THREE NUMERICAL PROCEDURES FOR THE POST-CRITICAL FLUTTER OF AN ORTHOTROPIC PLATE

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Abstract

It is of paramount importance to know the post-critical flutter behaviour of both orthotropic and isotropic plates in high supersonic flow to individuate the stability conditions of aircraft panels at very high speeds. The amplitude of the limit cycle fluttering plate can characterize the resistence of the panel to air flow at supersonic speeds, because the smaller this amplitude is, the higher this resistence is to the flutter phenomenon.

In the present work three different methods have been utilized for the integration on the panel surface to obtain a system of differential equations in time, which integrated by appropriate algorithms give the vibrating plate behaviour vs the time. Thus it is possible to determine with each method the permanent solution in post-critical conditions.

The knowledge of the flutter behaviour of a vibrating plate is useful also for multi-layered composite laminates, because it is well known that the dynamic analysis of a nearly symmetric and balanced composite vibrating structure can be simulated by an equivalent orthotropic plate, with appropriate values of its thickness and elastic parameters.

1. Introduction

Post-critical flutter behaviour of plates and shells under high supersonic flow has been a research subject for several authors, because of its particular importance for aerospace applications.

Von Karman's large deflection theory [1], which takes into account the presence of non-linear structural forces, has been employed by every author, together with the quasi-steady first order high supersonic theory [2]. The Galerkin method [3,4] has been utilized from Dowell [5,6] and Shiau et al. [7], and also the Rayleigh-Ritz method [3,4,8] by Ketter [9] and Eastep et al. [10], for the integration on the panel surface, and thus to reduce the mathematical problem to a system of non-linear ordinary differential equations in time, which are solved by numerical integration. Then other authors utilized the finite element method (FEM) [11,12] to integrate on the plate or shell surface and to derive a system of ordinary equations in-time [13-15]. Also the presence of piezoelectric actuators has been considered in the analysis of the fluttering panel dynamic behaviour, utilizing FEM, to suppress the non-linear panel flutter presence [16,17]. Further the effects due to the presence of thermal loads have also been taken into account [13,15,17].

The main purpose of the work focuses on setting-up particular procedures, based on the classical and well known Galerkin, Ritz and FEM methods, to integrate on the panel surface and derive ordinary differential equations in time.

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First the classical Galerkin method has been utilized as in the Dowell's model [5,6]. Then by applying FEM and the Ritz procedures [18-20] the Lagrangian functional expression has been formed, and from the variational principle [21], a system of ordinary differential equations in time is obtained. From the in-plane constitutive equations the dependence of the membrane dispacements on the out-of-plane one is evaluated. By substituting these dependence relations into the flutter constitutive equations, further vibration equations, where the alone in-time variable is the out-of-plane displacement, are derived in all the three methods. Thus the subsequent in-time integration operations are simplified.

In this paper also cases with in-plane boundary conditions, different from the out-of-plane ones, have been considered. For this reason in the Galerkin method two different models have been utilized; the first one employs the Airy function [22] to arrive at the classical von Karman's equations [1,22], while the second one considers the in-plane displacements behaviours as unknown variables.

If FEM or Ritz procedures are utilized the same model can be employed, with the presence of a parameter, which takes into account the different in-plane geometric boundary conditions.

It is well known that the dynamic behaviour of a nearly symmetric and balanced multi-layered laminate can be simulated by an equivalent orthotropic plate, with appropriate values of the thickness and of the elastic parameters [23,24]. It is then important to determine the influence of these parameters on the post-critical limit cycle amplitude, because this can be indicative of the resistence of the composite structure panel to air flowing at supersonic speeds.

A detailed description of the utilized Ritz and FEM procedures are shown in Appendices B and C, whereas the utilized elements of the series expansion of the Airy function in the free in-plane boundary conditions are introduced in Appendix A for the Galerkin method.

2. Mathematical model

A generic orthotropic plate exposed to a supersonic flow is shown in Fig. 1.

First the Galerkin method will be utilized to simulate the dynamic behaviour throughout the panel surface

2.1 *The Galerkin Method*

For the in-plane boundary conditions two different cases first will be considered.

1) A simply supported plate for the out-of-plane behaviour is supposed, but with the in-plane free borders, that is the membrane stresses vanish on the limit edges:

$$N_x = 0 \qquad N_y = 0 \qquad N_{xy} = 0 \tag{1}$$

Since the in-plane inertial forces are neglected, in this particular case it is better to utilize the Airy function [22], to descrive the in-plane dynamic behaviour and satisfy easily the boundary conditions (1). The in-plane compatibility relations [22], taking into account the in-plane elasticity relations of an orthotropic plate [23], together with the relations of the membrane stresses dependence on the Airy function [22], lead to the following classical von Karman's first constitutive equation [1,6,22], properly modified for an orthotropic plate:

$$\frac{1}{E_y}\frac{\partial^4\phi}{\partial x^4} + \left(\frac{1}{g_{xy}} - \frac{2\nu_{xy}}{E_x}\right)\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{1}{E_x}\frac{\partial^4\phi}{\partial y^4} = \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{2}$$

where $\phi(x,y,t)$ is the Airy function divided by the plate thickness h. Equation (2) can be rewritten in dimension-less form as:

$$a_1^4\frac{\partial^4\psi}{\partial\xi^4} + c_1^4\frac{\partial^4\psi}{\partial\xi^2\partial\eta^2} + b_1^4\frac{\partial^4\psi}{\partial\eta^4} = \gamma^2 a_1^2 b_1^2\frac{\sqrt{E_x E_y}}{E_r}\left[\left(\frac{\partial^2 W}{\partial\xi\partial\eta}\right)^2 - \frac{\partial^2 W}{\partial\xi^2}\frac{\partial^2 W}{\partial\eta^2}\right] \tag{3a}$$

where, if E_r is the Young's modulus of the reference isotropic plate, the Airy function, together with the in-plane coordinates and the flexural displacement, have been reformulated in non-dimensional form:

$$\psi = \frac{\phi}{E_r L^2} \qquad \xi = \frac{x}{a} \qquad \eta = \frac{y}{b} \qquad W = \frac{w}{L_w} \tag{3b}$$

and also the following non-dimensional parameters have been introduced:

$$a_1 = \sqrt[4]{\frac{E_r}{E_y}}\frac{L}{a} \qquad c_1 = \left[\left(\frac{1}{g_{xy}} - 2\frac{\nu_{xy}}{E_x}\right)\frac{L^4}{a^2 b^2}E_r\right]^{1/4} \qquad b_1 = \sqrt[4]{\frac{E_r}{E_x}}\frac{L}{b} \qquad \gamma = \frac{L_w}{L} \tag{3c}$$

The repeated indices rule will be utilized for the formulae in the paper. A series expansion is chosen for $\psi(\xi,\eta,\tau)$:

$$\psi(\xi,\eta,\tau) = a_{i_\psi}(\tau)\varphi_{i_\psi}(\xi,\eta) \tag{4}$$

where τ is the non-dimensional time, which will be defined in equation (11c), and each element $\varphi_{i_\psi}(\xi,\eta)$ can be written as:

$$\varphi_{i_\psi}(\xi,\eta) = \varphi_{i_{\psi x}}(\xi)\varphi_{i_{\psi y}}(\eta) \tag{5}$$

$$i_{\psi x} = 1,2\ldots N_{\psi x} \qquad i_{\psi y} = 1,2\ldots N_{\psi y} \qquad i_\psi = (i_{\psi x} - 1)N_{\psi y} + i_{\psi y} \qquad i_\psi = 1,2\ldots N_\psi \qquad N_\psi = N_{\psi x}N_{\psi y}$$

and $\varphi_{i_{\psi x}}(\xi), \varphi_{i_{\psi y}}(\eta)$ are orthonormal describing functions, which vanish with their first derivatives at the plate borders, so that, taking into account the membrane stresses dependence on the Airy function [22], the boundary conditions in equation (1) are satisfied. These are treated in Appendix A.

Also for the flexural displacement a series expansion is chosen, which satisfy the boundary conditions of a simply supported plate:

$$W(\xi,\eta,\tau) = W_{i_W}(\tau)\chi_{i_W}(\xi,\eta) \tag{6}$$

where each function element $\chi_{i_W}(\xi,\eta)$ can be written as:

$$\chi_{i_W}(\xi,\eta) = \sin(i_{Wx}\pi\xi)\sin(i_{Wy}\pi\eta) \tag{7a}$$

where

$$i_{Wx} = 1,2\ldots.N_{Wx} \quad i_{Wy} = 1,2\ldots.N_{Wy} \quad i_W = (i_{Wx}-1)N_{Wy} + i_{Wy} \quad i_W = 1,2\ldots.N_W \quad N_W = N_{Wx}N_{Wy} \tag{7b}$$

If the series expansions of equations (4) and (6) are substituted into equation (3a), which is pre-multiplied by the generic element φ_m of the Airy function series expansion and integrated, the following relation is derived::

$$\mathcal{I}_{mi_\psi}^{(\varphi\varphi)} a_{i_\psi} = \gamma^2 a_1^2 b_1^2 \frac{\sqrt{E_x E_y}}{E_r} \mathcal{I}_{mi_W j_W}^{(\chi\chi\varphi)} W_{i_W} W_{j_W} \qquad m = 1,2\ldots.N_\psi \tag{8a}$$

where:

$$\mathcal{I}_{mi_\psi}^{(\varphi\varphi)} = \int_\Sigma \left[a_1^4 \frac{\partial^4 \varphi_{i_\psi}}{\partial \xi^4} + c_1^4 \frac{\partial^4 \varphi_{i_\psi}}{\partial \xi^2 \partial \eta^2} + b_1^4 \frac{\partial^4 \varphi_{i_\psi}}{\partial \eta^4} \right] \varphi_m d\Sigma \tag{8b}$$

and:

$$\mathcal{I}_{mi_W j_W}^{(\chi\chi\varphi)} = \int_\Sigma \left[\frac{\partial^2 \chi_{i_W}}{\partial \xi \partial \eta} \frac{\partial^2 \chi_{j_W}}{\partial \xi \partial \eta} - \frac{\partial^2 \chi_{i_W}}{\partial \xi^2} \frac{\partial^2 \chi_{j_W}}{\partial \eta^2} \right] \varphi_m d\Sigma \tag{8c}$$

taking into account that the integrals extend over the whole nondimensional surface Σ of the fluttering plate $(d\Sigma = d\xi d\eta \,||\, 0 \le \xi \le 1; 0 \le \eta \le 1)$.

