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RESONANCE PHENOMENA AT THE INTERFACE OF TWO PERFECTLY BONDED, PRESTRESSED ELASTIC STRIPS

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The problem of vibration localized within the vicinity of the interface of two perfectly bonded semiinfinite elastic strips is investigated. The cases of free and forced vibration are both examined in strips composed of prestressed, incompressible elastic material. It is established that the localized interfacial vibration frequencies are functions of an associated interfacial wave speed. A consequence of the prestress is that interfacial waves exist only for certain regimes of primary deformation. For critical values of principal stretches the wave speed may approach either zero, corresponding to quasistatic interfacial deformations, or an associated body wave speed, corresponding to degeneration of the interfacial wave into a body wave. In the case of free vibration, approaching a critical principal stretch value is shown to result in a significant increase in the edge spectrum density. In the forced vibration problem, a corresponding significant decrease in the influence in the resonances is observed. The analysis is carried out within the most general appropriate constitutive framework, and includes a number of numerical illustrations involving neo-Hookean and Varga materials to illustrate the aforementioned phenomena.

1. Introduction

The subject of interfacial waves propagating along the interface of two perfectly bonded elastic half-spaces is a well-studied scientific area of long history. An important property of such waves is that the associated displacement and stress fields are localized within the vicinity of the interface. In the classical linear isotropic elastic case this problem was first studied by Stoneley [1924], with this type of wave later being termed a Stoneley wave. In more recent years, the study of this type of wave has been extended to more constitutively complicated elastic media; see, for example, [Dowaikh and Ogden 1991; Chadwick 1995a; 1995b].

In the case of two perfectly bonded prestressed incompressible elastic half-spaces, the existence criteria for interfacial waves has been investigated with respect to the most general isotropic strain energy function associated with incompressible elasticity [Dowaikh and Ogden 1991]. Within their two-dimensional study, the crucial influence of the principal stretches on the interfacial wave speed was established. A particular feature well worthy of remark is the existence of critical principal stretch values associated with the vanishing of the interfacial wave speed, indicating the existence of interfacial waves only within certain principal stretch domains. A further feature, which has also previously been investigated, is the problem of interfacial waves propagating along the interface of a half-space and a layer of finite thickness. This introduces the effect of dispersion, which is again highly sensitive to changes in the principal stretches. Such dependency has been investigated by Ogden and Sotiropoulos [1995] and Sotiropoulos and Sifniotopoulos [1995].

Keywords: interfacial vibration, prestress, elasticity.

In this present paper we are specifically interested in the type of so-called interfacial vibration that may occur when two layers, each infinite only in one direction, are perfectly bonded along a common boundary. Attention is restricted to a two-dimensional case; the geometrical setup may therefore be thought of as two perfectly bonded semiinfinite strips. In the case of one semiinfinite strip, vibrations localized within the vicinity of a free edge were investigated by Kaplunov et al. [2004], who established a link between the localized free edge vibration frequencies and the associated surface wave speed. These authors also showed that when a load is applied to the edge, various resonance phenomena may be observed. Within the framework of their study, a number of boundary conditions imposed on the upper and lower faces of the semiinfinite strip were considered and shown able to support localized edge vibrations. The problem was also shown to lead to a discrete spectrum of edge vibration frequencies which are in fact functions of the associated surface wave speed. In the case of surface waves, the wave speed is crucially dependent of the normal Cauchy prestress σ_2. A significant increase in the spectrum density was observed when the value of σ_2 was close to that associated with the vanishing of the surface wave speed.

Our aim is to extend the above investigations to elucidate the existence of localized vibration modes associated with two perfectly bonded, prestressed, semiinfinite incompressible elastic strips. In doing so we establish that localized interfacial vibration modes, associated with free interfacial vibrations, do exist and that their frequencies are functions of an associated interfacial wave speed. Moreover, in the case of vibrations arising from interfacial loading, a discrete spectrum of resonance frequencies is shown to exist, which are again functions of the interfacial wave speed. In contrast to the surface wave speed, the interfacial wave speed is independent of the normal (to the interface) static Cauchy stress. Specifically, we consider a plane problem and focus attention on the case in which both strips have coincident principal prestrain axes, one of which is normal to the interface. In the context of interfacial waves, the analogous problem has been previously investigated, largely for a class of strain energy functions epitomized by the Mooney–Rivlin strain energy function [Ogden 1984].

The layout of this paper is as follows. In Section 2 the specific interfacial vibration problem to be addressed is stated and the governing equations established. In Section 3 it is demonstrated that the discrete spectrum of interfacial vibration frequencies are functions of the associated interfacial wave speed. In Section 4 we review results related to the criteria for the existence of interfacial waves and investigate their consequences for interfacial vibration. In Section 5 the influence of the principal stretches, and in particular the passage towards critical principal stretches, is examined and numerically illustrated.

2. Statement of the problem

Consider two prestressed incompressible elastic plane semiinfinite strips, each of thickness $2h$, which are perfectly bonded along a common boundary. Relative to a Cartesian coordinate system, $\{x_1, x_2, x_3\}$, the two strips are bounded in the x_2 direction, with interface along $x_1 = 0$. A horizontal force of amplitude A acting parallel to x_1, is applied at the interface; see Figure 1. We also remark that the principal axes of the static primary deformation coincide with the chosen coordinate axes. The mixed boundary conditions applied to the faces $x_2 = \pm h$ will preclude any normal displacement, that is, displacement in the x_2 direction. These strips may therefore be thought of as two-dimensional and forming some sort of frictionless enclosure, terminology apparently first adopted by Kaplunov et al. [2004].

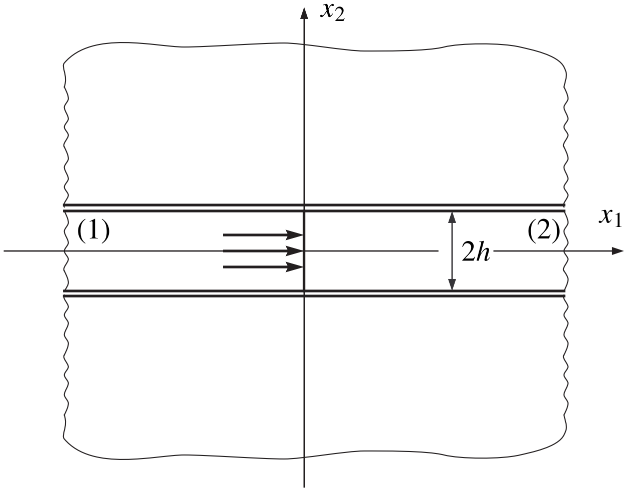


Figure 1. The geometry of the two semiinfinite strips.

