{d(w - w_1)},$$

for $0 \le w_1 \le w$. As $dg/dw_i \le 0$ and $a_i > 0$, it follows that H is concave and thus its minimum, in the interval $[0, w]$, is in the endpoints. This means that either $w_1 = w$ and $w_2 = 0$ or $w_1 = 0$ and $w_2 = w$. Hence in one of the points z_i there is no jump. Since the bulk energy does not depend on w, following the previous reasoning it is not hard to see that a minimizer can have only one fracture point. We will denote it by $\bar{z}$.

Remark 2. Note that by a similar argument it follows that the minimization problem in SBV is well posed even if the relaxed functional would be defined in the whole BV (see [Braides et al. 1999] and the references therein).

Denoting for notation convenience $U^{\eta \to \bar{\eta}^+} = \lim_{\eta \to \bar{\eta}^+} U^\eta$, the following proposition holds.

Proposition 3. Starting from an elastic state with $\xi_n = 0$, for an assigned value η of the imposed displacement, the governing equations can be obtained from the stationarity conditions of U^η. The position $\bar{z}$ of the activated crack can be obtained from the minimality of $U^{\eta \to \bar{\eta}^+}(u, 0)$.

Proof. Assume that u is a stationary point of U^η and consider the variations of the form $u + \lambda v$, where λ is a scalar variable and $v : [0, L] \to \Re$ a suitable test function. As $u + \lambda v$ must satisfy the boundary conditions, we assume that $v(0) = v(L) = 0$. Moreover, v may have a unit jump in S_v, which we denote by $[v] = 1$. Since interpenetration is not allowed, we must assume that $[u] + \lambda [v] \ge 0$ for λ sufficiently small. The stationarity condition is given, in terms of the first variation of U^η, by the inequality

$$\begin{aligned}
&\lim_{\lambda \to 0^+} \frac{1}{\lambda} \Big(U^\eta(u + \lambda v, 0) - U^\eta(u, 0) \Big) \\
&\quad = \lim_{\lambda \to 0^+} \frac{1}{2\lambda} \int_I \left[E\left(\frac{du}{dx} + \lambda \frac{dv}{dx}\right)^2 - E\left(\frac{du}{dx}\right)^2 \right] dx \\
&\qquad\qquad + \lim_{\lambda \to 0^+} \frac{1}{\lambda} \sum_{z \in S_u \cup S_v} \Big(a(z) G\big([u + \lambda v](z)\big) - a(z) G\big([u](z)\big) \Big) - \int_I bv\, dx \\
&\quad = \int_I \left(E \frac{du}{dx} \right) \frac{dv}{dx} dx + \lim_{\lambda \to 0^+} \frac{1}{\lambda} \sum_{z \in S_u \cup S_v} \Big(a(z) G\big([u + \lambda v](z)\big) - a(z) G\big([u](z)\big) \Big) - \int_I bv\, dx \ge 0. \qquad (9)
\end{aligned}$$

In general, the configuration $u + \lambda v$ may have two jumps, at $S_u = \{\bar{z}\}$ and $S_v = \{z_v\}$. If $S_u = S_v$, then $S_{u+\lambda v} = S_u$. In this case, the second term in Equation (9) can be written as

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda} \Big(a(\bar{z}) G\big([u + \lambda v](\bar{z})\big) - a(\bar{z}) G\big([u](\bar{z})\big) \Big) = a(\bar{z}) g([u](\bar{z}))[v] \ge 0.$$

If $S_u \neq S_v$, then $S_{u+\lambda v} = S_u \cup S_v$ and the previous condition becomes

$$\lim_{\lambda\to 0^+} \frac{1}{\lambda}\Big(a(\bar{z})G\big([u+\lambda v](\bar{z})\big) + a(z_v)G\big([u+\lambda v](z_v)\big) - a(\bar{z})G\big([u](\bar{z})\big)\Big)$$
$$= \lim_{\lambda\to 0^+} \frac{1}{\lambda}\Big(a(\bar{z})G\big([u](\bar{z})\big) + a(z_v)G\big([\lambda v](z_v)\big) - a(\bar{z})G\big([u](\bar{z})\big)\Big) = \lim_{\lambda\to 0^+} \frac{1}{\lambda} a(z_v)G\big([\lambda v](z_v)\big)$$
$$= a(z_v)g(0)[v] \geq 0.$$

Now we consider some particular variations v in order to obtain the governing equations.

- Assume first that $S_u = \varnothing$ and that $v(\bar{z}) = 0$. Integrating by parts the first integral in Equation (9), the inequality becomes

$$E\frac{du}{dx}v\Big|_0^{\bar{z}} - \int_0^{\bar{z}} \Big(E\frac{d^2u}{dx^2} + b\Big)v\,dx + E\frac{du}{dx}v\Big|_{\bar{z}}^{L} - \int_{\bar{z}}^{L} \Big(E\frac{d^2u}{dx^2} + b\Big)v\,dx$$
$$= -\int_0^{\bar{z}} \Big(E\frac{d^2u}{dx^2} + b\Big)v\,dx - \int_{\bar{z}}^{L} \Big(E\frac{d^2u}{dx^2} + b\Big)v\,dx \geq 0,$$

  which gives $(E\frac{d^2u}{dx^2} + b) = 0$, a.e. in $[0, L]$. This is the condition of equilibrium in the bulk.

- Consider now a variation v such that $v(\bar{z})^+ = v(\bar{z})^- \neq 0$. Taking into account that $(E\frac{d^2u}{dx^2} + b) = 0$, Equation (9) then becomes

$$\Big(E\frac{du}{dx}\Big)^{-} v^- - \Big(E\frac{du}{dx}\Big)^{+} v^+ \geq 0,$$

  which gives easily $\sigma^+(\bar{z}) = \sigma^-(\bar{z})$. This is the condition of equilibrium across the crack.