The matrices with elements $\mathcal{I}_{mi_\psi}^{(\varphi\varphi)}$ and $\mathcal{I}_{mi_W j_W}^{(\chi\chi\varphi)}$ are introduced and denoted by $[\mathbf{I}^{(\varphi\varphi)}]$ and $[\mathbf{I}^{(\chi\chi\varphi)}]$ (in this second matrix the two indices i_W, j_W have been contracted in an alone i_{Wc2}), whose dimensions are $N_\psi \times N_\psi$ and $N_\psi \times N_W^2$, respectively, together with the column vectors $[\mathbf{A}]$ and $[\mathbf{W}^{(2)}]$, whose elements are a_{i_ψ} and the product $p_{i_{Wc2}}^{(2)} = W_{i_W} W_{j_W}$, with dimensions N_ψ and N_W^2, respectively. Thus equation (8a) can be rewritten in matrix form:

$$[\mathbf{A}] = [\mathbf{Z}][\mathbf{W}^{(2)}] \tag{9a}$$

where the matrix $[\mathbf{Z}]$ expression reads:

$$[\mathbf{Z}] = [\mathbf{I}^{(\varphi\varphi)}]^{-1} [\mathbf{I}^{(\chi\chi\varphi)}] \gamma^2 a_1^2 b_1^2 \frac{\sqrt{E_x E_y}}{E_r} \tag{9b}$$

whose elements are denoted by $\zeta_{mi_{Wc2}}$.

The out-of-plane translational equilibrium relations [22], together with the elasticity relations between flexural-torsional moments and bending-twisting curvatures for orthotropic plates [23], according to the Kirchoff theory [22], and taking into account the membrane stress dependence on the Airy function [22], give the second von Karman's constitutive equation of the flutter vibration [1,6,22], properly modified for an orthotropic plate:

$$D_x \frac{\partial^4 w}{\partial x^4} + 2\left(\nu_{xy} D_y + 2D_t\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = \left(\frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)h$$
$$- p_z - \mu \frac{\partial^2 w}{\partial t^2} \quad (10a)$$

where the dependence of the membrane stresses on ϕ is the same of equation (2), and p_z is the aerodynamic force per unity length, which according to the first order "Piston Theory' [2][5] is equal to:

$$p_z = \frac{2q}{\beta}\left(\frac{\partial w}{\partial x} + \frac{1}{U_a}\frac{\beta^2 - 1}{\beta^2}\frac{\partial w}{\partial t}\right) \quad (10b)$$

with $\beta^2 = M_{ach}^2 - 1$ and U_a is the high supersonic flow speed, $q = \rho U_a^2 / 2$ is the air flowing dynamic pressure, and:

$$D_x = \frac{E_x h^3}{1 - \nu_{xy}\nu_{yx}} \qquad D_y = \frac{E_y h^3}{1 - \nu_{xy}\nu_{yx}} \qquad D_t = g_{xy}\frac{h^3}{12} \quad (10c)$$

are the orthotropic plate flexural-torsional rigidity parameters.
Eq.(10a) can be rewritten in non-dimensional form, as:

$$\alpha_1^4 \frac{\partial^4 W}{\partial \xi^4} + \gamma_1^4 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + \beta_1^4 \frac{\partial^4 W}{\partial \eta^4} + \pi^4\left[\sigma L_a \frac{\partial W}{\partial \xi} + \sqrt{\sigma \vartheta}\frac{\partial W}{\partial \tau} + \frac{\partial^2 W}{\partial \tau^2}\right] =$$
$$= hL^2 a_1^2 b_1^2 \frac{\sqrt{E_x E_y}}{D_r}\left[\frac{\partial^2 \psi}{\partial \eta^2}\frac{\partial^2 W}{\partial \xi^2} - 2\frac{\partial^2 \psi}{\partial \xi \partial \eta}\frac{\partial^2 W}{\partial \xi \partial \eta} + \frac{\partial^2 \psi}{\partial \xi^2}\frac{\partial^2 W}{\partial \eta^2}\right] \quad (11a)$$

where:

$$\alpha_1^4 = \frac{D_x L^4}{D_r a^4} \qquad \beta_1^4 = \frac{D_y L^4}{D_r b^4} \qquad \gamma_1^4 = \frac{2\left(\nu_{xy} D_y + 2D_t\right)}{D_r}\frac{L^4}{a^2 b^2} \qquad L_a = \frac{L}{a} \quad (11b)$$

$$W = \frac{w}{L_w} \qquad \sigma = \frac{2qL^3}{\beta \pi^4 D_r} \qquad \vartheta = \left(\frac{\beta^2 - 1}{\beta^2}\right)^2 \frac{2qL}{\beta \mu U_a^2} \qquad \tau = \sqrt{\frac{D_r \pi^4}{\mu L^4}}t \quad (11c)$$

and D_r is the flexural rigidity modulus of the isotropic fluttering reference plate.

Equation (11a) is pre-multiplied by the generic element $\chi_m = \boldsymbol{sin}\left(m_x \pi \xi\right)\boldsymbol{sin}\left(m_y \pi \eta\right)$ of the out-of-plane displacement W series expansion in Eq. (7.a), and taking into account the properties of the trigonometric functions, for which it is true that:

$$\int_{\Sigma} \chi_m(\xi,\eta)\chi_{i_W}(\xi,\eta)d\Sigma = \frac{\delta_{mi_W}}{4} \quad (12)$$

the following relation is obtained:

$$\ddot{W}_m + \sqrt{\sigma\vartheta}\dot{W}_m + i_{3W}^4 W_m = -4\sigma L_a \mathcal{I}_{mi_W}^{(\chi\chi)} W_{i_W} + \frac{4}{\pi^4} hL^2 a_1^2 b_1^2 \frac{\sqrt{E_x E_y}}{E_r} \mathcal{I}_{mi_\psi i_W}^{(\varphi\chi\chi)} a_{i_\psi} W_{i_W} \qquad m = 1,2\ldots. N_W \tag{13a}$$

where:

$$i_{3W}^4 = \alpha_1^4 i_{Wx}^4 + \gamma_1^4 i_{Wx}^2 i_{Wy}^2 + \beta_1^4 i_{Wy}^4 \qquad \dot{W}_{i_W} = \frac{\partial W_{i_W}}{\partial \tau} \qquad \ddot{W}_{i_w} = \frac{\partial^2 W_{i_w}}{\partial \tau^2} \qquad \mathcal{I}_{mi_W}^{(\chi\chi)} = \int_\Sigma \frac{\partial \chi_{i_W}}{\partial \xi} \chi_m d\Sigma \tag{13b}$$

$$\mathcal{I}_{mi_\psi i_W}^{(\varphi\chi\chi)} = \int_\Sigma \left[\frac{\partial^2 \varphi_{i_\psi}}{\partial \eta^2} \frac{\partial^2 \chi_{i_W}}{\partial \xi^2} - 2 \frac{\partial^2 \varphi_{i_\psi}}{\partial \xi \partial \eta} \frac{\partial^2 \chi_{i_W}}{\partial \xi \partial \eta} + \frac{\partial^2 \varphi_{i_\psi}}{\partial \xi^2} \frac{\partial^2 \chi_{i_W}}{\partial \eta^2} \right] \chi_m d\Sigma \tag{13c}$$

The coefficients a_{i_ψ} are connected with the products of the coefficients of out-of-plane displacement series expansion in equation (6), by the relation (9a), and if $\zeta_{i_\psi i_{Wc2}}$ is the generic element of the matrix $[\mathbf{Z}]$, it follows that:

$$a_{i_\psi} = \zeta_{i_\psi i_{Wc2}} p_{i_{Wc2}} \tag{14}$$

and thus the last term in equation (13a) becomes:

$$4hL^2 a_1^2 b_1^2 \frac{\sqrt{E_x E_y}}{E_r} \mathcal{I}_{mi_\psi i_W}^{(\varphi\chi\chi)} \zeta_{i_\psi j_W k_W} W_{i_W} W_{j_W} W_{k_W} \tag{15}$$

Then equation (13a) can be also written in matrix form:

$$[\ddot{\mathbf{W}}] + \sqrt{\sigma\vartheta}[\dot{\mathbf{W}}] + i_{3W}^4 [\mathbf{W}] + [\mathbf{H}][\mathbf{W}] - [\Lambda][\mathbf{W}^{(3)}] = 0 \tag{16a}$$

where $[\mathbf{W}]$ is the column vector of the series expansions coefficients of the out-of-plane displacement, $[\mathbf{W}^{(3)}]$ is the column vector with elements the triple products of the same coefficients $p_{i_{Wc3}}^{(3)} = W_{i_W} W_{j_W} W_{k_W} = W_{i_W} p_{j_{Wc2}}$ (i_{Wc3} is the contraction of the three indices i_W, j_W, k_W or i_W, j_{Wc2}), $[\mathbf{H}]$ and $[\Lambda]$ are the matrices, with dimensions $N_W \times N_W$ and $N_W \times N_W^3$, respectively, whose elements are:

$$h_{mi_W} = -4\sigma L_a \mathcal{I}_{mi_W}^{(\chi\chi)} \qquad \lambda_{mi_{Wc3}} = 4hL^2 a_1^2 b_1^2 \frac{\sqrt{E_x E_y}}{E_r} \mathcal{I}_{mi_\psi i_W}^{(\varphi\chi\chi)} \zeta_{i_\psi j_W k_W} \tag{16b}$$

Thus a system of non-linear differential equations in time is obtained, with W as alone unknown variable, which can be integrated by appropriate algorithms.