Prior to proceeding with the establishment of the appropriate governing equations, it will prove helpful to introduce some notation and conventions. Parameters associated the left and right strips are assigned the superscripts (1) and (2), respectively. We shall also let $\lambda_j^{(i)}$, $i = 1, 2$, $j = 1, 2, 3$, denote the principal stretches along the x_j axes in the indicated strip. The squares of these principal stretches are the eigenvalues of the left, or right, Cauchy–Green strain tensor. In view of our assumption of incompressibility, the product of the principal stretches within each strip must obey the constraint $\lambda_1^{(i)}\lambda_2^{(i)}\lambda_3^{(i)} = 1$. The strain-energy function in each strip will be assumed to be a function of the three appropriate principal stretches. The subsequent analysis will be carried out within a general theoretical framework. However, for numerical calculations particular cases of the Ogden strain-energy function will be employed. This particularly general strain energy function may be represented in the form

$$W\left(\lambda_1^{(i)}, \lambda_2^{(i)}, \lambda_3^{(i)}\right) = \frac{\mu^{(i)}}{s}\left[\left(\lambda_1^{(i)}\right)^s + \left(\lambda_2^{(i)}\right)^s + \left(\lambda_3^{(i)}\right)^s - 3\right], \qquad s \in \mathbb{R}, \tag{1}$$

with $\mu^{(i)}$ denoting the shear modulus [Ogden 1984]. Through a combination of the problem's two-dimensional nature and incompressibility, we will also assume that

$$\lambda_3^{(i)} = 1, \qquad \lambda_2^{(i)} = 1/\lambda_1^{(i)}, \qquad \lambda_1^{(i)} \equiv \lambda^{(i)}. \tag{2}$$

We remark that the two particular cases of the strain energy (1) associated with $s = 1$ and $s = 2$ correspond to what are commonly known as Varga and neo-Hookean materials, respectively. These two strain energy functions will be later employed for the purposes of numerical illustration.

We now consider small time-dependent motions superimposed upon the finite, static primary deformation. In view of the fact that our problem is two-dimensional, we take the infinitesimal displacement v_3 along x_3 to be identically zero and assume that the other two displacement components, v_1 and v_2, are independent of x_3. The two nontrivial equations of motion, assuming time variation of the form $\exp(i\omega t)$, are then given by

$$B^{(i)}_{1111}v^{(i)}_{1,11} + \big(B^{(i)}_{1122} + B^{(i)}_{1221}\big)v^{(i)}_{2,12} + B^{(i)}_{2121}v^{(i)}_{1,22} - p^{(i)}_{,1} = -\rho^{(i)}\omega^2 v^{(i)}_1,$$
$$B^{(i)}_{1212}v^{(i)}_{2,11} + \big(B^{(i)}_{1122} + B^{(i)}_{1221}\big)v^{(i)}_{1,12} + B^{(i)}_{2222}v^{(i)}_{2,22} - p^{(i)}_{,2} = -\rho^{(i)}\omega^2 v^{(i)}_2, \tag{3}$$

with the corresponding two-dimensional linearised incompressibility condition requiring that

$$v^{(i)}_{1,1} + v^{(i)}_{2,2} = 0. \tag{4}$$

For full details concerning the derivation of these governing equations, the reader is referred to [Dowaikh and Ogden 1991]. In (3), B_{ijkl} denote components of the fourth order elasticity tensor, which possesses the symmetries $B_{ijji} = B_{jiij}$ and $B_{iijj} = B_{jjii}$, with $i, j, k, l = 1, 2, 3$. Additionally, a comma indicates differentiation with respect to the indicated component of $\mathbf{x}$, ω is the circular frequency and $\rho^{(i)}$, $i = 1, 2$, are the mass densities.

As alluded to briefly earlier in the paper, the problem we will consider involves imposing conditions on the faces $x_2 = \pm h$, together with some possible loading at the interface $x_1 = 0$. The mixed conditions, imposed on the faces of the strips at $x_2 = \pm h$, are prescribed by

$$v^{(i)}_2\big|_{x_2=\pm h} = 0, \qquad \tau^{(i)}_{21}\big|_{x_2=\pm h} = 0, \tag{5}$$

while the perfectly bonded interface conditions take the form

$$v^{(1)}_j\big|_{x_1=0} = v^{(2)}_j\big|_{x_1=0}, \qquad \tau^{(1)}_{12}\big|_{x_1=0} = \tau^{(2)}_{12}\big|_{x_1=0}, \qquad \tau^{(1)}_{11}\big|_{x_1=0} = \tau^{(2)}_{11}\big|_{x_1=0} - A\phi(x_2), \tag{6}$$

with traction components

$$\tau^{(i)}_{11} = \big(B^{(i)}_{1111} + B^{(i)}_{1212} - B^{(i)}_{1221} - \sigma^{(i)}_1\big)\, v^{(i)}_{1,1} + B^{(i)}_{1122}v^{(i)}_{2,2} - p^{(i)},$$
$$\tau^{(i)}_{12} = B^{(i)}_{1212}v^{(i)}_{2,1} + \big(B^{(i)}_{1212} - \sigma^{(i)}_1\big)\, v^{(i)}_{1,2}, \qquad \tau^{(i)}_{21} = B^{(i)}_{2121}v^{(i)}_{1,2} + \big(B^{(i)}_{1212} - \sigma^{(i)}_1\big)\, v^{(i)}_{2,1},$$

where $\sigma^{(i)}_1$ denote the appropriate principal Cauchy stresses along x_1 in the prestressed equilibrium states within each semistrip. The principal Cauchy stress must also be continuous across the interface, namely, $\sigma^{(1)}_1 = \sigma^{(2)}_1$ at $x_1 = 0$.