- Now we can choose a variation v having a unit discontinuity at $z_v \in [0, L]$. According to this definition $[v](z_v) = 1$ and $\int_0^L \frac{dv}{dx}dx = -[v](z_v) = -1$. Let us consider first the case $\bar{z} \neq z_v$. Making reference to the expression in Equation (8) of the energy, the stationarity condition can be written as

$$\int_I \Big(E\frac{du}{dx} + B\Big)\frac{dv}{dx}\,dx + B(z_v)[v] + a(z_v)g(0)[v] \geq 0. \tag{10}$$

  Note that, in view of the bulk equilibrium ($E\frac{d^2u}{dx^2} + b = 0$), the stress $\sigma^\eta = E\frac{du}{dx} + B$, which depends on the imposed boundary displacement η, is constant along x. Therefore, one can write

$$-\sigma^\eta + B(z_v) + a(z_v)g(0) \geq 0. \tag{11}$$

  Taking into account that $\sigma^\eta - B(z) = \sigma(z)$ is the stress in the bar due to the dead load and the imposed boundary displacement and that $a(z_v)g(0) = \bar{p}(z_v)$, one has

$$\sigma - \bar{p} \leq 0 \qquad \text{for } z_v \neq \bar{z}. \tag{12}$$

  When $S_u = S_v$, that is, for $z_v = \bar{z}$, since $[u] > 0$ a perturbed configuration of the form $u - \lambda v$ can be considered (as for λ small enough $[u - \lambda v](\bar{z}) \geq 0$). Computing again the limit for $\lambda \to 0^+$, and combining with Equation (10), the stationarity requires that

$$\int_I \Big(E\frac{du}{dx}+B\Big)\frac{dv}{dx}\,dx+B(\bar z)[v]+a(\bar z)g([u](\bar z))[v]=0. \tag{13}$$

Since $\xi_n\equiv 0$ at crack initiation, $\Delta\xi(\bar z)=[u](\bar z)$ at the end of the step and the traction p in the crack follows the softening branch. Therefore, equation Equation (13) becomes

$$\sigma-p=0 \qquad \text{for } z_v=\bar z.$$

It is now easy to see that the governing equations (2)–(4) are all satisfied. □

Finally, we remark that the position $\bar z$ of the crack may depend on the amplitude of the time-step of the incremental solution, as it will be further discussed in Section 6. The correct position is recovered for $\Delta t\to 0$ or equivalently for $\eta\to\bar\eta^+$, where $\bar\eta$ is the critical value of the imposed displacement for which $p(\bar z)=\bar p(\bar z)$. For $\eta=\bar\eta$, the crack will activate at a position $\bar z$ where the stress $\sigma(z)=\sigma^{\bar\eta}-B(z)$ reaches for the first time the critical value $\sigma(\bar z)=\bar p(\bar z)$, which implies that $B(\bar z)+\bar p(\bar z)=\sigma^{\bar\eta}$. At all other points, one has $\sigma(z)=\sigma^{\bar\eta}-B(z)\le\bar p(z)$, which implies that $B(z)+\bar p(z)\ge\sigma^{\bar\eta}$. One can conclude that the position $\bar z$ of the crack is a minimizer of the function $B(z)+\bar p(z)$. If now one considers the case $\eta\to\bar\eta^+$, since $[u]\to 0$ in correspondence of the crack initiation, $G([u])$ behaves like $g(0)[u]$. Thus the energy given by Equation (8) can be written as

$$U^{\eta\to\bar\eta^+}(u,0)=\int_I\frac{du}{dx}\Big(\frac{1}{2}E\frac{du}{dx}+B\Big)\,dx+(\bar p(z)+B(z))[u].$$

As the elastic energy does not depend on the position of cracks, it is clear that a minimizer of $U^{\eta\to\bar\eta^+}(u,0)$ will concentrate the jump $[u]$ in a point where $(\bar p(z)+B(z))$ reaches its minimum.

5. Crack opening problem

The same problem defined in Section 4 is considered at time t_n for an imposed displacement $\eta>\bar\eta$. However, this time the crack remains fixed in the position $\bar z$ of the first activation for $\eta=\bar\eta$, since healing is not permitted by the assumed model, and the opening of a new crack is not energetically convenient, in the sense that will be specified in Section 6. A load step is considered where the imposed displacement is incremented by a quantity $\Delta\eta$. The functional defined in Equation (6) is considered for $\xi_n>0$. The following proposition holds.

Proposition 4. For fixed crack position $\bar z$, consider an evolutionary problem, discretized in time steps with finite increment Δt. The displacement u_{n+1} at time $t_{n+1}=t_n+\Delta t$ solution of the finite-step problem presented in Equations (2)–(4), is obtained minimizing the energy $U^\eta(u,\xi_n)$ with the boundary conditions $u(0)=0$ and $u(L)=\eta$. Note that, with the simple one dimensional cohesive law here adopted, the incremental problem is explicit with respect to the internal variable ξ, therefore this variable can be updated independently at the end of the step, when u_{n+1} is known.