2) A second case of plate likewise simply supported at the borders for the out-of-plane behaviour, but clamped at the four edges for the in-plane displacements, so that in place of Eq. (1) there are the boundary conditions for in-plane displacements:

$$u = 0 \qquad\qquad v = 0 \tag{17}$$

In this case it is not convenient to utilize the Airy function, but set-up the in-plane constitutive equations utilizing the functions of the displacements along the axes x and y. The equilibrium equation along the axis x [22], taking into account the elastic dependence of the membrane stresses of an orthotropic plate on the in-plane strain [23] and the kinematic relations [22], give the following constitutive equation:

$$A_x\left[\frac{\partial^2 u}{\partial x^2} + \nu_{yx}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + \nu_{yx}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y}\right] + G_{xy}\left[\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial^2 y}\right] = 0 \tag{18a}$$

$$A_x = \frac{E_x h}{\left(1 - \nu_{xy}\nu_{yx}\right)} \qquad\qquad G_{xy} = g_{xy} h \tag{18b}$$

where, like in the case of the in-plane free plate, the in-plane inertial forces are neglected. Equation (18) can be rewritten as:

$$\gamma_a^2 \frac{\partial^2 U}{\partial \xi^2} + \nu_{yx}\gamma_b^2 \frac{\partial^2 V}{\partial \xi \partial \eta} + \gamma_a^2 \frac{\partial W}{\partial \xi}\frac{\partial^2 W}{\partial \xi^2} + \nu_{yx}\gamma_b^2 \frac{\partial W}{\partial \eta}\frac{\partial^2 W}{\partial \xi \partial \eta} +$$

$$+ G_{Ax}\gamma_b^2\left[\frac{\partial^2 V}{\partial \xi \partial \eta} + \frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 W}{\partial \xi \partial \eta}\frac{\partial W}{\partial \eta} + \frac{\partial W}{\partial \xi}\frac{\partial^2 W}{\partial \eta^2}\right] = 0 \tag{19a}$$

where together with $W = w/L_w$, as in Eq. (3b), also the in-plane displacements u, v have been reformulated in non-dimensional form:

$$U = \frac{u}{a}\frac{1}{\gamma_a^2} \qquad V = \frac{v}{b}\frac{1}{\gamma_b^2} \qquad \gamma_a = \frac{L_w}{a} \qquad \gamma_b = \frac{L_w}{b} \qquad G_{Ax} = G_{xy} / A_x \tag{19b}$$

considering that the order of magnitude of the in-plane displacements u, v is the same of w^2, so that the same order corresponds to U, V, W.

We choose also for the in-plane displacements an appropriate series expansion to be employed with Galerkin method:

$$U(\xi, \eta, \tau) = U_{i_U}(\tau)\chi_{i_U}(\xi, \eta) \qquad\qquad V(\xi, \eta, \tau) = V_{i_V}(\tau)\chi_{i_V}(\xi, \eta) \tag{20a}$$

where:

$$\chi_{i_U}(\xi, \eta) = \sin(i_{Ux}\pi\xi)\sin(i_{Uy}\pi\eta) \qquad\qquad \chi_{i_V}(\xi, \eta) = \sin(i_{Vx}\pi\xi)\sin(i_{Vy}\pi\eta) \tag{20b}$$

$$i_{Ux}, i_{Vx} = 1,2....N_{Ux}, N_{Vx} \qquad i_{Uy}, i_{Vy} = 1,2....N_{Uy}, N_{Vy} \qquad i_U, i_V = \left(i_{Ux}, i_{Vx} - 1\right)N_{Uy}, N_{Vy} + i_{Uy}, i_{Vy}$$
$$i_U, i_V = 1,2....N_U, N_V \qquad i_U, i_V = 1,2....N_U, N_V \qquad N_U, N_V = N_{Ux}, N_{Vx} \times N_{Uy}, N_{Vy} \tag{20c}$$

For practical reasons it is convenient to choose $N_{Ux} = N_{Vx}$, $N_{Uy} = N_{Vy}$ and consequently $N_U = N_V$. Equation (19a) is pre-multiplied by $\chi_m(\xi,\eta) = sin(m_x \pi \xi) sin(m_y \pi \eta)$ and integrated, and the following relation is derived:

$$\gamma_a^2 \mathcal{I}_{mi_U}^{(Ux2)} U_{i_U} + \nu_{yx}\gamma_b^2 \mathcal{I}_{mi_V}^{(Vxy)} V_{i_V} + \gamma_a^2 \mathcal{I}_{mi_W j_W}^{(WxWx2)} W_{i_W} W_{j_W} + \nu_{yx}\gamma_b^2 \mathcal{I}_{mi_W j_W}^{(WyWxy)} W_{i_W} W_{j_W}$$
$$+ \quad G_{Ax}\gamma_b^2 \left[\mathcal{I}_{mi_V}^{(Vxy)} V_{i_V} + \mathcal{I}_{mi_U}^{(Uy2)} U_{i_U} + \mathcal{I}_{mi_W j_W}^{(WyWxy)} W_{i_W} W_{j_W} + \mathcal{I}_{mi_W j_W}^{(WxWy2)} W_{i_W} W_{j_W}\right] = 0 \quad (21a)$$

where $m = 1,2....N_U$ and:

$$\mathcal{I}_{mi_U}^{(Ux2)} = \int_\Sigma \chi_m \frac{\partial^2 \chi_{i_U}}{\partial \xi^2} d\Sigma \qquad \mathcal{I}_{mi_V}^{(Vxy)} = \int_\Sigma \chi_m \frac{\partial^2 \chi_{i_V}}{\partial \xi \partial \eta} d\Sigma \qquad \mathcal{I}_{mi_W j_W}^{(WxWx2)} = \int_\Sigma \chi_m \frac{\partial \chi_{i_W}}{\partial \xi} \frac{\partial^2 \chi_{j_W}}{\partial \xi^2} d\Sigma$$

$$\mathcal{I}_{mi_W j_W}^{(WyWxy)} = \int_\Sigma \chi_m \frac{\partial \chi_{i_W}}{\partial \eta} \frac{\partial^2 \chi_{j_W}}{\partial \xi \partial \eta} d\Sigma \qquad \mathcal{I}_{mi_U}^{(Uy2)} = \int_\Sigma \chi_m \frac{\partial^2 \chi_{i_U}}{\partial \eta^2} d\Sigma \qquad \mathcal{I}_{mi_W j_W}^{(WxWy2)} = \int_\Sigma \chi_m \frac{\partial \chi_{i_W}}{\partial \xi} \frac{\partial^2 \chi_{j_W}}{\partial \eta^2} d\Sigma \quad (21b)$$

The matrices $\left[\mathbf{I}^{(Ux2)}\right]$, $\left[\mathbf{I}^{(Vxy)}\right]$ and $\left[\mathbf{I}^{(Uy2)}\right]$, with dimensions $N_U \times N_U$, whose elements are $\mathcal{I}_{mi_U}^{(Ux2)}$, $\mathcal{I}_{mi_V}^{(Vxy)}$ and $\mathcal{I}_{mi_U}^{(Uy2)}$ respectively, are then introduced, together with the matrices $\left[\mathbf{I}^{(WxWx2)}\right]$, $\left[\mathbf{I}^{(WyWxy)}\right]$ and $\left[\mathbf{I}^{(WxWy2)}\right]$, with dimensions $N_U \times N_W^2$, whose element are $\mathcal{I}_{mi_{Wc2}}^{(WxWx2)}$, $\mathcal{I}_{mi_{Wc2}}^{(WxWxy)}$ and $\mathcal{I}_{mi_{Wc2}}^{(WxWy2)}$, where i_{Wc2} is the contraction of the indices i_W, j_W. If also the column vectors $[\mathbf{U}]$ and $[\mathbf{V}]$, whose elements are the series expansions coefficients U_{i_U} and V_{i_V}, respectively, are introduced, and we recall the previously utilized column vector $\left[\mathbf{W}^{(2)}\right]$, Eq. (21a) can be rewritten in matrix form:

$$\gamma_a^2\left[\mathbf{I}^{(Ux2)}\right][\mathbf{U}] + \nu_{yx}\gamma_b^2\left[\mathbf{I}^{(Vxy)}\right][\mathbf{V}] + \gamma_a^2\left[\mathbf{I}^{(WxWx2)}\right]\left[\mathbf{W}^{(2)}\right] + \nu_{yx}\gamma_b^2\left[\mathbf{I}^{(WyWxy)}\right]\left[\mathbf{W}^{(2)}\right]$$
$$+ G_{Ax}\gamma_b^2\left\{\left[\mathbf{I}^{(Vxy)}\right][\mathbf{V}] + \left[\mathbf{I}^{(Uy2)}\right][\mathbf{U}] + \left[\mathbf{I}^{(WyWxy)}\right]\left[\mathbf{W}^{(2)}\right] + \left[\mathbf{I}^{(WxWy2)}\right]\left[\mathbf{W}^{(2)}\right]\right\} = 0 \quad (22)$$

The constitutive equilibrium equation along the axis y can be written in dual form:

$$\gamma_b^2\left[\mathbf{I}^{(Vy2)}\right][\mathbf{V}] + \nu_{xy}\gamma_a^2\left[\mathbf{I}^{(Uxy)}\right][\mathbf{U}] + \gamma_b^2\left[\mathbf{I}^{(WyWy2)}\right]\left[\mathbf{W}^{(2)}\right] + \nu_{xy}\gamma_a^2\left[\mathbf{I}^{(WxWxy)}\right]\left[\mathbf{W}^{(2)}\right]$$
$$+ G_{Ay}\gamma_a^2\left\{\left[\mathbf{I}^{(Uxy)}\right][\mathbf{U}] + \left[\mathbf{I}^{(Vx2)}\right][\mathbf{V}] + \left[\mathbf{I}^{(WxWxy)}\right]\left[\mathbf{W}^{(2)}\right] + \left[\mathbf{I}^{(WyWx2)}\right]\left[\mathbf{W}^{(2)}\right]\right\} = 0 \quad (23)$$

where: $G_{Ay} = G_{xy} / A_y$ and $A_y = E_y h / (1 - \nu_{xy}\nu_{yx})$.