We also note that in the perfectly bonded interface conditions in Equation (6), $A\,\phi(x_2)$ denotes a force of amplitude A, which, following Kaplunov et al. [2004], we assume to be

$$\phi(x_2) = \pi^2\Big(\frac{1}{3} - \frac{x_2^2}{h^2}\Big). \tag{7}$$

Finally, in this section it will prove convenient to introduce the two nondimensional parameters $\epsilon = \mu^{(2)}/\mu^{(1)}$ and $\kappa = \rho^{(2)}\mu^{(1)}/\rho^{(1)}\mu^{(2)}$. These will aid numerical investigation and in particular allow us to easily contrast the relative parameters of both semiinfinite strips.

3. The discrete spectrum of the problem solution

Taking into account the boundary conditions imposed upon the upper and lower faces at $x_2 = \pm h$ — see (5) — the solution of equations (3) together with the incompressibility conditions permits the two possible forms of linear combinations

$$\big(v_1^{(i)}, v_2^{(i)}, p^{(i)}\big) = \sum_{n=1}^{\infty} \big[V_1^{(i)} \cos(k_n x_2),\, V_2^{(i)} \sin(k_n x_2),\, k_n P^{(i)} \cos(k_n x_2)\big] \times \exp\big[(-1)^{i+1} q^{(i)} x_1\big], \tag{8}$$

$$\big(\tilde{v}_1^{(i)}, \tilde{v}_2^{(i)}, \tilde{p}^{(i)}\big) = \sum_{n=1}^{\infty} \big[\tilde{V}_1^{(i)} \sin(\tilde{k}_n x_2),\, \tilde{V}_2^{(i)} \cos(\tilde{k}_n x_2),\, \tilde{k}_n \tilde{P}^{(i)} \sin(\tilde{k}_n x_2)\big] \times \exp\big[(-1)^{i+1} \tilde{q}^{(i)} x_1\big], \tag{9}$$

composed of harmonics characterized by two discrete sets of wave numbers $k_n = \Lambda_n/h = \pi n/h$, and $\tilde{k}_n = \tilde{\Lambda}_n/h = \pi(n - \frac{1}{2})/h$, with $n \in \mathbb{N}$, which allow us to define an associated speed of wave propagation using the circular frequency ω as

$$c = \frac{\omega h}{\Lambda_n}, \qquad \tilde{c} = \frac{\omega h}{\tilde{\Lambda}_n}. \tag{10}$$

Without loss of generality, the speed given by the first of Equation (10) is used below as the main unknown parameter. It will be further assumed that both the velocity and circular frequency are real-valued.

To reduce the number of similar and relatively routine calculations, we restrict attention to the first harmonic given in Equation (8). The general form of the problem solution, for both the left and right half-strips, may now be substituted into the equations of motion and incompressibility condition, Equations (3) and (4), giving us

$$\alpha^{(i)} - \rho^{(i)} c^2 - (2\beta^{(i)} - \rho^{(i)} c^2)\Big(\frac{q^{(i)}}{k_n}\Big)^2 + \Big(\frac{q^{(i)}}{k_n}\Big)^4 \gamma^{(i)} = 0, \tag{11}$$

where

$$\alpha^{(i)} = B_{2121}^{(i)}, \qquad \beta^{(i)} = \tfrac{1}{2}\big(B_{1111}^{(i)} + B_{2222}^{(i)} - 2B_{1122}^{(i)} - 2B_{1221}^{(i)}\big), \qquad \gamma^{(i)} = B_{1212}^{(i)}.$$

For solutions to be localized close to the interface at $x_1 = 0$, we require decay of the displacement components and incremental pressure, within both strips, as $|x_1| \to \infty$. As a consequence, the roots of the biquadratic equation (11) must be chosen so that they have positive real parts. Only two roots, denoted by $q_j^{(i)}(c)$, for $i, j = 1, 2$, are able to satisfy this requirement; these may be represented in the explicit forms

$$q_j^{(i)}(c) = k_n \sqrt{\frac{2\beta^{(i)} - \rho^{(i)} c^2 + (-1)^j \sqrt{\big(2\beta^{(i)} - \rho^{(i)} c^2\big)^2 - 4\gamma^{(i)}\big(\alpha^{(i)} - \rho^{(i)} c^2\big)}}{2\gamma^{(i)}}} \tag{12}$$

defined through

$$\big(q_1^{(i)}\big)^2 + \big(q_2^{(i)}\big)^2 = \frac{k_n^2}{\gamma^{(i)}}\big(2\beta^{(i)} - \rho^{(i)} c^2\big), \qquad \big(q_1^{(i)} q_2^{(i)}\big)^2 = \frac{k_n^4}{\gamma^{(i)}}\big(\alpha^{(i)} - \rho^{(i)} c^2\big). \tag{13}$$

Each harmonic in (8) can now be represented through a two term summation, with the arguments of the exponents taking appropriate roots of (11).

To proceed further, we need to represent the function ϕ, defined in (7), as a Fourier cosine series expanded in terms of $\pi x_2/h$, which is readily shown to take the form

$$\phi(x_2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cos(\Lambda_n x_2/h)}{n^2}.$$

The continuity conditions in (6) may now be employed to determine the constants in (8) associated with the n-th harmonic. After doing this we arrive at the following form of the solution for displacement components and pressure increment

$$\big(v_1^{(i)}, v_2^{(i)}, p^{(i)}\big) = \sum_{n=1}^{\infty}\sum_{j=1}^{2} \frac{A(-1)^{n+j}\big(q_2^{(i)}-q_1^{(i)}\big)\big(w_j^{(i)}+y_j^{(i)}\big)e^{(-1)^{i+1}q_j^{(i)}x_1}}{n^2\big(q_2^{(1)}-q_1^{(1)}\big)\big(q_2^{(2)}-q_1^{(2)}\big)f(c)} \times \big(k_n^2\cos(k_n x_2), k_n q_j^{(i)}\sin(k_n x_2), -q_j^{(i)} z_j^{(i)}\cos(k_n x_2)\big),$$

within which the functions occurring in the numerator are given by

$$w_1^{(l)}(c) = \gamma^{(l)}\big(q_1^{(l)}q_2^{(l)} + q_1^{(l)}q_2^{(m)} + q_2^{(l)}q_2^{(m)} - k_n^2\big), \quad w_2^{(l)}(c) = \gamma^{(l)}\big(q_1^{(l)}q_2^{(l)} + q_1^{(m)}q_2^{(l)} + q_1^{(l)}q_1^{(m)} - k_n^2\big),$$
$$y_1^{(l)}(c) = \gamma^{(m)}\Big(k_n^2 + \big(q_2^{(m)}\big)^2\Big), \quad y_2^{(l)}(c) = \gamma^{(m)}\Big(k_n^2 + \big(q_1^{(m)}\big)^2\Big),$$
$$z_j^{(l)}(c) = k_n^2\Big[B_{1221} - B_{2222} + B_{1122} + B_{1212}\big(q_j^{(l)}/k_n\big)^2 + \rho^{(l)}c^2\Big],$$