Proof. Assume that u is a stationary point of $U^\eta(u,\xi_n)$ and consider the variations of the form $u+\lambda v$, where λ is a scalar variable and $v:[0,L]\to\Re$ a test function, satisfying homogeneous boundary conditions, which may have a jump in $z_v\in S_v$. The stationarity condition is given in terms of the first variation of $U^\eta(u, \xi_n)$ by the inequality

$$\lim_{\lambda\to 0^+} \frac{1}{\lambda}\big(U^\eta(u+\lambda v, \xi_n) - U^\eta(u, \xi_n)\big) = \lim_{\lambda\to 0^+} \frac{1}{2\lambda}\int_I \Big[E\Big(\frac{du}{dx}+\lambda\frac{dv}{dx}\Big)^2 - E\Big(\frac{du}{dx}\Big)^2\Big]dx + \lim_{\lambda\to 0^+} \frac{1}{\lambda}\sum_{z\in S_u\cup S_v}\Big(a(z)\tilde{G}\big([u+\lambda v], \xi_n\big) - a(\bar{z})\tilde{G}([u], \xi_n)\Big) - \int_I bv\,dx \geq 0.$$

Making use of the same arguments discussed in the previous section, the previous condition can be rewritten as

$$\int_I \Big(E\frac{du}{dx}\Big)\frac{dv}{dx}\,dx + a(z_v)\frac{d\tilde{G}([u], \xi_n)}{d[u]}[v] - \int_I bv\,dx \geq 0.$$

Now we consider some particular variations v in order to obtain the governing equations.

- Assuming $[v] = 0$, following the same path of the previous section one can obtain equilibrium conditions in the bulk and across the interface in $\bar{z}$.
- Now we can choose a variation v having a unit discontinuity at $\bar{z}$. Consider again the form of the functional given in Equation (8) obtained by integrating by parts the body force integral. Taking into account the bulk equilibrium, the stationarity conditions of Equation (8) read

$$-\sigma^\eta + B(\bar{z}) + a(\bar{z})\frac{d\tilde{G}([u], \xi_n)}{d[u]} \geq 0. \tag{14}$$

  Accounting for the definition of $\tilde{G}([u], \xi_n)$ in Equation (5) and noting that $\sigma^\eta - B(\bar{x}) = \sigma(\bar{z}) = p$ is the stress acting on the crack, from Equation (14) one obtains

$$-p + a(\bar{z})\frac{g(\xi_n)}{\xi_n}[u] \geq 0, \qquad \text{for } [u] \leq \xi_n, \tag{15}$$

$$-p + a(\bar{z})g([u]) \geq 0, \qquad \text{for } [u] \geq \xi_n. \tag{16}$$

  When $[u] = 0 < \xi_n$, condition in Equation (15) gives $p \leq 0$, meaning that, if the crack has been already activated ($\xi_n > 0$), a complete closure of the crack ($[u] = 0$) corresponds to zero or negative stress. Assume now $[u] > 0$, in this case also a perturbation $u - \lambda v$ is admissible for small λ, and inequalities opposite to Equations (15)–(16) are obtained. Therefore one has

$$p = a(\bar{z})\frac{g(\xi_n)}{\xi_n}[u], \qquad \text{for } [u] \leq \xi_n, \tag{17}$$

$$p = a(\bar{z})g([u]), \qquad \text{for } [u] \geq \xi_n. \tag{18}$$

  From Equation (17) one obtains

$$p \leq a(\bar{z})g(\xi_n), \qquad \text{for } [u] \leq \xi_n. \tag{19}$$

- Consider a variation v having a unit discontinuity in $z_v \notin S_u$. Having in mind that, for $x \neq \bar{z}$, $[u](x) = \xi_n(x) = 0$, one has

$$a(z_v)\frac{d\tilde{G}([u](z_v), \xi_n(z_v))}{d[u]} = a(z_v)g(0) = \bar{p}(z_v)$$

and the stationarity conditions simply give the consistency condition in the bulk $-\sigma(x)+\bar{p}(x)\geq 0$.

Noting that, by definition, at the end of the step $\Delta\xi=0$ if $[u]\leq\xi_n$ while $\Delta\xi=[u]-\xi_n$ if $[u]\geq\xi_n$, one obtains that : (i) Equations (17) and (18) define the traction-crack opening displacement relation (3); (ii) Equations (18) and (19) can be rewritten in the Kuhn–Tucker form seen in Equation (4) and thus provide the finite-step loading-unloading conditions for the cohesive crack. □

6. Algorithmic aspects

One of the important features of the present finite-step formulation is that the governing equations and the nonreversibility of damage growth are enforced only at the end of the step. In view of the nonconvex character of the energy functional, this implies that solutions may exist which minimize the energy but could not be reached in a continuous process due to the existence of energy barriers. These solutions appears to be an artifact of the algorithmic formulation of the finite-step problem and does not reflect the physical behavior. Therefore, it seems necessary to complement the algorithm with criteria for the exclusion of nonphysical solutions.

Let us consider first the crack initiation problem. At the end of Section 4, it has been proven that for $\eta\to\bar{\eta}^+$, the position of the first crack minimizes the functional $U^{\eta\to\bar{\eta}^+}(u,0)$. We now consider the more general case where η can assume arbitrary values with respect to $\bar{\eta}$. The solution of the step problem is sought by searching for minimizers of the energy $U^\eta(u,0)$. A classical approach consists of computing first an elastic trial, that is, a tentative solution assuming unlimited elastic behavior. If this is not a minimizer, then a better solution is found minimizing the energy along a descent direction, defined by the local gradient. The procedure is repeated until a minimizer is found.