Then the following matrices are introduced:

$$\left[\mathbf{P}^{(UU)}\right] = \gamma_a^2\left[\mathbf{I}^{(Ux2)}\right] + G_{Ax}\gamma_b^2\left[\mathbf{I}^{(Uy2)}\right]$$

$$\left[\mathbf{P}^{(UV)}\right] = \gamma_b^2\left[\mathbf{I}^{(Vxy)}\right](\nu_{yx} + G_{Ax}) \quad (24)$$

$$\left[\mathbf{P}^{(UW)}\right] = \gamma_a^2\left[\mathbf{I}^{(WxWx2)}\right] + \nu_{yx}\gamma_b^2\left[\mathbf{I}^{(WyWxy)}\right] + G_{Ax}\gamma_b^2\left\{\left[\mathbf{I}^{(WyWxy)}\right] + \left[\mathbf{I}^{(WxWy2)}\right]\right\}$$

and:

$$\left[\mathbf{P}^{(VV)}\right] = \gamma_b^2\left[\mathbf{I}^{(Vy2)}\right] + G_{Ay}\gamma_a^2\left[\mathbf{I}^{(Vx2)}\right]$$

$$\left[\mathbf{P}^{(VU)}\right] = \gamma_a^2\left[\mathbf{I}^{(Uxy)}\right]\left(\nu_{xy} + G_{Ay}\right) \tag{25}$$

$$\left[\mathbf{P}^{(VW)}\right] = \gamma_b^2\left[\mathbf{II}^{(WyWy2)}\right] + \nu_{xy}\gamma_a^2\left[\mathbf{I}^{(WxWxy)}\right] + G_{Ay}\gamma_a^2\left\{\left[\mathbf{I}^{(WxWxy)}\right] + \left[\mathbf{I}^{(WyWx2)}\right]\right\}$$

so that Eqs. (22) and (23) can be newly rewritten as:

$$\left[\mathbf{P}^{(UU)}\right]\left[\mathbf{U}\right] + \left[\mathbf{P}^{(UV)}\right]\left[\mathbf{V}\right] + \left[\mathbf{P}^{(UW)}\right]\left[\mathbf{W}^{(2)}\right] = 0$$

$$\left[\mathbf{P}^{(VU)}\right]\left[\mathbf{U}\right] + \left[\mathbf{P}^{(VV)}\right]\left[\mathbf{V}\right] + \left[\mathbf{P}^{(VW)}\right]\left[\mathbf{W}^{(2)}\right] = 0 \tag{26}$$

Then the following matrices are introduced:

$$\left[\mathbf{Q}^{(UU)}\right] = \left[\mathbf{P}^{(UU)}\right] - \left[\mathbf{P}^{(UV)}\right]\left[\mathbf{P}^{(VV)}\right]^{-1}\left[\mathbf{P}^{(VU)}\right]$$

$$\left[\mathbf{Q}^{(UW)}\right] = \left[\mathbf{P}^{(UV)}\right]\left[\mathbf{P}^{(VV)}\right]^{-1}\left[\mathbf{P}^{(VW)}\right] - \left[\mathbf{P}^{(UW)}\right] \tag{27}$$

and their dual ones:

$$\left[\mathbf{Q}^{(VV)}\right] = \left[\mathbf{P}^{(VV)}\right] - \left[\mathbf{P}^{(VU)}\right]\left[\mathbf{P}^{(UU)}\right]^{-1}\left[\mathbf{P}^{(UV)}\right]$$

$$\left[\mathbf{Q}^{(VW)}\right] = \left[\mathbf{P}^{(VU)}\right]\left[\mathbf{P}^{(UU)}\right]^{-1}\left[\mathbf{P}^{(UW)}\right] - \left[\mathbf{P}^{(VW)}\right] \tag{28}$$

so that equations (26) can be also written as:

$$\left[\mathbf{U}\right] = \left[\mathbf{R}^{(UW)}\right]\left[\mathbf{W}^{(2)}\right]$$

$$\left[\mathbf{V}\right] = \left[\mathbf{R}^{(VW)}\right]\left[\mathbf{W}^{(2)}\right] \tag{29a}$$

where:

$$\left[\mathbf{R}^{(UW)}\right] = \left[\mathbf{Q}^{(UU)}\right]^{-1}\left[\mathbf{Q}^{(UW)}\right] \quad \text{and} \quad \left[\mathbf{R}^{(VW)}\right] = \left[\mathbf{Q}^{(VV)}\right]^{-1}\left[\mathbf{Q}^{(VW)}\right] \tag{29b}$$

The out-of-plane translational von Karman's equilibrium equation (10a), if the Airy function is not utilized and the elastic dependence of the membrane stresses on the in-plane strain [23] of an orthotropic plate and the kinematic relations [22] are considered, becomes [7]:

$$D_x \frac{\partial^4 w}{\partial x^4} + 2\left(\nu_{xy} D_y + 2D_t\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - A_x\left[\frac{\partial u}{\partial x} + \nu_{yx}\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\nu_{yx}\left(\frac{\partial w}{\partial y}\right)^2\right]\frac{\partial^2 w}{\partial x^2}$$

$$-2G_{xy}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\frac{\partial^2 w}{\partial x \partial y} - A_y\left[\frac{\partial v}{\partial y} + \nu_{xy}\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\nu_{xy}\left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial^2 w}{\partial y^2} =$$

$$= -p_z - \mu\frac{\partial^2 w}{\partial t^2} \tag{30}$$

where the expression of p_z has been given in Eq. (10b). The same foregoing equation can be rewritten in dimension-less form:

$$\alpha_1^4 \frac{\partial^4 W}{\partial \xi^4} + \gamma_1^4 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + \beta_1^4 \frac{\partial^4 W}{\partial \eta^4} - \alpha_a\left[\gamma_a^2 \frac{\partial U}{\partial \xi} + \nu_{yx}\gamma_b^2 \frac{\partial V}{\partial \eta} + \frac{1}{2}\gamma_a^2\left(\frac{\partial W}{\partial \xi}\right)^2 + \frac{1}{2}\nu_{yx}\gamma_b^2\left(\frac{\partial W}{\partial \eta}\right)^2\right]\frac{\partial^2 W}{\partial \xi^2}$$

$$-2\gamma_{as}\gamma_a\gamma_b\left[\frac{\partial V}{\partial \xi} + \frac{\partial U}{\partial \eta} + \frac{\partial W}{\partial \xi}\frac{\partial W}{\partial \eta}\right]\frac{\partial^2 W}{\partial \xi \partial \eta} - \beta_a\left[\gamma_b^2 \frac{\partial V}{\partial \eta} + \nu_{xy}\gamma_a^2 \frac{\partial U}{\partial \xi} + \frac{1}{2}\gamma_b^2\left(\frac{\partial W}{\partial \eta}\right)^2 + \frac{1}{2}\nu_{xy}\gamma_a^2\left(\frac{\partial W}{\partial \xi}\right)^2\right]\frac{\partial^2 W}{\partial \eta^2} =$$

$$= -\pi^4\left[\sigma L_a \frac{\partial W}{\partial \xi} + \sqrt{\sigma\vartheta}\frac{\partial W}{\partial \tau} + \frac{\partial^2 W}{\partial \tau^2}\right] \tag{31a}$$

where both the displacements and the in-plane coordinates have been reformulated in non-dimensional form, as in Eqs. (3b) and (19b), and further the same dimension-less parameters of Eqs. (11b-c) and (19b) have been utilized, with also:

$$\alpha_a = \frac{A_x L^4}{D_r a^2} \qquad \beta_a = \frac{A_y L^4}{D_r b^2} \qquad \gamma_{as} = \frac{G_{xy} L^4}{D_r ab} \tag{31b}$$

The same series expansions for W and U,V of equations (6) and (20a) are choosen,and, according the Galerkin method, Eq. (31a) is pre-multiplied by $\chi_m(\xi,\eta) = \sin(m_x\pi\xi)\sin(m_y\pi\eta)$ and integrated, so that, taking into account Eq. (12), the following relation is obtained:

$$\ddot{W}_m + \sqrt{\sigma\vartheta}\dot{W}_m + i_{3W}^4 W_m = -4\sigma L_a I_{mi_W}^{(\chi\chi)} W_{i_W} + \frac{4}{\pi^4}\alpha_a\gamma_a^2\left(I_{mi_U i_W}^{(UxWx2)} U_{i_U} W_{i_W} + \frac{1}{2} I_{mi_W i_W k_W}^{(WxWxWx2)} W_{i_W} W_{j_W} W_{k_W}\right)$$

$$+\frac{4}{\pi^4}\alpha_a\nu_{yx}\gamma_b^2\left(I_{mi_V i_W}^{(VyWx2)} V_{i_V} W_{i_W} + \frac{1}{2} I_{mi_W j_W k_W}^{(WyWyWx2)} W_{i_W} W_{j_W} W_{k_W}\right)$$

$$+\frac{8}{\pi^4}\gamma_{as}\gamma_a\gamma_b\left[I_{mi_V i_W}^{(VxWxy)} V_{i_V} W_{i_W} + I_{mi_V i_W}^{(UyWxy)} U_{i_U} W_{i_W} + I_{mi_W j_W k_W}^{(WxWyWxy)} W_{i_W} W_{j_W} W_{k_W}\right]$$

$$+\frac{4}{\pi^4}\beta_a\gamma_b^2\left(I_{mi_V i_W}^{(VyWy2)} V_{i_V} W_{i_W} + \frac{1}{2} I_{mi_W j_W k_W}^{(WyWyWy2)} W_{i_W} W_{j_W} W_{k_W}\right)$$

$$+\frac{4}{\pi^4}\beta_a\nu_{xy}\gamma_a^2\left(I_{mi_U i_W}^{(UxWy2)} U_{i_U} W_{i_W} + \frac{1}{2} I_{mi_W i_W k_W}^{(WxWxWy2)} W_{i_W} W_{j_W} W_{k_W}\right) \qquad m = 1,2....N_W \tag{32a}$$

where the in-time derivatives, the parameter i_{3W}^4 and the integral $I_{mi_W}^{(\chi\chi)}$ have been previously defined in Eq. (13b), and further:

$$\mathcal{I}_{mi_U i_W}^{(UxWx2)} = \int_\Sigma \chi_m \frac{\partial \chi_{i_U}}{\partial \xi} \frac{\partial^2 \chi_{i_W}}{\partial \xi^2} d\Sigma \qquad \mathcal{I}_{mi_U i_W}^{(VyWx2)} = \int_\Sigma \chi_m \frac{\partial \chi_{i_V}}{\partial \eta} \frac{\partial^2 \chi_{i_W}}{\partial \xi^2} d\Sigma \qquad \mathcal{I}_{mi_V i_W}^{(VxWxy)} = \int_\Sigma \chi_m \frac{\partial \chi_{i_V}}{\partial \xi} \frac{\partial^2 \chi_{i_W}}{\partial \xi \partial \eta} d\Sigma$$

$$\mathcal{I}_{mi_U i_W}^{(UyWxy)} = \int_\Sigma \chi_m \frac{\partial \chi_{i_U}}{\partial \eta} \frac{\partial^2 \chi_{i_W}}{\partial \xi \partial \eta} d\Sigma \qquad \mathcal{I}_{mi_V i_W}^{(VyWy2)} = \int_\Sigma \chi_m \frac{\partial \chi_{i_V}}{\partial \eta} \frac{\partial^2 \chi_{i_W}}{\partial \eta^2} d\Sigma \qquad \mathcal{I}_{mi_U i_W}^{(UxWy2)} = \int_\Sigma \chi_m \frac{\partial \chi_{i_U}}{\partial \xi} \frac{\partial^2 \chi_{i_W}}{\partial \eta^2} d\Sigma$$