with $f(c)$ given by

$$f(c) = {\gamma^{(1)}}^2 g^{(1)}(c) - \gamma^{(1)}\gamma^{(2)}\Big[2k_n^2\big(k_n^2 - q_1^{(1)}q_2^{(1)} - q_1^{(2)}q_2^{(2)}\big) + 2q_1^{(1)}q_2^{(1)}q_1^{(2)}q_2^{(2)} + \big(q_1^{(1)}+q_2^{(1)}\big)\big(q_1^{(2)}+q_2^{(2)}\big)\big(q_1^{(1)}q_2^{(1)} + q_1^{(2)}q_2^{(2)}\big)\Big] + {\gamma^{(2)}}^2 g^{(2)}(c), \tag{14}$$

where $g^{(i)}(c) = k_n^4 - 2q_1^{(i)}q_2^{(i)}k_n^2 - q_1^{(i)}q_2^{(i)}(q_1^{(i)}q_2^{(i)} + (q_1^{(i)})^2 + (q_2^{(i)})^2)$, $l \neq m \in \{1, 2\}$ and $i, j = 1, 2$. In accordance with formulae (12), the wave number k_n can be eliminated from $f(c)$. As a result, Equation (14) essentially becomes the phase speed equation associated with interfacial waves propagating along, and localized close to, the interface of two perfectly bonded incompressible prestressed elastic half-spaces. These two half-spaces have the same geometrical setup and prestressed states as our two semiinfinite strips. A similar equation has previously been derived within the interfacial wave context by Dowaikh and Ogden [1991]. Therefore, by making use of the theory of the interfacial waves we are able to use the existence conditions for the zeros of $f(c)$ interpreting them as free vibrational frequencies or, alternatively, resonance frequencies of our problem with respect to the wave number k_n.

4. Prestressed domain within which resonance exists

Before proceeding to reviewing the existence conditions for interfacial waves, we first note the strong ellipticity conditions, which for the two-dimensional problem considered here take the form

$$\alpha^{(i)} > 0, \qquad \gamma^{(i)} > 0, \qquad \beta^{(i)} > -\sqrt{\alpha^{(i)}\gamma^{(i)}}. \tag{15}$$

Results relating to the existence of interfacial waves will be stated without proof. For further details the reader is referred to [Dowaikh and Ogden 1991].

Clearly, the solutions of the equation $f(c) = 0$ have to be consistent with the decay conditions $\Re\{q_j^{(i)}(c)\} > 0$. As may readily be established from (12), these inequalities introduce the strict condition that the real part of $q_j^{(i)}$ exists. In order to analyse the domain within which interfacial vibration frequencies, or resonance frequencies, may exist, the two roots given in (12) of the characteristic equation (11) are represented in the form $q_j^{(i)} = k_n(a^{(i)} \pm \sqrt{b^{(i)}})^{1/2}$, within which $a^{(i)}$ and $b^{(i)}$ are readily determined from Equation (12). Investigation of the decay condition above may be carried out with respect to the two cases

$$a^{(i)} > 0, \qquad b^{(i)} < 0. \tag{16}$$

Due to the fact that the decay conditions dictate that only real or complex conjugate values of characteristic roots $q_j^{(i)}$ may arise, these inequalities help us to uniquely define the conditions that will guarantee the existence of a positive real part of the solutions for $q_j^{(i)}$. In addition, it is then possible to consider $f(c)$ as a real function, this being readily established by considering the resulting specific forms of (14). As expected, whenever the equation $f(c) = 0$ possesses real solutions, associated free interfacial (or resonance) vibrations will exist.

Within the framework of the incompressibility requirement introduced in (2), conditions (16) may be analysed in terms of regions in the $\lambda^{(1)}\lambda^{(2)}$-plane. The two inequalities respectively define the two regions as

$$2\beta^{(i)} \geqslant \alpha^{(i)}, \qquad 2\beta^{(i)} < \alpha^{(i)}, \tag{17}$$

where we recall that for specific strain energy functions, α and β are themselves known functions of λ.

The region defined by the first inequality in (17) introduces an upper bound to ensure a nonnegative squared wave speed, hence defining the domain of existence of interfacial vibration frequencies. Using Equation (13) together with the first inequality in (17), the wave speed is bounded as follows

$$0 \leqslant c \leqslant \min_{i=1,2} \{c_L^{(i)}\}, \qquad c_L^{(i)} = \sqrt{\alpha^{(i)}/\rho^{(i)}}. \tag{18}$$

These are the so-called limiting wave speeds for appropriate half strip. They may also be identified as the plane shear wave speeds at which interfacial waves degenerate into shear waves.

Crucially, the upper bound in (18) may in certain cases be not compatible with the required decay condition for some range of $\alpha^{(i)}$. In such cases the upper bound must be redefined; see [Ogden and Sotiropoulos 1995; Sotiropoulos and Sifniotopoulos 1995] for more details. This leads to the second inequality in (17), where the limit of the wave speed in interval given in Equation (18) must be replaced by zeros of $b^{(i)}$, implying that

$$0 \leqslant c < \min_{i=1,2} \{c_b^{(i)}\}, \qquad c_b^{(i)} = \max_{j=1,2} \left\{ \frac{2\beta^{(i)} - 2\gamma^{(i)} + (-1)^j 2\sqrt{-2\beta^{(i)}\gamma^{(i)} + {\gamma^{(i)}}^2 + \gamma^{(i)}\alpha^{(i)}}}{\rho^{(i)}} \right\}^{1/2}, \tag{19}$$

and if the second inequality in (17) holds, then we have $\min_{i=1,2}\{c_b^{(i)}\} < \min_{i=1,2}\{c_L^{(i)}\}$.