Let u^{tr} be the elastic (trial) solution, that is, the minimizer of U^η restricted to the space of admissible displacements without cracks. Equilibrium in the bulk is clearly satisfied, that is, $(E(u^{tr})'+B)'=(\sigma^{\eta,tr})'=0$; hence for every admissible variation v with jump $[v](z_v)>0$, from Equation (10) one has

$$\lim_{\lambda\to 0^+}\frac{1}{\lambda}\big(U^\eta(u^{tr}+\lambda v,0)-U^\eta(u^{tr},0)\big)=\big(-\sigma^{\eta,tr}+B(z_v)+a(z_v)g(0)\big)[v]. \tag{20}$$

If the variation of U^η given in Equation (20), computed in u^{tr} and with respect to every variation v, is positive, that is, if

$$\lim_{\lambda\to 0^+}\frac{1}{\lambda}\big(U^\eta(u^{tr}+\lambda v,0)-U^\eta(u^{tr},0)\big)\geq 0,$$

then u^{tr} is a local minimizer. As discussed in Section 4, (Equations (11) and (12)), since $[v](z_v)>0$, from Equation (20) it follows that $-\sigma^{\eta,tr}+B(z_v)+a(z_v)g(0)>0$ and thus $\sigma^{\eta,tr}(z_v)<\bar{p}(z_v)$. This means that the stress associated to u^{tr} is everywhere below the limit strength and therefore $u_{n+1}=u^{tr}$ will be the solution of the step. On the contrary, if there exists a variation v such that

$$\lim_{\lambda\to 0^+}\frac{1}{\lambda}\big(U^\eta(u^{tr}+\lambda v,0)-U^\eta(u^{tr},0)\big)=\big(-\sigma^{\eta,tr}+B(z_v)+a(z_v)g(0)\big)[v]<0, \tag{21}$$

then a new solution $u_{n+1}\neq u^{tr}$ will be computed along a descent direction. From the physical point of view it seems reasonable to consider that a fracture could appear in any of the points z_v where the limit strength is exceeded, that is, where Equation (21) holds (for a variation v with a single discontinuity in z_v).

In Section 4 it has been shown that for crack initiation (that is, for $\eta = \bar{\eta}$) it is energetically convenient to open just one crack. It is shown below that the same result is obtained also in a computational procedure starting from the trial solution. Let us assume to be using a solution algorithm capable to deal with multiple cracks. Starting from the trial solution, it may happen that for a suitable variation v with two jumps in z_v^1 and z_v^2, we get

$$\lim_{\lambda\to 0^+} \frac{1}{\lambda}\big(U^\eta(u^{tr}+\lambda v, 0) - U^\eta(u^{tr}, 0)\big)$$
$$= \big(-\sigma^{\eta,tr} + B(z_v^1) + a(z_v^1)g(0)\big)[v](z_v^1) + \big(-\sigma^{\eta,tr} + B(z_v^2) + a(z_v^2)g(0)\big)[v](z_v^2) < 0$$

with

$$\big(-\sigma^{\eta,tr} + B(z_v^2) + a(z_v^2)g(0)\big) > 0. \tag{22}$$

This means that a solution with multiple cracks may appear as energetically convenient with respect to the trial solution, and therefore reachable along a descent direction, even if at one of the points of activation (z_v^2 in this case) a local energy barrier (that is, a local stress below the limit strength) prevents the actual opening of the crack. Such a solution seems not acceptable from a physical point of view and since fracture is a phenomenon governed by the local state of the material, it seems more reasonable to exclude from the search those locations where Equation (22) holds. Therefore, starting from the trial solution, only those points z_v where

$$\big(-\sigma^{\eta,tr} + B(z_v) + a(z_v)g(0)\big) < 0, \tag{23}$$

that is, points where the trial stress exceeds the local strength, will be considered as possible locations of the cracks in the search algorithm. Note that this argument in general does not exclude solutions with multiple cracks. There may be cases where solutions with multiple cracks can be reached along descending paths where condition in Equation (23) is satisfied at each crack location. These solutions are ruled out by the minimality, as for the crack initiation problem the concavity of the inelastic potential $G([u])$ makes it convenient to open only one crack. Its correct position $\bar{z}$ can be obtained searching for the solution along the steepest descent direction v from u^{tr}, that is, looking for the lowest value of the derivative of U^η computed in u^{tr}. It can be readily seen that this is obtained when the crack position minimizes the quantity $B(z_v) + a(z_v)g(0)$.

Let us consider now the crack opening problem. Assume that at time t_n only the first crack at $\bar{z}$ is open. A displacement $\eta > \eta_n$ be assigned at the bar end.

As in the previous case, let u^{tr} be the "elastic" trial solution. Since a damage model is used for the description of the cohesive crack behavior, it is appropriate to compute the trial solution using a secant elastic modulus. This allows to obtain the exact solution in the case of unloading. The following unlimited elastic behavior is assumed for the active cohesive crack (see Figure 4):

$$p^{tr}(\bar{z}) = a(\bar{z})\frac{g(\xi_n)}{\xi_n}[u^{tr}](\bar{z}).$$

In this case, given $\eta > \eta_n$ we have $[u^{tr}] > \xi_n$ and

$$p^{tr}(\bar{z}) = \sigma^{\eta,tr} - B(\bar{z}) > a(\bar{z})g\big([u^{tr}](\bar{z})\big). \tag{24}$$

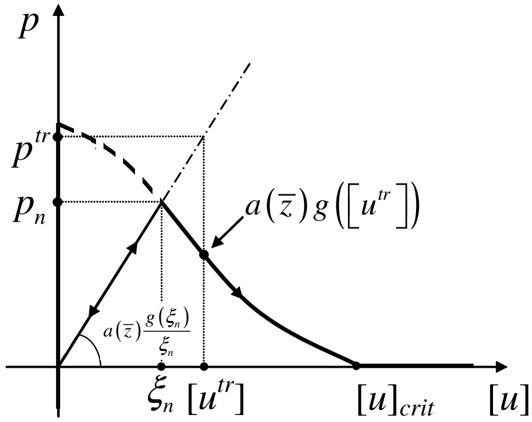

Figure 4. Unlimited elastic cohesive law used for computing the trial solution.