$$\mathcal{I}_{mi_W j_W k_W}^{(WxWxWx2)} = \int_\Sigma \chi_m \frac{\partial \chi_{i_W}}{\partial \xi} \frac{\partial \chi_{j_W}}{\partial \xi} \frac{\partial^2 \chi_{k_W}}{\partial \xi^2} d\Sigma \qquad \mathcal{I}_{mi_W j_W k_W}^{(WyWyWx2)} = \int_\Sigma \chi_m \frac{\partial \chi_{i_W}}{\partial \eta} \frac{\partial \chi_{j_W}}{\partial \eta} \frac{\partial^2 \chi_{k_W}}{\partial \xi^2} d\Sigma$$

$$\mathcal{I}_{mi_W j_W k_W}^{(WxWyWxy)} = \int_\Sigma \chi_m \frac{\partial \chi_{i_W}}{\partial \xi} \frac{\partial \chi_{j_W}}{\partial \eta} \frac{\partial^2 \chi_{k_W}}{\partial \xi \partial \eta} d\Sigma \tag{32b}$$

$$\mathcal{I}_{mi_W j_W k_W}^{(WyWyWy2)} = \int_\Sigma \chi_m \frac{\partial \chi_{i_W}}{\partial \eta} \frac{\partial \chi_{j_W}}{\partial \eta} \frac{\partial^2 \chi_{k_W}}{\partial \eta^2} d\Sigma \qquad \mathcal{I}_{mi_W j_W k_W}^{(WxWxWy2)} = \int_\Sigma \chi_m \frac{\partial \chi_{i_W}}{\partial \xi} \frac{\partial \chi_{j_W}}{\partial \xi} \frac{\partial^2 \chi_{k_W}}{\partial \eta^2} d\Sigma$$

There are within square brackets terms as U_{i_U} and V_{i_V}, which are connected with the product $p_{j_{Wc2}} = W_{j_W} W_{k_W}$ (as above mentioned j_{Wc2} is the contraction of the two indices j_W and k_W), by the relations:

$$U_{i_U} = r_{i_U j_{Wc2}}^{(UW)} W_{j_W} W_{k_W} \qquad V_{i_V} = r_{i_V j_{Wc2}}^{(VW)} W_{j_W} W_{k_W} \tag{33}$$

where $r_{i_U j_{Wc2}}^{(UW)}$ and $r_{i_V j_{Wc2}}^{(VW)}$ are elements of the matrices $\left[\mathbf{R}^{(UW)}\right]$ and $\left[\mathbf{R}^{(VW)}\right]$, as in equations (29a). Thus all the elements within square brackets in equation (32a) contain products as $p_{i_{Wc3}} = W_{i_W} W_{j_W} W_{k_W}$ (i_{Wc3}, as above mentioned, is the contraction of i_W, j_W and k_W, or i_W and j_{Wc2}), and consequently their sum is equal to one term:

$$t_{mi_{Wc3}} W_{i_W} W_{j_W} W_{k_W} \tag{34}$$

Thus if the matrix with elements $t_{mi_{Wc3}}$ is introduced and denoted by $[\mathbf{T}]$, equation (32a) can be rewritten in matrix form:

$$\left[\ddot{\mathbf{W}}\right] + \sqrt{\sigma \vartheta}\left[\dot{\mathbf{W}}\right] + i_{3W}^4 [\mathbf{W}] + [\mathbf{H}][\mathbf{W}] - [\mathbf{T}]\left[\mathbf{W}^{(3)} = 0\right] \tag{35}$$

as in equation (16a), for which there exist appropriate algorithms for integration in time.

2.2 *Ritz and FEM procedures*

Also a procedure which arises from the Rayleigh-Ritz method, together with the FEM [18-20], can be utilized to find a solution of the problem, as for a beam fluttering case [20].

Both procedures arise from differential operations on an energetic functional, whose stationary conditions lead to the dynamic constitutive equations. Since with FEM or Ritz method one is not obliged to satisfy the natural boundary conditions in the free in-plane plate case, as in Eq. (1), the same model can be utilized in both fluttering plate cases, but a parameter, which takes into account the different in-plane geometric boundary conditions, has to be introduced, as it will be shown in the Appendices B and C.

The strain energy expression of the in-plane and out-of-plane linear structural forces can be written in the classical form:

$$\mathcal{U}_l^{(in)} = \frac{1}{2} k_{ij}^{(in)} q_i^{(in)} q_j^{(in)} \qquad \mathcal{U}_l^{(op)} = \frac{1}{2} k_{ij}^{(op)} q_i^{op} q_j^{(op)} \tag{36}$$

where $k_{ij}^{(in)}$ and $k_{ij}^{(op)}$ are the in-plane and out-of-plane stiffness matrix elements of the orthotropic plate, which have been determined for the Ritz procedure and FEM, in Appendices B and C, respectively, while $q_i^{(in)}$, $q_j^{(in)}$ and $q_i^{(op)}$, $q_j^{(op)}$ are in-plane and out-of-plane d.o.f. of both methods, whose meaning is illustrated in the same above mentioned Appendices.

Also the contribution of the mixed and non-linear structural forces to the in-plane strain energy can be evaluated and expressed as:

$$\mathcal{U}_m^{(in)} = \frac{1}{2} d_{ijk}^{(3)} q_i^{(in)} q_j^{(op)} q_k^{(op)} \qquad \mathcal{U}_{nl}^{(in)} = \frac{1}{2} d_{ijkl}^{(4)} q_i^{(op)} q_j^{(op)} q_k^{(op)} q_l^{(op)} \tag{37}$$

where $d_{ijk}^{(3)}$ and $d_{ijkl}^{(4)}$ are tensor elements, determined in Appendices A and B, for Ritz procedure and FEM, respectively.

The in-plane and out-of-plane kinetic energy expressions can be written in the classical form:

$$\mathcal{T}^{(in)} = \frac{1}{2} m_{ij}^{(in)} \dot{q}_i^{(in)} \dot{q}_j^{(in)} \qquad \mathcal{T}^{(op)} = \frac{1}{2} m_{ij}^{(op)} \dot{q}_i^{(op)} \dot{q}_j^{(op)} \tag{38}$$

where $m_{ij}^{(in)}$ and $m_{ij}^{(op)}$ are in-plane and out-of-plane mass matrix elements, evaluated for both methods in the above mentioned Appendices.

The Lagrangian $\mathcal{L}$ functional is introduced:

$$\mathcal{L} = \mathcal{T}^{(in)} + \mathcal{T}^{(op)} - \mathcal{U}_l^{(in)} - \mathcal{U}_l^{(op)} - \mathcal{U}_m^{(in)} - \mathcal{U}_{nl}^{(in)} \tag{39}$$

Thus the generic $i\text{-}th$ in-plane constitutive equation, corresponding to the d.o.f. $q_i^{(in)}$, can be determined [21]:

$$\frac{d\left(\partial\mathcal{L}/\partial\dot{q}_i^{(in)}\right)}{d\tau} - \frac{\partial\mathcal{L}}{\partial q_i^{(in)}} = 0 \tag{40}$$

which, if equations (36), (37) and (38) are taken into account, gives:

$$k_{ij}^{(in)} q_j^{(in)} + \frac{1}{2} d_{ijk}^{(3)} q_j^{(op)} q_k^{(op)} = 0 \tag{41}$$

considering that, as in Galerkin model, the in-plane inertial forces have been neglected.

The in-plane stiffness matrix $\left[\mathbf{K}^{(in)}\right]$ with elements $k_{ij}^{(in)}$ and the matrix $\left[\mathbf{D}^{(3)}\right]$ with elements $d_{ij_{c2}}^{(3)}$, where j_{c2} is the contraction of j and k, are introduced, And the column vectors $\left[\mathbf{Q}^{(in)}\right]$ and $\left[\mathbf{Q}^{(op,2)}\right]$, of the in-plane degrees of freedom and the products between out-of-plane degrees of freedom $p_{j_{c2}}^{(op)} = q_j^{(op)} q_k^{(op)}$, respectively, are introduced too. Then equation (41) can be written in matrix form:

$$\left[\mathbf{K}^{(in)}\right]\left[\mathbf{Q}^{(in)}\right] + \frac{1}{2}\left[\mathbf{D}^{(3)}\right]\left[\mathbf{Q}^{(op,2)}\right] = 0 \tag{42}$$

which can be written also as:

$$\left[\mathbf{Q}^{(in)}\right] = \left[\mathbf{H}\right]\left[\mathbf{Q}^{(op,2)}\right] \tag{43a}$$

where:

$$\left[\mathbf{H}\right] = -\frac{1}{2}\left[\mathbf{K}^{(in)}\right]^{-1}\left[\mathbf{D}^{(3)}\right] \tag{43b}$$

which is similar to Eqs (29a).

Thus the generic in-plane degree of freedom is connected with the out-of-plane degrees of freedom by the relation:

$$q_i^{(in)} = h_{ij_{c2}} q_{j_{c2}}^{(op)} = h_{ijk} q_j^{(op)} q_k^{(op)} \tag{44}$$

where $h_{ij_{c2}}$ are elements of the matrix $\left[\mathbf{H}\right]$ if the multiple indices symbolism is returned.

Also the generic i-th out-of-plane dynamic constitutive equation can be determined by the same differential operation as in Eq. (40), but with $\dot{q}_i^{(op)}$ and $q_i^{(op)}$ in place of $\dot{q}_i^{(in)}$ and $q_i^{(in)}$, and similar relations are obtained, but with the presence of external generalized forces in the numerical model:

$$m_{ij}^{(op)} \ddot{q}_j^{(op)} + k_{ij}^{(op)} q_j^{(op)} + d_{jik}^{(3)} q_j^{(in)} q_k^{(op)} + 2 d_{ijkl}^{(4)} q_j^{(op)} q_k^{(op)} q_l^{(op)} + F_i^{(a)} = 0 \tag{45}$$

where $F_i^{(a)}$ are the generalized aerodynamic forces acting on the d.o.f. $q_i^{(op)}$, depending on the aerodynamic force p_z per unity surface, introduced in Eq.(10b), which has been obtained by the "Piston Theory" [2][5]. This is formed by the component $F_i^{(a,x)}$ with derivative with respect to x, responsable for coupling between different natural vibrating modes, and the damping component $F_i^{(a,t)}$ with time derivative, as follows:

$$F_i^{(a)} = F_i^{(a,x)} + F_i^{(a,t)} \tag{46}$$

like p_z in equation (10b). These components can be written as:

$$F_i^{(a,x)} = f_{ij}^{(x)} q_j^{(op)} \qquad F_i^{(a,t)} = f_{ij}^{(t)} \dot{q}_j^{(op)} \tag{47}$$

where the coefficients $f_{ij}^{(x)}$ and $f_{ij}^{(t)}$ have been evaluated for both methods in the above mentioned Appendices.