The existence of real $c_b^{(i)}$ is in general dependent on both the strain-energy function and the prestress. In the case of linear isotropic, neo-Hookean or Mooney–Rivlin materials, real $c_b^{(i)}$ can never exist. For Varga materials, however, real $c_b^{(i)}$ may exist for certain prestressed states. In passing we remark that

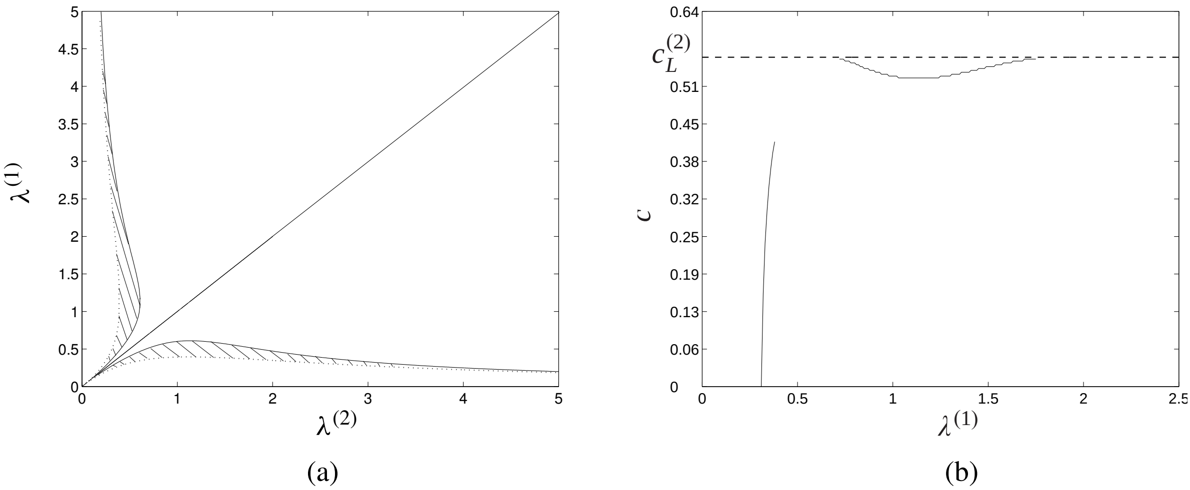


Figure 2. Plots of Equation (14) for a neo-Hookean material with identical material parameters: (a) dependency of $f(c)|_{c=0,c_L^{(i)}}=0$ on principal stretches $\lambda^{(i)}$, $i=1,2$; (b) dependency of $f(c)=0$ on principal stretch $\lambda^{(1)}$ and velocity c when $\lambda^{(2)}=0.5$.

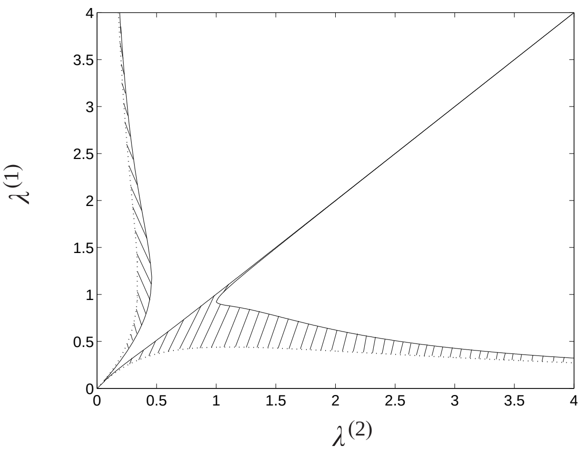
the existence of real $c_b^{(i)}$ also indicates that the associated slowness section may be nonconvex. This has important consequences for reflection and transmission of waves [Ogden and Sotiropoulos 1997]. In view of the two speeds given in Equations (18) and (19), the two regions in (17) may be expressed as $2\beta^{(l)} \geq \alpha^{(m)}$ and $2\beta^{(l)} < \alpha^{(m)}$, respectively, with $l, m = 1, 2$.

4.1. ***Neo-Hookean material: s=2.*** In this section results are presented for the case of a neo-Hookean strain energy function. The main parameters of the problem in this case are given by

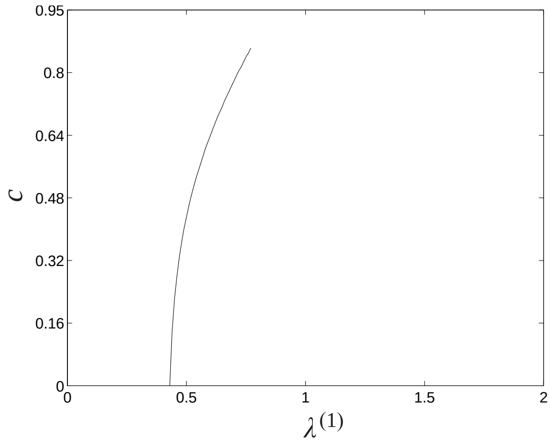
$$\alpha^{(i)} = \mu^{(i)}\big(\lambda^{(i)}\big)^2, \qquad \gamma^{(i)} = \frac{\mu^{(i)}}{\big(\lambda^{(i)}\big)^2}, \qquad \beta^{(i)} = \frac{\alpha^{(i)}+\gamma^{(i)}}{2}.$$

These definitions and the strong ellipticity conditions (15) dictate that only the one region, given by the first inequality in (17), need be considered. As a result, the interval of resonance existence is uniquely determined merely by consideration of inequality (18).

Our graph of the $\lambda^{(1)}\lambda^{(2)}$-plane is similar to plots previously seen in [Dowaikh and Ogden 1991; Sotiropoulos and Sifniotopoulos 1995]. Specifically, the numerical illustrations which we present relate to two different scenarios. The first case, presented in Figure 2, is for identical materials in both half-strips, that is, $\epsilon = 1$, but with differing prestress $\lambda^{(i)}$. Within this figure, the prestressed states associated with resonance are indicated as the shaded regions within the narrow domains that lie between the dotted and solid curves in Figure 2(a). The solid and dotted curves define solutions of the equation $f(0)=0$ and $f(c_L)=0$, respectively, where c_L is the smallest value of $c_L^{(i)}$, $i=1,2$. The potential resonance existence region may also be confirmed by considering Figure 2 (b), illustrating resonance with respect to $\lambda^{(1)}$ at the fixed value of $\lambda^{(2)}=0.5$. For this value of $\lambda^{(2)}$ we must take into account both parts of Figure 2(a), that is, that below and above the line $\lambda^{(1)}=\lambda^{(2)}$. The domain $\lambda^{(1)}>\lambda^{(2)}$ is of particular interest in our numerical analysis, where both a decrease and an increase of the resonance velocity, bounded by $c_L^{(2)}$,



(a)



(b)

Figure 3. Plots of Equation (14) for a neo-Hookean material with contrasting material parameters ($\epsilon = 0.6$): (a) dependency of $f(c)|_{c=0,c_L^{(i)}} = 0$ on principal stretches $\lambda^{(i)}$, $i = 1, 2$; (b) dependency of $f(c) = 0$ on principal stretch $\lambda^{(1)}$ and velocity c when $\lambda^{(2)} = 0.9$.

may be observed. In fact, the resonance velocity will tend to zero as the principal stretch $\lambda^{(2)}$ approaches the point at which the tangent to the upper curve $f(0) = 0$ is parallel to the $\lambda^{(1)}$-axis in Figure 2(a).