Hence, u^{tr} is not stable with respect to variations v with $[v](\bar{z}) > 0$. Indeed

$$\lim_{\lambda\to 0^+} \frac{1}{\lambda}\big(U^\eta(u^{tr}+\lambda v, \xi_n) - U^\eta(u^{tr}, \xi_n)\big) = \big(-\sigma^{\eta,tr} + B(\bar{z}) + a(\bar{z})g([u^{tr}](\bar{z}))\big)[v](\bar{z}) < 0. \tag{25}$$

A possible minimizer has to be sought within one of the following situations that may arise: (i) the crack at $\bar{z}$ closes and one or more new cracks open; (ii) the crack at $\bar{z}$ grows and one or more new cracks initiate; (iii) the crack at $\bar{z}$ remains constant and one or more new cracks initiate; (iv) the crack at $\bar{z}$ grows and no other cracks initiate.

As done for the crack initiation case, the physical feasibility of possible solutions is tested considering variations with a single jump. Variations v which close the crack, that is, with $[v](\bar{z}) < 0$, lead to a positive derivative in Equation (25) and therefore are not energetically convenient with respect to the trial solution. For this reason, case (i) can be ruled out and only displaced configurations such that $[u](\bar{z}) \geq [u^{tr}](\bar{z})$ will be considered henceforth.

Now we will show that, starting from the trial solution, both closing the active crack and/or initiating new cracks are energetically less convenient than further opening the existing crack (that is, situation (iv)). Let us consider first the situation (ii). According to the criterion in Equation (21) of negative derivative for the selection of possible locations $z_v \neq \bar{z}$ of new cracks, variations with $[v](z_v) > 0$ lead to the local condition $\sigma^{\eta,tr} > a(z_v)g(0) + B(z_v)$. Denote by $w_1 \geq 0$ the incremental opening of the crack at $\bar{z}$, that is, $[u](\bar{z}) = [u^{tr}](\bar{z}) + w_1$, and by $w_2 = [v](z_v) = w - w_1 \geq 0$ the opening of a new crack at $z_v \neq \bar{z}$, w being the total incremental opening with respect to the trial solution. As $\tilde{G}([u^{tr}](\bar{z}) + w_1, \xi_n(\bar{z}))$ is concave in the variable w_1, the argument used in Section 4 allows to state that a single active fracture (corresponding to $w_1 = 0$ or $w_2 = 0$) is always energetically more convenient, which rules out case (ii).

It remains to show that the minimizer is given by $w_2 = 0$ and hence that also case (iii) can be ruled out. Assume by contradiction that u is a minimizer with $w_1 = 0$, so that

$$[u](\bar{z}) = [u^{tr}](\bar{z}) \quad \text{and} \quad [u](z_v) = w_2 = w > 0.$$

In this case, the total energy (see Equation (8)) would be given by

$$U^\eta(u,\xi_n)=\int_I \frac{du}{dx}\Big(\tfrac{1}{2}E\frac{du}{dx}+B\Big)\,dx+a(\bar z)\tilde G([u^{tr}](\bar z),\xi_n(\bar z))+B(\bar z)[u^{tr}](\bar z)+a(z_v)G(w_2)+B(z_v)w_2-B(L)\eta. \quad (26)$$

In view of the results of Proposition 4, assuming that u is a minimizer implies that equilibrium has to be satisfied. Imposing equilibrium we obtain $a(z_v)g(w_2)+B(z_v)=a(\bar z)g([u^{tr}](\bar z))+B(\bar z)=\sigma^\eta$. Being $a(z_v)g(w)+B(z_v)$ a nonincreasing function, in the variable w, we get

$$a(z_v)g(w_2)+B(z_v)=a(\bar z)g([u^{tr}](\bar z))+B(\bar z)\ge a(\bar z)g([u^{tr}](\bar z)+\hat w_1)+B(\bar z), \quad (27)$$

for every choice of $\hat w_1\ge 0$. Taking $\hat w_1=w_2$, one has

$$\begin{aligned} a(z_v)G(w_2)+B(z_v)w_2&=\int_0^{w_2}\Big(a(z_v)g(w)+B(z_v)\Big)\,dw\ge\int_{[u^{tr}](\bar z)}^{[u^{tr}](\bar z)+w_2}\Big(a(\bar z)g(w)+B(\bar z)\Big)\,dw\\ &=a(\bar z)\Big(G([u^{tr}](\bar z)+w_2)-G([u^{tr}](\bar z))\Big)+B(\bar z)w_2. \end{aligned}$$

By this inequality and taking into account that from Equation (5) one has $G([u^{tr}](\bar z)+w_2)-G([u^{tr}](\bar z))=\tilde G([u^{tr}](\bar z)+w_2,\xi_n(\bar z))-\tilde G([u^{tr}](\bar z),\xi_n(\bar z))$, it turns out that the total energy corresponding to the opening $w_2=w$ concentrated in the first crack, that is, with $[u](\bar z)=[u^{tr}](\bar z)+w$ and $[u](z_v)=0$, is

$$U^\eta(u,\xi_n)=\int_I \frac{du}{dx}\Big(\tfrac{1}{2}E\frac{du}{dx}+B\Big)\,dx+a(\bar z)\tilde G([u^{tr}](\bar z)+w_2,\xi_n(\bar z))+B(\bar z)([u^{tr}](\bar z)+w_2)-B(L)\eta,$$

and is smaller than the energy in Equation (26). This proves that the situation (iv) is the one which minimizes the energy.