In the third term of equation (45) the in-plane generic degree of freedom $q_j^{(in)}$ is connected with the out-of-plane degrees of freedom by the relation (44) which can be written also as:

$$q_j^{(in)} = \hbar_{jlm} q_l^{(op)} q_m^{(op)} \tag{48}$$

Then this term then becomes:

$$d_{jik}^{(3)} \hbar_{jlm} q_l^{(op)} q_m^{(op)} q_k^{(op)} \tag{49}$$

which can be also written as:

$$e_{iklm} q_l^{(op)} q_m^{(op)} q_k^{(op)} \tag{50}$$

because, taking into account the repeated indices rule for tensor elements, it is true that:

$$d_{jik}^{(3)} \hbar_{jlm} \rightarrow e_{iklm} \tag{51}$$

Then Eq. (45) becomes:

$$m_{ij}^{(op)} \ddot{q}_j^{(op)} + f_{ij}^{(t)} \dot{q}_j^{(op)} + \left(k_{ij}^{(op)}\right)^* q_j^{(op)} + e_{iklm} q_k^{(op)} q_l^{(op)} q_m^{(op)} + 2 d_{ijkl}^{(4)} q_j^{(op)} q_k^{(op)} q_l^{(op)} = 0 \tag{52a}$$

where the elements:

$$\left(k_{ij}^{(op)}\right)^* = k_{ij}^{(op)} + f_{ij}^{(x)} \tag{52b}$$

take into account both the linear out-of-plane structural and aerodynamic coupling forces [18]. The out-of-plane mass matrix $\left[\mathbf{M}^{(op)}\right]$ and the aerodynamic-structural forces matrix $\left[\mathbf{K}^{(op)}\right]^*$ are introduced, together with the non-linear forces matrices $\left[\mathbf{E}\right]$ and $\left[\mathbf{D}^{(4)}\right]$, with elements $e_{ij_{c3}}$ and $d^{(4)}_{ij_{c3}}$ (j_{c3} is the contraction of the three indices k,l,m or j,k,l). The column vectors $\left[\mathbf{Q}^{(op)}\right]$ of the out-of-plane degrees of freedom $q_i^{(op)}$, and $\left[\mathbf{Q}^{(op,3)}\right]$ containing the products $p^{(op)}_{j_{c3}} = q_k^{(op)} q_l^{(op)} q_m^{(op)}$ or $p^{(op)}_{j_{c3}} = q_j^{(op)} q_k^{(op)} q_l^{(op)}$, are introduced too. The elements $f_{ij}^{(t)}$ of the damping aerodynamic forces are proportional to the out-of-plane mass matrix elements $m_{ij}^{(op)}$ [20]:

$$f_{ij}^{(t)} = \gamma_d m_{ij}^{(op)} \tag{53}$$

and the matrix with elements $f_{ij}^{(t)}$ is coincident with the out-of-plane mass matrix $\left[\mathbf{M}^{(op)}\right]$, but a scale factor γ_d, as shown in Appendices B and C, for Ritz and FEM methods, respectively. Thus equation (52a) can be rewritten in matrix form:

$$\left[\mathbf{M}^{(op)}\right]\left\{\left[\ddot{\mathbf{Q}}^{(op)}\right] + \gamma_d \left[\dot{\mathbf{Q}}^{(op)}\right]\right\} + \left[\mathbf{K}^{(op)}\right]^* \left[\mathbf{Q}^{(op)}\right] + \left\{\left[\mathbf{E}\right] + 2\left[\mathbf{D}^{(4)}\right]\right\}\left[\mathbf{Q}^{(op,3)}\right] = 0 \tag{54}$$

A non-linear equations system is obtained, similar to the one in equation (35), which likewise can be integrated by good appropriate algorithms.

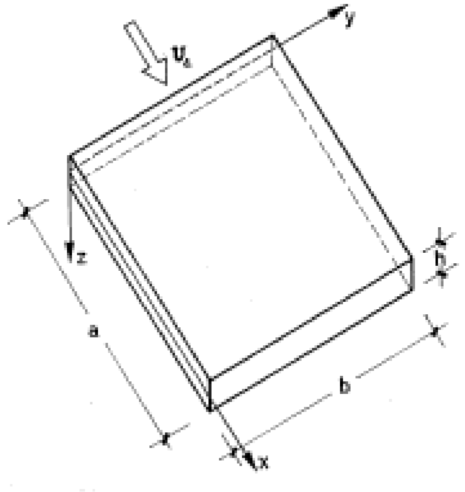

Fig. 1. Plate exposed to an air flowing at supersonic speed.   .

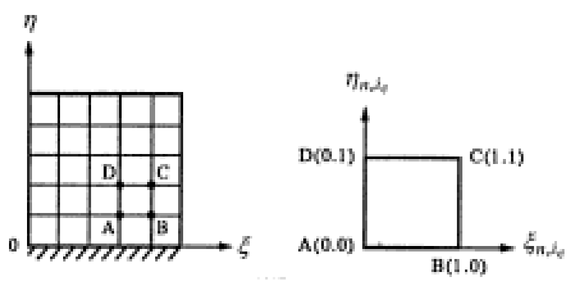

Fig. 2. A particular grid mesh of the FEM model.

Table 1

Values of $p_{i_{\psi x}}, p_{i_{\psi y}}$ and N_d, N_p vs $i_{\psi x}, i_{\psi y}$

$i_{\psi x}, i_{\psi y}$	$p_{i_{\psi x}}, p_{i_{\psi y}}$	N_d	N_p
1	4.712389	3.829886	
2	7.853982		17.923313
3	10.995574	86.324865	
4	14.137167		415.2424
5	17.278760	1997.5148	
6	20.420352		9608.99
7	23.561945	46223.87	
8	26.703538		222358.8966

Table 2

Values of i_x, i_y vs i_c and i_d

i_c	1				2				3				4			
i_d	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
i_x	1	2	1	2	3	4	3	4	3	4	3	4	1	2	1	2
i_y	1	1	2	2	1	1	2	2	3	3	4	4	3	3	4	4

Appendix A

The constitutive elements of the non-dimensional Airy function $\psi(\xi,\eta)$ in Eq. (4), are formed by two separate components, depending on ξ and η, respectively:

$$\varphi_{i_\psi}(\xi,\eta) = \varphi_{i_{\psi x}}(\xi)\varphi_{i_{\psi y}}(\eta) \tag{A.1}$$

where both components $\varphi_{i_{\psi x}}(\xi)$ and $\varphi_{i_{\psi y}}(\eta)$ vanish at the rectangular edges, along with their first derivatives:

$$\begin{aligned} &\varphi_{i_{\psi x}}(0) = (\varphi_{i_{\psi x}})'(0) = \varphi_{i_{\psi x}}(1) = (\varphi_{i_{\psi x}})'(1) \qquad ( )' = \frac{\partial( )}{\partial \xi} \\ &\varphi_{i_{\psi y}}(0) = (\varphi_{i_{\psi y}})'(0) = \varphi_{i_{\psi y}}(1) = (\varphi_{i_{\psi y}})'(1) \qquad ( )' = \frac{\partial( )}{\partial \eta} \end{aligned} \tag{A.2}$$

because, taking into account the membrane stresses dependence on the Airy function [22], the in-plane boundary conditions in Eq. (1) are satisfied. For convenience functions are utilized which satisfy further conditions:

$$\frac{d^4\varphi_{i_{\psi x}}}{d\xi^4} = p_{i_{\psi x}}^4 \varphi_{i_{\psi x}} \qquad \frac{d^4\varphi_{i_{\psi y}}}{d\eta^4} = p_{i_{\psi y}}^4 \varphi_{i_{\psi y}} \tag{A.3}$$

$$\int_0^1 \varphi_{i_{\psi x}}\varphi_{j_{\psi x}} d\xi = \delta_{i_{\psi x} j_{\psi x}} \qquad \int_0^1 \varphi_{i_{\psi y}}\varphi_{j_{\psi y}} d\eta = \delta_{i_{\psi y} j_{\psi y}} \tag{A.4}$$

Consequently the expressions of $\varphi_{i_{\psi x}}$ can be written as:

$$\varphi_{i_{\psi x}}(\xi) = \frac{1}{N_d}\left\{\cos\left(\frac{p_{i_{\psi x}}}{2}\right)\cosh\left[p_{i_{\psi x}}\left(\xi-\frac{1}{2}\right)\right] - \cosh\left(\frac{p_{i_{\psi x}}}{2}\right)\cos\left[p_{i_{\psi x}}\left(\xi-\frac{1}{2}\right)\right]\right\} \qquad i_{\psi x} = 1,3,5....$$

(A.5a)

$$\varphi_{i_{\psi x}}(\xi) = \frac{1}{N_p}\left\{\sin\left(\frac{p_{i_{\psi x}}}{2}\right)\sinh\left[p_{i_{\psi x}}\left(\xi-\frac{1}{2}\right)\right] - \sinh\left(\frac{p_{i_{\psi x}}}{2}\right)\sin\left[p_{i_{\psi x}}\left(\xi-\frac{1}{2}\right)\right]\right\} \qquad i_{\psi x} = 2,4,6....$$

where:

$$p_{i_{\psi x}} = (2i_{\psi x}+1)\frac{\pi}{2} \tag{A.5b}$$

and:

$$N_d^2 = \left[\textbf{cosh}\left(\frac{p_{i_{\psi x}}}{2} \right) \right]^2 \left[\frac{1}{2} - \frac{\textbf{cos}\left(\frac{p_{i_{\psi x}}}{2} \right) \textbf{sin}\left(\frac{p_{i_{\psi x}}}{2} \right)}{p_{i_{\psi x}}} \right] + \left[\textbf{cos}\left(\frac{p_{i_{\psi x}}}{2} \right) \right]^2 \left[\frac{1}{2} - \frac{\textbf{cosh}\left(\frac{p_{i_{\psi x}}}{2} \right) \textbf{sinh}\left(\frac{p_{i_{\psi x}}}{2} \right)}{p_{i_{\psi x}}} \right]$$

(A.5c)

$$N_p^2 = \left[\textbf{sinh}\left(\frac{p_{i_{\psi x}}}{2} \right) \right]^2 \left[\frac{1}{2} - \frac{\textbf{cos}\left(\frac{p_{i_{\psi x}}}{2} \right) \textbf{sin}\left(\frac{p_{i_{\psi x}}}{2} \right)}{p_{i_{\psi x}}} \right] + \left[\textbf{sin}\left(\frac{p_{i_{\psi x}}}{2} \right) \right]^2 \left[\frac{1}{2} - \frac{\textbf{cosh}\left(\frac{p_{i_{\psi x}}}{2} \right) \textbf{sinh}\left(\frac{p_{i_{\psi x}}}{2} \right)}{p_{i_{\psi x}}} \right]$$

and similar for $\varphi_{i_{\psi y}}$, with $p_{i_{\psi y}}$ in place of $p_{i_{\psi x}}$ and η in place of ξ. The conditions in Eq. (A.2) referring to the first derivatives, are satisfied if it is taken into account that:

$$\textbf{tanh}\left(\frac{p_{i_{\psi x}}, p_{i_{\varphi y}}}{2} \right) \cong 1 \qquad \text{(A.6)}$$

for the values utilized of $p_{i_{\psi x}}, p_{i_{\varphi y}}$.