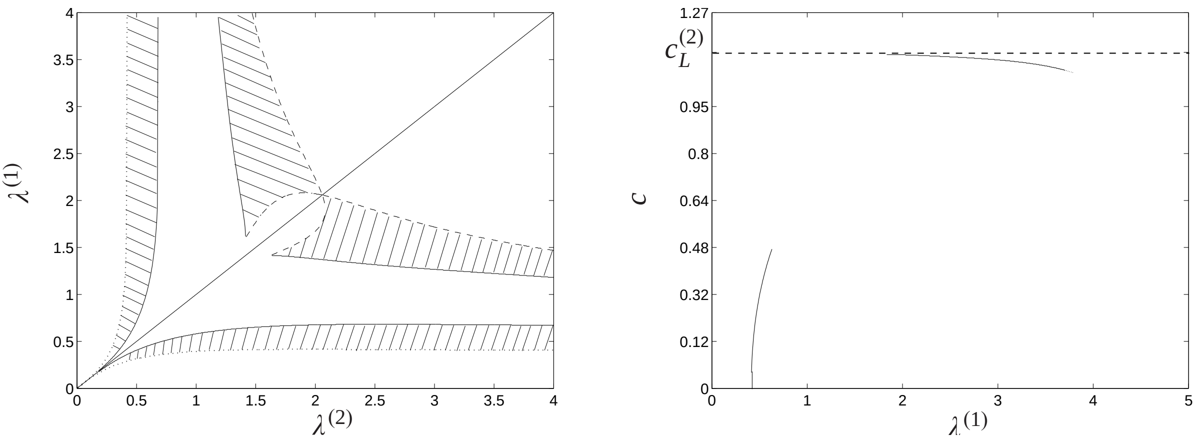
A contrast case with $\epsilon = 0.6$ and $\kappa = 1$ is presented in Figure 3. The resonance existence characteristics associated with the region below the line $\lambda^{(1)} = \lambda^{(2)}$ in Figure 3(a) differ considerably from the previous example and from those above the line in this example. The upper existence domain is again indicated as a shaded region between the solid and dotted curves, defined by $f(0) = 0$ and $f(c_L) = 0$. However, the lower existence region is bounded above by the line $\lambda^{(1)} = \lambda^{(2)}$, to which the curve $f(c_L^{(1)}) = 0$ asymptotes. In this case along the line $\lambda^{(1)} = \lambda^{(2)}$ we have $c_L^{(1)} = c_L^{(2)}$. We additionally note that the lower existence region is in fact significantly larger in size than both the upper region and the upper and lower regions of the previous example. The lower existence interval is shown in Figure 3(b) for the fixed value $\lambda^{(2)} = 0.9$. The associated velocity approaches that corresponding to the point on the line $\lambda^{(1)} = \lambda^{(2)} = 0.9$ in Figure 3(a), and there exists no other branch of the curve unlike the previous example.

4.2. ***Varga material: s=1.*** The problem parameters for the case of a Varga material are

$$\alpha^{(i)} = \frac{\mu^{(i)}\left(\lambda^{(i)}\right)^2}{\lambda^{(i)} + 1/\lambda^{(i)}}, \qquad \gamma^{(i)} = \frac{\mu^{(i)}}{\left(\lambda^{(i)}\right)^2\left(\lambda^{(i)} + 1/\lambda^{(i)}\right)}, \qquad \beta^{(i)} = \sqrt{\alpha^{(i)}\gamma^{(i)}}.$$

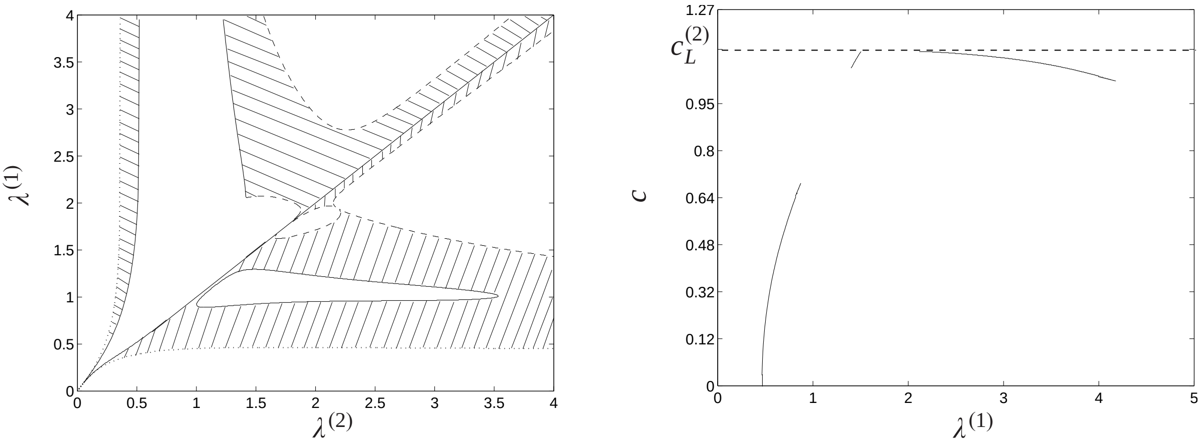
For the Varga material, analysis of the secular equation $f(c) = 0$ allows either of the two inequalities in (17). We may therefore expect more complicated existence domains within the $\lambda^{(1)}\lambda^{(2)}$-plane.