In conclusion, provided that criterion in Equation (23) is used at each step to assess the feasibility of a point as location of a new crack, it can be stated that also in the present finite-step computational approach, for monotonically increasing imposed displacement, it is always energetically more convenient to open only one crack.

7. An explicit computation

The quasistatic evolution, defined in terms of local minimizers of the potential energy, is verified on a simple one-dimensional example whose step-by-step solution can be obtained analytically.

Consider a bar of length $L=10$ mm and uniform elastic modulus $E=1$ MPa, subject to a constant body force $b=0.2E/L$ and a monotonically increasing imposed displacement $\eta(t)$. The bar is assumed to have a fracture strength $\bar p(x)=a(x)g(0)=a(x)\bar g$ varying along the bar with $a(x)=1+100(x/L-1/2)^2$ and $\bar p(L/2)=\bar g=0.1E$. Denoting by w the displacement discontinuity, a linear cohesive crack model is considered

$$g(w)=\begin{cases}\bar g\big(1-\frac{1}{w_{\text{crit}}}w\big), & \text{for } w\le w_{\text{crit}},\\ 0, & \text{for } w>w_{\text{crit}},\end{cases}$$

where $w_{\text{crit}} = 0.15L$ is the critical opening beyond which no traction can be transmitted across the crack. The potential $G(w)$ is then given by

$$G(w) = \begin{cases} \bar{g}\big(w - \frac{1}{2w_{\text{crit}}} w^2\big), & \text{for } w \le w_{\text{crit}}, \\ \bar{g}\frac{w_{\text{crit}}}{2}, & \text{for } w \ge w_{\text{crit}}. \end{cases}$$

Let $\mathcal{A}(\eta, w)$ be the set of displacements such that $u(0) = 0$, $u(L) = \eta$ and $[u] = w$. Assuming a holonomic process ($\xi = 0$) and imposing equilibrium in the bulk, one can express the displacements along the bar in terms of the assigned end displacement η, the crack opening w and the crack position $\bar{z}$. Equivalently one can minimize U^η with respect to $u \in \mathcal{A}(\eta, w)$, thus obtaining the energy function $\mathcal{U}(\eta, w, \bar{z})$

$$\mathcal{U}(\eta, w, \bar{z}) = \min\{U^\eta(u, 0) \,:\, u \in \mathcal{A}(\eta, w)\}. \tag{28}$$

For this simple example the solution $u(x)$ can be explicitly computed and is given by

$$u(x) = \begin{cases} -\frac{b}{2E}x^2 + \Big(\eta - w + \frac{bL^2}{2E}\Big)\frac{x}{L}, & \text{for } x < \bar{z}, \\ -\frac{b}{2E}x^2 + \Big(\eta - w + \frac{bL^2}{2E}\Big)\frac{x}{L} + w, & \text{for } x > \bar{z}. \end{cases} \tag{29}$$

The energy function is then obtained substituting Equation (29) into Equation (8) and working out the integrals

$$\mathcal{U}(\eta, w, \bar{z}) = \frac{E\,(\eta - w)^2}{2L} + \frac{b(2w\bar{z} - L\,(\eta + w))}{2} - \frac{b^2L^3}{24E} + \Big[1 + 100\Big(\frac{\bar{z}}{L} - \frac{1}{2}\Big)^2\Big]G(w). \tag{30}$$

Note that $\mathcal{U}(\eta, w, \bar{z})$ is differentiable with respect to w and $\bar{z}$. Local minimizers are found from

$$\begin{aligned} \frac{\partial \mathcal{U}}{\partial w} &= -\frac{E\,(\eta - w)}{L} + \frac{b(2\,\bar{z} - L\,)}{2} + \Big[1 + 100\Big(\frac{\bar{z}}{L} - \frac{1}{2}\Big)^2\Big]g(w) = 0, \\ \frac{\partial \mathcal{U}}{\partial \bar{z}} &= bw + \frac{100}{L}\Big(\frac{2\bar{z}}{L} - 1\Big)G(w) = 0. \end{aligned} \tag{31}$$

The second stationarity condition reflects the requirement that the crack must initiate in a position which minimizes the potential energy. At crack initiation ($w \to 0^+$), $G(w)$ behaves like $g(0)w$ and the second condition in Equation (31) requires that

$$b + \frac{100}{L}\Big(\frac{2\bar{z}}{L} - 1\Big)g(0) = 0,$$

which is solved for $\bar{z} = 0.49L$. From the first condition in Equation (31), for $\bar{z} = 0.49L$ and $w = 0$, one obtains the value $\bar{\eta} = 0.099L$ of the imposed displacement at crack initiation.

Figure 5a shows the plot of $\mathcal{U}$ as a function of the crack position and opening displacement for $\eta = \bar{\eta}$ (for representation convenience values of $\mathcal{U}$ >0.2 have been cut in this plot). It should be noted that for $\eta = \bar{\eta}$, $\bar{z} = 0.49L$ is the position of the point where the curve representing the stress along the bar is tangent to the curve representing the fracture strength $\bar{p}(x) = \bar{g}[1 + (x - \frac{1}{2}L)^2]$, see Figure 5b.