The values of $p_{i_{\psi x}}, p_{i_{\psi y}}$ connected with $i_{\psi x}, i_{\psi y}$, together with N_d, N_p, are summarized in Table 1, for $i_{\psi x}, i_{\psi y} = 1,2\ldots 8$.

Appendix B

The numerical approach, based on the Rayleigh-Ritz method, has to be explained. Series expansions for non-dimensional in-plane displacements functions $U(\xi,\eta,\tau)$ and $V(\xi,\eta,\tau)$ can be chosen:

$$U(\xi,\eta,\tau) = U_{i_U}(\tau)\chi_{i_U}(\xi,\eta) \qquad V(\xi,\eta,\tau) = V_{i_V}(\tau)\chi_{i_V}(\xi,\eta) \qquad \text{(B.1a)}$$

where:

$$\chi_{i_U}(\xi,\eta) = \left(\xi^{i_{Ux}+i_p} - i_p \xi^{i_{Ux}+i_p+1} \right)\left(\eta^{i_{Uy}+i_p} - i_p \eta^{i_{Uy}+i_p+1} \right)$$
$$\chi_{i_V}(\xi,\eta) = \left(\xi^{i_{Vx}+i_p} - i_p \xi^{i_{Vx}+i_p+1} \right)\left(\eta^{i_{Vy}+i_p} - i_p \eta^{i_{Vy}+i_p+1} \right) \qquad \text{(B.1b)}$$

$$i_{Ux}, i_{Vx} = 0,1,2\ldots N_{Ux}-1, N_{Vx}-1 \qquad i_{Uy}, i_{Vy} = 0,1,2\ldots N_{Uy}-1, N_{Vy}-1 \qquad i_U, i_V = i_{Ux}N_{Uy}, i_{Vx}N_{Vy} + i_{Uy}, i_{Vy} + 1$$

$$i_U, i_V = 1,2\ldots N_U, N_V \qquad N_U, N_V = N_{Ux}, N_{Vx} \times N_{Uy}, N_{Vy} \qquad \text{(B.1c)}$$

and for the out-of-plane non-dimensional displacement $W(\xi,\eta,\tau)$:

$$W(\xi,\eta,\tau) = W_{i_W}(\tau)\chi_{i_W}(\xi,\eta) \qquad \text{(B.2a)}$$

where:

$$\chi_{i_W}(\xi,\eta) = \left(\xi^{i_{Wx}} - \xi^{i_{Wx}+1}\right)\left(\eta^{i_{Wy}} - \eta^{i_{Wy}+1}\right) \tag{B.2b}$$

$$\begin{gathered} i_{Wx} = 1,2....N_{Wx} \qquad i_{Wy} = 1,2....N_{Wy} \qquad i_W = (i_{Wx}-1)N_{Wy} + i_{Wy} \\ i_W = 1,2....N_W \qquad N_W = N_{Wx}N_{Wy} \end{gathered} \tag{B.2c}$$

as in Eq.(7b).

It is evident that the series expansions elements of U and V in Eq. (B.1a) satisfy only the geometric boundary conditions, as in the Ritz method [3][4][8]. In fact if $i_p = 0$ there are not particular geometric in-plane boundary conditions to be satisfied for U and V, and this corresponds to the first case with in-plane free edges of the rectangular orthotropic plate, whilst for $i_p = 1$ both displacements vanish at the plate borders, as requested by the geometric boundary conditions of the in-plane clamped panel. Concerning W the series describing elements in Eq. (B.2a) satisfy the geometric boundary conditions, for which it vanishes at the rectangular delimiting edges; this series expansion is coincident with the ones of U and V with $i_p = 1$.

As in Galerkin method it is convenient to choose $N_{Ux} = N_{Vx}$ and $N_{Uy} = N_{Vy}$, so that it is true that $N_U = N_V$. Therefore the generic degree of freedom q_i can be defined as:

$$\begin{aligned} &q_i^{(in)} = U_{i_U} \quad \text{with} \quad i_1 = i = i_U \quad \text{for} \quad i \le N_U \\ &q_i^{(in)} = V_{i_V} \quad \text{with} \quad i_2 = i = i_V + N_U \quad \text{for} \quad N_U < i \le 2N_U \\ &q_i^{(op)} = W_{i_W} \quad \text{with} \quad i_3 = i = i_W + 2N_U \quad \text{for} \quad 2N_U < i \le N_W + 2N_U \end{aligned} \tag{B.3}$$

The in-plane elasticity costitutive equations of an orthotropic plate are well known [22], and also the corresponding kinematic relations, from which it is possible to determine the in-plane strain energy expression:

$$\begin{aligned} U^{(in)} = \frac{1}{2}\int_S \Bigg\{ & A_x(\,)\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2 + A_y\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]^2 + 2A_x\nu_{yx}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\Bigg\} dS \\ & + \frac{1}{2}\int_S \left\{ G_{xy}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]^2\right\} dS \end{aligned} \tag{B.4}$$

which, divided by the flexural rigidity parameter $D_r = E_r h^3 / (1-0.09) = A_r h^2$ of the reference isotropic plate, can be re-written in non-dimensional form:

$$\begin{aligned} \mathcal{U}^{(in)} = \frac{1}{2}\int_\Sigma \Bigg\{ & A_x^* \gamma_a^4 \left[\frac{\partial U}{\partial \xi} + \frac{1}{2}\left(\frac{\partial W}{\partial \xi}\right)^2\right]^2 + A_y^*\gamma_b^4\left[\frac{\partial V}{\partial \eta} + \frac{1}{2}\left(\frac{\partial W}{\partial \eta}\right)^2\right]^2 + 2A_x^*\nu_{yx}\gamma_a^2\gamma_b^2\left[\frac{\partial U}{\partial \xi} + \frac{1}{2}\left(\frac{\partial W}{\partial \xi}\right)^2\right]\left[\frac{\partial V}{\partial \eta} + \frac{1}{2}\left(\frac{\partial W}{\partial \eta}\right)^2\right]\Bigg\} d\Sigma \\ & + \frac{1}{2}\int_\Sigma \left\{ G_{xy}^*\gamma_a^2\gamma_a^2\left[\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi} + \frac{\partial W}{\partial \xi}\frac{\partial W}{\partial \eta}\right]^2\right\} d\Sigma \end{aligned} \tag{B.5a}$$

where the in-plane coordinates and all the displacements have been reformulated in non-dimensional form:

$$\xi = \frac{x}{a} \qquad \eta = \frac{y}{b} \qquad U = \frac{u}{a}\frac{1}{\gamma_a^2} \qquad V = \frac{v}{b}\frac{1}{\gamma_b^2} \qquad W = \frac{w}{L_w} \tag{B.5b}$$

as in Eqs. (3b) and (19b), together with the in-plane rigidity parameters:

$$A_x^* = \frac{A_x ab}{D_r} \qquad A_y^* = \frac{A_y ab}{D_r} \qquad G_{xy}^* = \frac{G_{xy} ab}{D_r} \tag{B.5c}$$

taking into account that the surface element dS is connected with the corresponding non-dimensional one $d\Sigma$ by the following relation:

$$dS = dxdy = abd\xi d\eta = abd\Sigma \tag{B.5d}$$

and further the following non-dimensional parameters have been introduced:

$$\gamma_a = \frac{L_w}{a} \qquad \gamma_b = \frac{L_w}{b} \tag{B.5e}$$

Thus the contribution to the in-plane strain energy only of the linear structural forces, is derived by retaining only the terms with U and V, which in dimension-less form reads:

$$\mathcal{U}_l^{(in)} = \frac{1}{2}\int_\Sigma \left\{ A_x^*\gamma_a^4\left(\frac{\partial U}{\partial \xi}\right)^2 + A_y^*\gamma_b^4\left(\frac{\partial V}{\partial \eta}\right)^2 + 2A_x^*\nu_{yx}\gamma_a^2\gamma_b^2\frac{\partial U}{\partial \xi}\frac{\partial V}{\partial \eta} + G_{xy}^*\gamma_{ab}^4\left(\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi}\right)^2 \right\} d\Sigma \tag{B.6}$$

Thence, if series expansions in Eq. (B1) are taken into account, it is possible to evaluate the integrals:

$$\int_\Sigma \left(\frac{\partial U}{\partial \xi}\right)^2 d\Sigma = U_{i_U} U_{j_U} \mathcal{I}_{i_U j_U}^{(Ux2)} \tag{B.7a}$$

where:

$$\mathcal{I}_{i_U j_U}^{(Ux2)} = \left[\frac{(i_{Ux}+i_p)(j_{Ux}+i_p)}{i_{Ux}+j_{Ux}+2i_p-1} - i_p\frac{(i_{Ux}+i_p)(j_{Ux}+i_p+1)+(i_{Ux}+i_p+1)(j_{Ux}+i_p)}{i_{Ux}+j_{Ux}+2i_p} + i_p\frac{(i_{Ux}+i_p+1)(j_{Ux}+j_p+1)}{i_{Ux}+j_{Ux}+2i_p+1}\right]$$

$$\times\left[\frac{1}{i_{Uy}+j_{Uy}+2i_p+1} - i_p\frac{2}{i_{Uy}+j_{Uy}+2i_p+2} + i_p\frac{1}{i_{Uy}+j_{Uy}+2i_p+3}\right] \tag{B.7b}$$