Figure 4 presents an illustration for a noncontrast case, with $\epsilon = 1$ and $\kappa = 1$. Observation of Figure 4(a) clearly shows two types of domains within which resonance may exist. The first of these is a natural analogue of the previously observed neo-Hookean type case. In this case, these two regions are the shaded regions between the dotted and solid curves which approach the origin. The two additional



(a) (b)

Figure 4. Plots of Equation (14) for a Varga material with identical material parameters: (a) dependency of $f(c)|_{c=0,c_L^{(i)},c_b^{(i)}} = 0$ on principal stretches $\lambda^{(i)}$, $i = 1, 2$; (b) dependency of $f(c) = 0$ on principal stretch $\lambda^{(1)}$ and velocity c when $\lambda^{(2)} = 1.5$.



(a) (b)

Figure 5. Plots of Equation (14) for a Varga material with contrasting material parameters ($\epsilon = 0.6$): (a) dependency of $f(c)|_{c=0,c_L^{(i)},c_b^{(i)}} = 0$ on principal stretches $\lambda^{(i)}$, $i = 1, 2$, (b) dependency of $f(c) = 0$ on principal stretch $\lambda^{(1)}$ and velocity c when $\lambda^{(2)} = 1.5$.

shaded areas occur because of the existence of real values for $c_b^{(i)}$. For these domains, their extremities are the solid curve, defined by $f(c_L^{(i)}) = 0$, and the dashed curve, defined by $f(c_b^{(i)}) = 0$.

Figure 4(b) presents the resonances associated with both the above mentioned domain types for the fixed value of $\lambda^{(2)} = 1.5$. As might be expected for this value of $\lambda^{(2)}$, the upper branch emanates from close to the line $c = c_L^{(2)}$, specifically originating at the point $c = c_b^{(2)} \approx 1.13$ and then monotonically decreasing to the value $c = c_b^{(1)}$.

The final numerical example in this section concerns the contrast case $\epsilon = 0.6$ and $\kappa = 1$. As with the immediately previous example, this produces significant changes in the existence characteristics which may be observed in Figure 5(a). In this case a closed solid curve is observed below the line $\lambda^{(1)} = \lambda^{(2)}$. Within this region, the boundary of which corresponds to the solution of $f(c_L^{(1)}) = 0$, resonances may not exist. Moreover, an additional closed curve exists with boundary $f(c_b^{(1)}) = 0$ and represented by a dashed curve. Again, within this region no resonances may exist.

Figure 5(b) presents the resonances associated with both the above mentioned domain types for the fixed value of $\lambda^{(2)} = 1.5$. In this case the upper branch starts at a point associated with the solution of $f(c_L^{(1)}) = 0$ and monotonically increases to a point on the line $\lambda^{(1)} = \lambda^{(2)}$. After this, it monotonically decreases to the lowest possible value of $c_b^{(1)}$ associated with existence of resonances. We also note that for this example the value of $c_b^{(2)}$ is well within 1% of $c_L^{(2)}$.

5. Critical values of stretches

In this section we numerically examine the cases in which the associated interfacial wave either degenerates into a shear wave or a quasistatic interfacial deformation. In the first graph shown in Figure 6 the variation of $v_1^{(2)}$ with velocity is presented for the fixed value of $\lambda^{(2)} = 1.5$ and an appropriate range of $\lambda^{(1)}$. The range of $\lambda^{(1)}$ chosen is that indicated by the lower branch in Figure 4(b). The degeneration of localized interfacial vibration is indicated by the gradual reduction in the width of the resonance spikes as the speed, and corresponding $\lambda^{(1)}$, approach those values associated with degeneration into a shear wave. This is in fact the upper-most point on the lower branch of Figure 4(b).

In Figure 7, a scenario in which the interfacial wave degenerates into quasistatic interfacial deformations is presented. Specifically, these graphs show an appropriate approximation of $v_1^{(2)}$ against scaled frequency Ω over the range from $0 \le \Omega \le 1.0$. For all three presented graphs within this figure we take $\lambda^{(2)} = 1.5$ and three specific values of $\lambda^{(1)}$ associated with those approaching the critical state. It is immediately noted that, as the critical state is approached, by moving from (a) to (b) to (c), a significant increase in the interfacial spectrum density is observed.

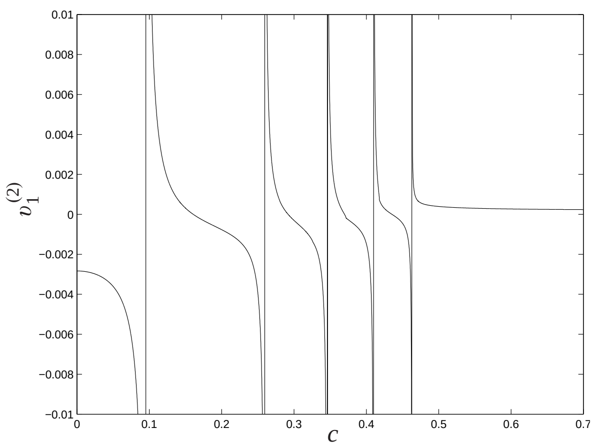


Figure 6. Degeneration of the interfacial resonances for $\lambda^{(1)} \in [0.42, 0.65]$, with $\lambda^{(2)} = 1.5$.