For $\eta > \bar{\eta}$ the minimizers of Equation (30) give a position of the crack different from $\bar{z} = 0.49L$, which is not feasible for the real problem: once the material is broken at a certain position it cannot heal and

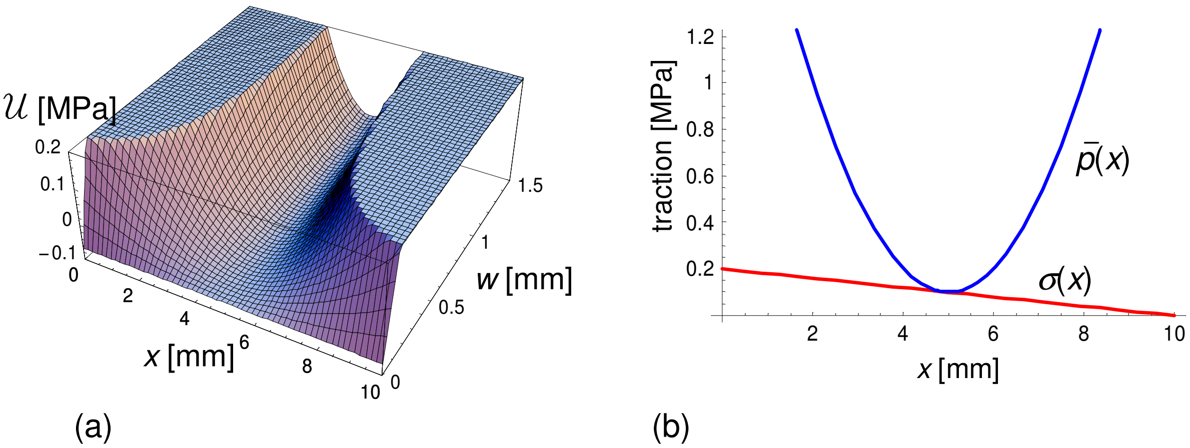

Figure 5. (a) Energy function for $\eta = \bar{\eta}$; (b) crack position.

the crack position cannot change. In this case, the solution has to be sought solving the first condition of Equation (31) for $\bar{z} = 0.49L$.

The history of the assigned end displacement is shown in Figure 6a together with the corresponding computed crack opening. After initiation, the crack position is kept fixed at $\bar{z} = 0.49L$. The damage variable ξ is updated at the end of each step. The solution at each step is computed analytically minimizing the updated functional $U^{\eta}(u, \xi_n)$ according to the step-by-step procedure outlined in Section 5. Since in each step the bar is either monotonically loaded or unloaded, the computed and exact solutions coincide. The computed history of cohesive traction is shown in Figure 6b. The values of the crack opening displacement and the explicit expressions of the energy functionals to be minimized at each step are reported in Table 1.

Contour plots of the minimized functionals are shown in Figure 7. The computed solutions are denoted by a white dot. As expected, the dots do not coincide with the absolute minimum of the functionals, since the crack position has to remain fixed.

The optimal value of the energy $\mathcal{U}$ as a function of the imposed displacement η is plotted in Figure 8. The curve from point 0 to point 1 represents the bulk energy, while the curve from 1 to 2 corresponds to

step	η [mm]	functional to be minimized	w [mm]
1	0.99	$U^{\eta=0.99}(u, \xi = 0) =$ bulk energy $+a(\bar{z})G(w)$	0.00
2	1.20	$U^{\eta=1.20}(u, \xi = 0) =$ bulk energy $+a(\bar{z})G(w)$	0.64
3	0.90	$U^{\eta=0.90}(u, \xi = 0.64) =$ bulk energy $+a(\bar{z})\frac{g(0.64)}{2\cdot 0.64}w^2$	0.48
4	1.30	$U^{\eta=1.30}(u, \xi = 0.64) =$ bulk energy $+a(\bar{z})\big(G(w)-G(0.64)+0.64\frac{g(0.64)}{2}\big)$	0.95

Table 1. Energy functionals to be minimized at each step.

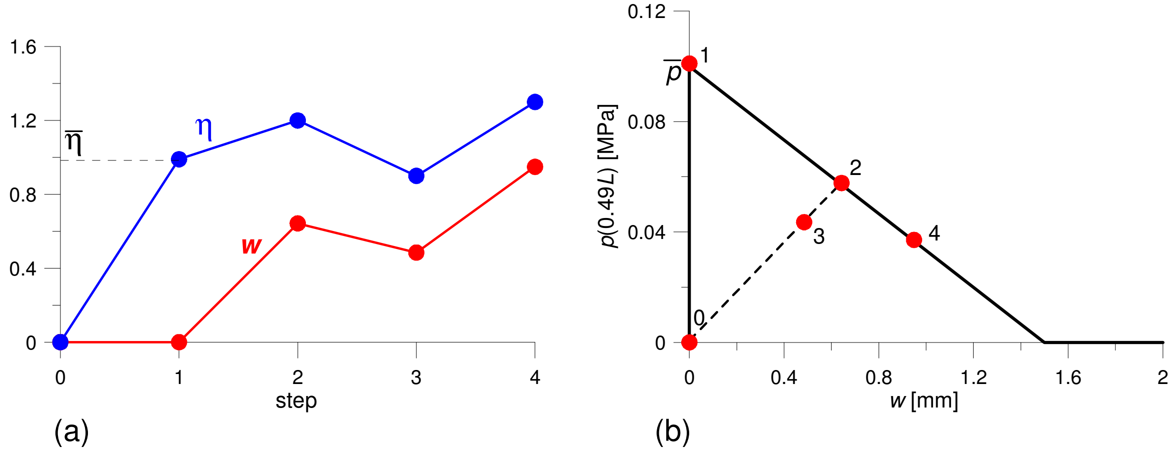

Figure 6. (a) History of assigned end displacement and corresponding computed crack opening; (b) history of cohesive traction across the crack.

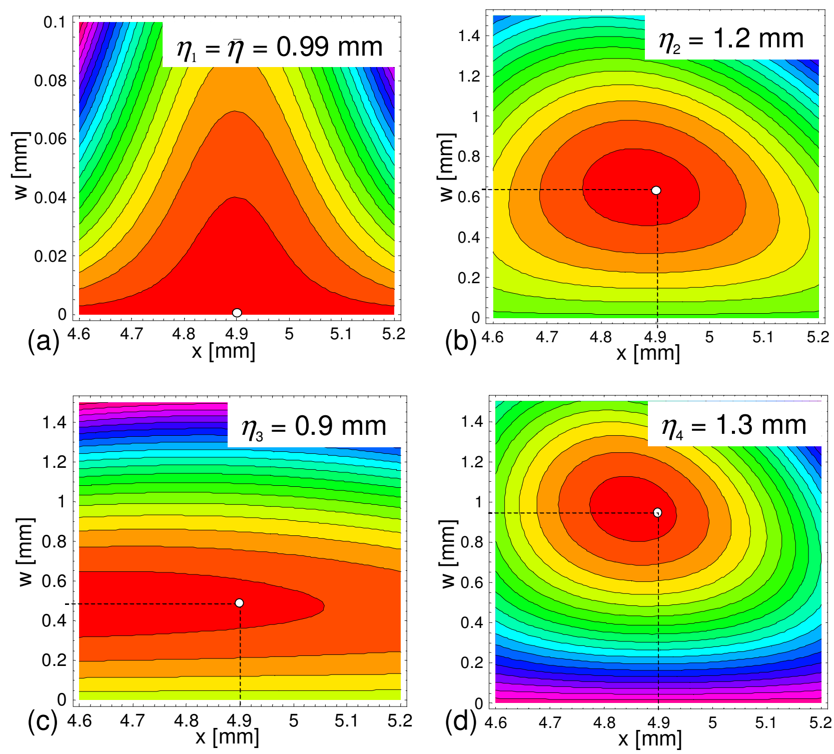

Figure 7. Contour plots of functionals to be minimized at each step. White dots denote computed solutions.

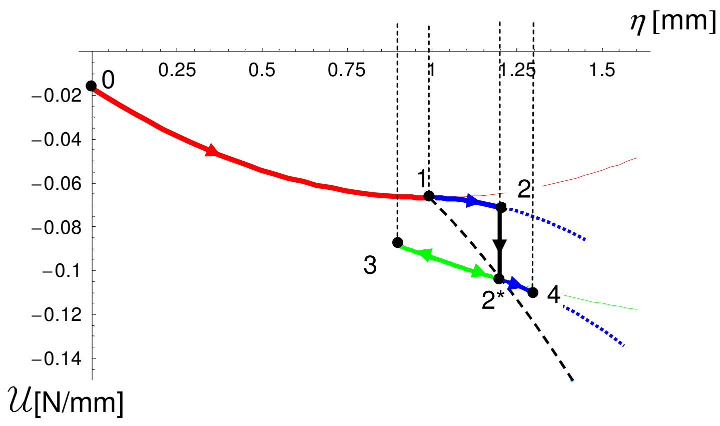

Figure 8. Evolution of energy $\mathcal{U}$ as a function of imposed displacement η. Solid curves denote real path.

the sum of bulk and surface energies, defined only for $\eta \geq \bar{\eta}$. The picture shows clearly how, for $\eta \geq \bar{\eta}$, it becomes energetically more convenient to activate the crack at $\bar{z}$. When $\eta_2 = 1.2$ mm is reached, the functional is modified updating damage and its value passes from point 2 to point 2*. The dashed curve from 1 to 2* represents the exact solution corresponding to continuous increase of damage ξ, as it occurs in the real process. As already pointed out, the computed and exact solution coincide at the end of the step, after damage updating. From 2* to 3, the imposed displacement decreases (unloading) while from 3 to 4 further loading occurs.

8. Conclusions

The problem of a bar constrained at both ends and subject to an axial body force and an imposed displacement at one end has been studied. A linear elastic bulk material with limited strength, varying along the bar axis, has been assumed. When the axial stress exceeds the material strength at a point, a cohesive damaging crack opens producing a localized displacement jump. The possibility of a variational characterization of the solution has been investigated leading to the following results.

Assuming an ideally reversible material behavior (the material heals upon unloading and crack closure), it has been shown that the problem solution minimizes the total potential energy, which includes a bulk term (elastic energy) and a surface term (cohesive fracture energy). It has also been shown that the position of crack initiation can be obtained from a minimality condition for zero crack opening.

If the evolution problem is approximated as a sequence of finite holonomic steps, where the nondecreasing damage internal variable is updated only at the end of each step, it is shown that the finite-step solution can be obtained from the minimum conditions of a suitable functional.

The issue of the opening of multiple cracks has also been addressed. In the finite-step formulation, after crack initiation it is not possible to state that the opening of multiple cracks is not energetically convenient. However, solutions with multiple cracks appear to be a consequence of the finite-step context, and not reachable in a continuous process. An algorithmic criterion has been proposed for the exclusion of solutions which are classified as unphysical. If solutions which do not satisfy this criterion are excluded, then it has been shown that the energy is always minimized by opening only one crack. In our opinion, this discussion may provide a useful insight also in view of possible extensions of the present computational approach to 2 or 3-dimensional problems, where, however, the appropriate underlying mathematical setting is still debated in the current literature on the subject. A simple example with a uniform axial body force has been used to illustrate the theoretical results obtained.

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