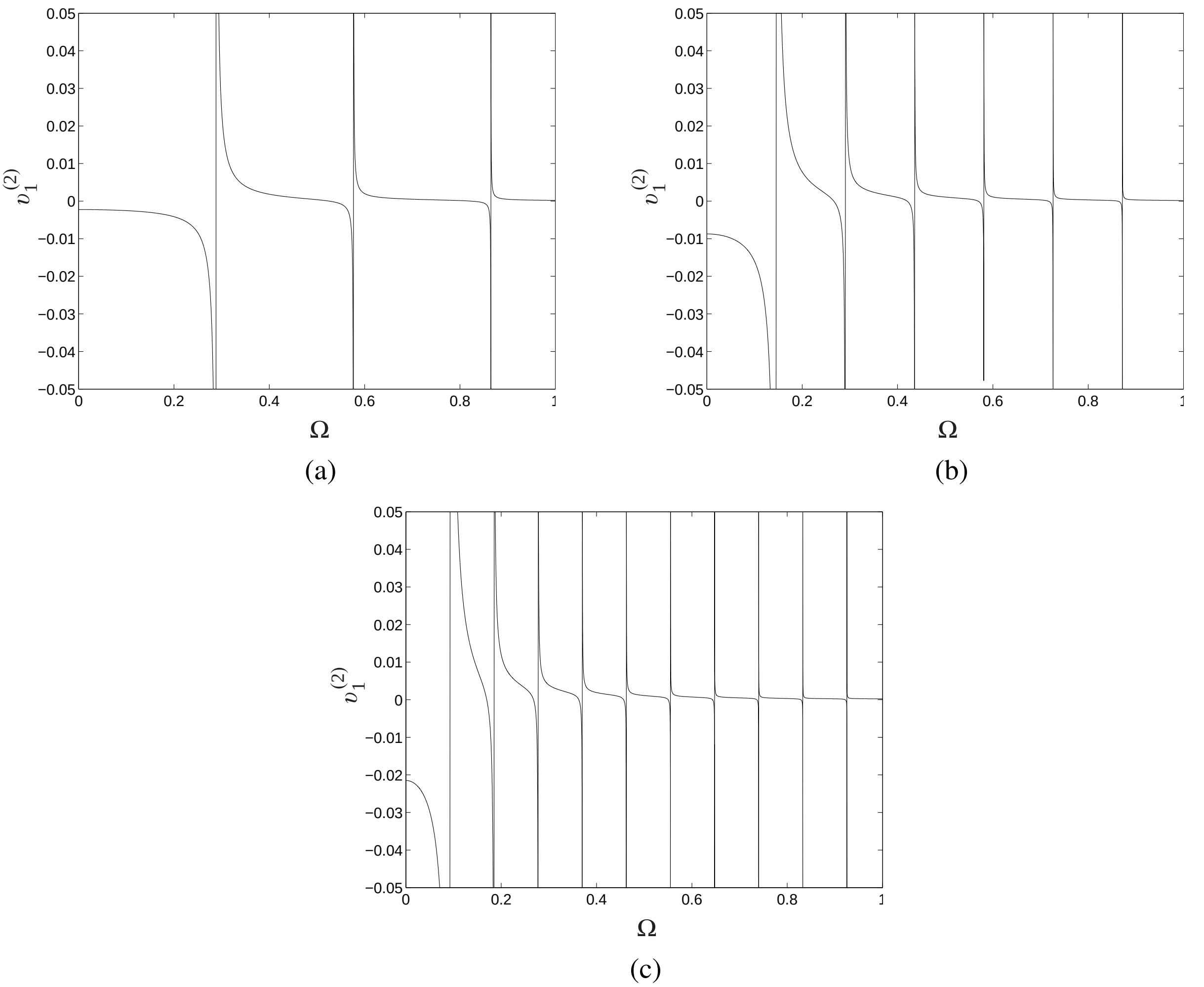


Figure 7. Singularities of displacement v_1, as functions of frequency $\Omega = c\,k_n$, for Varga materials with $\lambda^{(2)} = 1.5$: (a) $\lambda^{(1)} = 0.4$, (b) $\lambda^{(1)} = 0.395$, (c) $\lambda^{(1)} = 0.394$.

Finally, in Figure 8, plots of the real and imaginary parts of $v_1^{(i)}$ are presented against x_1, the in-plane spatial variation, for a number of values of $\lambda^{(1)}$ and fixed $\lambda^{(2)} = 1.5$. These plots clearly show that the displacement is in general localized within the vicinity of the interface $x_1 = 0$, with this localization being least pronounced when the velocity is close to the shear velocity. In particular, we note that it is quite possible to have a situation in which significant localization may occur within one semistrip, with little localization visible in the other. This is particularly the case when $\lambda^{(1)} = 0.62$, in which case there is significant localization in the right strip and little in the left.

6. Conclusion

For certain upper and lower face boundary conditions it has been established that localized vibrations may exist within the vicinity of a common plane boundary of two perfectly bonded semiinfinite strips.

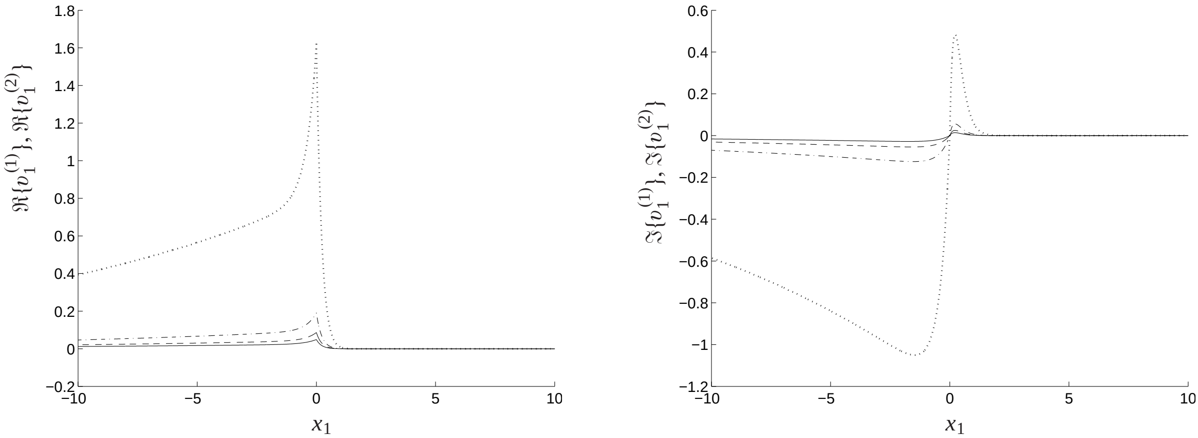


Figure 8. Localisation of the interfacial vibrations, showing displacements $v_1^{(i)}$, $i = 1, 2$, against x_1 as the velocity moves closer to the limiting wave speed $c_L^{(1)}$ for a fixed $\lambda^{(2)} = 1.5$: (a) real part, (b) imaginary part.

The frequencies of these so-called interfacial vibrations have been shown to be functions of the associated interfacial wave speed. The presence of an underlying finite primary deformation makes it quite possible for the interfacial wave speed to either vanish or for an interfacial wave to degenerate into a shear wave. These scenarios have been shown to have important consequences with regard to types of localized interfacial vibrations considered. In the case of the wave speed vanishing, a significant increase in the spectrum density has been observed, whereas degeneration into a body wave has been shown to be associated with the diminishing influence of the resonances within the overall dynamic response.

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