The first term between the first brackets disappears if $i_p = 0$, and $i_{Ux} = 0$ or $j_{Ux} = 0$, and the following one if both indices vanish. Its dual integral can be easily determined:

$$\int_\Sigma \left(\frac{\partial V}{\partial \eta}\right)^2 d\Sigma = V_{i_V} V_{j_V} \mathcal{I}_{i_V j_V}^{(Vy2)} \tag{B.8a}$$

where:

$$\mathcal{I}_{i_V j_V}^{(Vy2)} = \left[\frac{1}{i_{Vx} + j_{Vx} + 2i_p + 1} - i_p \frac{2}{i_{Vx} + j_{Vx} + 2i_p + 2} + i_p \frac{1}{i_{Vx} + j_{Vx} + 2i_p + 3} \right] \quad \text{(B.8b)}$$

$$\times \left[\frac{(i_{Vy} + i_p)(j_{Vy} + i_p)}{i_{Vy} + j_{Vy} + 2i_p - 1} - i_p \frac{(i_{Vy} + i_p)(j_{Vy} + i_p + 1) + (i_{Vy} + i_p + 1)(j_{Vy} + i_p)}{i_{Vy} + j_{Vy} + 2i_p} + i_p \frac{(i_{Vy} + i_p + 1)(j_{Vy} + j_p + 1)}{i_{Vy} + j_{Vy} + 2i_p + 1} \right]$$

where the first term between the second brackets disappears if $i_p = 0$, and $i_{Vy} = 0$ or $j_{Vy} = 0$, and the following one disappears if both indices vanish.

The mixed product of the coupling term of the in-plane energy expression (B.6a) has to be considered, for which it is necessary to evaluate the following integral:

$$\int_{\Sigma} \left(\frac{\partial U}{\partial \xi} \frac{\partial V}{\partial \eta} \right) d\Sigma = U_{i_U} V_{j_V} \mathcal{I}_{i_U j_V}^{(UxVy)} \quad \text{(B.9a)}$$

with:

$$\mathcal{I}_{i_U j_V}^{(UxVy)} = \left[\frac{i_{Ux} + i_p}{i_{Ux} + j_{Vx} + 2i_p} - i_p \frac{2i_{Ux} + 2i_p + 1}{i_{Ux} + j_{Vx} + 2i_p + 1} + i_p \frac{i_{Ux} + i_p + 1}{i_{Ux} + j_{Vx} + 2i_p + 2} \right]$$

$$\times \left[\frac{j_{Vy} + i_p}{i_{Uy} + j_{Vy} + 2i_p} - i_p \frac{2j_{Vy} + 2i_p + 1}{i_{Uy} + j_{Vy} + 2i_p + 1} + i_p \frac{j_{Vy} + i_p + 1}{i_{Uy} + j_{Vy} + 2i_p + 2} \right] \quad \text{(B.9b)}$$

where the first term between the first brackets disappears if $i_p = 0$ and $i_{Ux} = 0$, and the same between the second brackets disappears if $i_p = 0$ and $j_{Vy} = 0$.

The mixed product of the shear term of the in-plane strain energy expression (B.6a) leads to the the following integral:

$$\int_{\Sigma} \left(\frac{\partial U}{\partial \eta} \frac{\partial V}{\partial \xi} \right) d\Sigma = U_{i_U} V_{j_V} \Gamma_{i_U j_V}^{(UyVx)} \quad \text{(B.10a)}$$

with:

$$\mathcal{I}_{i_U j_V}^{(UyVx)} = \left[\frac{j_{Vx} + i_p}{i_{Ux} + j_{Vx} + 2i_p} - i_p \frac{2j_{Vx} + 2i_p + 1}{i_{Ux} + j_{Vx} + 2i_p + 1} + i_p \frac{j_{Vx} + i_p + 1}{i_{Ux} + j_{Vx} + 2i_p + 2} \right]$$

$$\times \left[\frac{i_{Uy} + i_p}{i_{Uy} + j_{Vy} + 2i_p} - i_p \frac{2i_{Uy} + 2i_p + 1}{i_{Uy} + j_{Vy} + 2i_p + 1} + i_p \frac{i_{Uy} + i_p + 1}{i_{Uy} + j_{Vy} + 2i_p + 2} \right] \quad \text{(B.10b)}$$

where the first term between the first brackets disappears if $i_p = 0$ a $j_{Vx} = 0$, and the same between the second brackets disappears if $i_p = 0$ and $i_{Uy} = 0$.

In the same way by the series expansions in Eq.(1) two other integrals in the shear term of the strain energy expression (B.6a), can be determined:

$$\int_{\Sigma} \left(\frac{\partial U}{\partial \eta} \right)^2 d\Sigma = U_{i_U} U_{j_U} \mathcal{I}_{i_U j_U}^{(Uy2)} \quad \text{(B.11a)}$$

with:

$$\mathcal{I}_{i_U j_U}^{(Uy2)} = \left[\frac{1}{i_{Ux} + j_{Ux} + 2i_p + 1} - i_p \frac{2}{i_{Ux} + j_{Ux} + 2i_p + 2} + i_p \frac{1}{i_{Ux} + j_{Ux} + 2i_p + 3} \right] \quad \text{(11b)}$$

$$\times \left[\frac{(i_{Uy} + i_p)(j_{Uy} + i_p)}{i_{Uy} + j_{Uy} + 2i_p - 1} - i_p \frac{(i_{Uy} + i_p)(j_{Uy} + i_p + 1) + (i_{Uy} + i_p + 1)(j_{Uy} + i_p)}{i_{Uy} + j_{Uy} + 2i_p} + i_p \frac{(i_{Uy} + i_p + 1)(j_{Uy} + j_p + 1`)}{i_{Uy} + j_{Uy} + 2i_p + 1} \right]$$

where the first term between the second brackets disappears if $i_p = 0$, and $i_{Uy} = 0$ or $j_{Uy} = 0$, and the following one disappears if both indices vanish, and also:

$$\int_\Sigma \left(\frac{\partial V}{\partial \xi} \right)^2 d\Sigma = V_{i_V} V_{j_V} \mathcal{I}_{i_V j_V}^{(Vx2)} \quad \text{(B.12a)}$$

with:

$$\mathcal{I}_{i_V j_V}^{(Vx2)} = \left[\frac{(i_{Vx} + i_p)(j_{Vx} + i_p)}{i_{Vx} + j_{Vx} + 2i_p - 1} - i_p \frac{(i_{Vx} + i_p)(j_{Vx} + i_p + 1) + (i_{Vx} + i_p + 1)(j_{Vx} + i_p)}{i_{Vx} + j_{Vx} + 2i_p} + i_p \frac{(i_{Vx} + i_p + 1)(j_{Vx} + j_p + 1)}{i_{Vx} + j_{Vx} + 2i_p + 1} \right]$$

$$\times \left[\frac{1}{i_{Vy} + j_{Vy} + 2i_p + 1} - i_p \frac{2}{i_{Vy} + j_{Vy} + 2i_p + 2} + i_p \frac{1}{i_{Vy} + j_{Vy} + 2i_p + 3} \right] \quad \text{(B.12b)}$$

where the first term between the first brackets disappears if $i_p = 0$, and $i_{Vx} = 0$ or $j_{Vx} = 0$, and the following one if both indices vanish.

If the in-plane strain energy expression in the classical form in Eq. (36) is considered, and taking into account the integrals previously evaluated, the in-plane stiffness matrix elements can be determined:

$$\begin{aligned} k_{i_1 j_1}^{(in)} &= A_x^* \gamma_a^4 \mathcal{I}_{i_U j_U}^{(Ux2)} + G_{xy}^* \gamma_a^2 \gamma_b^2 \mathcal{I}_{i_U j_U}^{(Uy2)} \\ k_{i_1 j_2}^{(in)} &= A_x^* \nu_{yx} \gamma_a^2 \gamma_b^2 \mathcal{I}_{i_U j_V}^{(UxVy)} + G_{xy}^* \gamma_a^2 \gamma_b^2 \mathcal{I}_{i_U j_V}^{(UyVx)} \\ k_{i_2 j_2}^{(in)} &= A_y^* \gamma_b^4 \mathcal{I}_{i_V j_V}^{(Vy2)} + G_{xy}^* \gamma_a^4 \gamma_b^4 \mathcal{I}_{i_V j_V}^{(Vx2)} \end{aligned} \quad \text{(B.13)}$$

Finally the symmetry of this in-plane stiffness matrix has to be imposed:

$$k_{j_2 i_1} = k_{i_1 j_2} \quad \text{(B.14)}$$

which corresponds to the second element in the double mixed product of the coupling and shear terms of the in-plane strain energy expression (B.6a).

Then the contribution to the in-plane strain energy due only to the mixed structural forces (linear together with non-linear) has to be considered. In the complete in-plane strain energy expression (B.5a), if only the terms containing the out-of-plane displacement W, together with U and V, are taken into account, the following non-dimensional energy expression is derived:

$$\mathcal{U}_m^{(in)} = \frac{1}{2}\int_\Sigma \left\{ A_x^* \gamma_a^4 \frac{\partial U}{\partial \xi}\left(\frac{\partial W}{\partial \xi}\right)^2 + A_y^* \gamma_b^4 \frac{\partial V}{\partial \eta}\left(\frac{\partial W}{\partial \eta}\right)^2 + A_x^* \nu_{yx}\gamma_a^2\gamma_b^2\left[\frac{\partial U}{\partial \xi}\left(\frac{\partial W}{\partial \eta}\right)^2 + \frac{\partial V}{\partial \eta}\left(\frac{\partial W}{\partial \xi}\right)^2\right]\right\} d\Sigma$$

$$+ \frac{1}{2}\int_\Sigma \left\{ 2G_{xy}^* \gamma_a^2\gamma_b^2 \left(\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi}\right)\frac{\partial W}{\partial \xi}\frac{\partial W}{\partial \eta}\right\} d\Sigma \qquad \text{(B.15)}$$

First it is necessary to evaluate the square of derivatives:

$$\left(\frac{\partial W}{\partial \xi}\right)^2 = W_{j_W} W_{k_W}\left[j_{Wx}\xi^{j_{Wx}-1} - (j_{Wx}+1)\xi^{j_{Wx}}\right]\left[k_{Wx}\xi^{k_{Wx}-1} - (k_{Wx}+1)\xi^{j_{Wx}}\right]\left[\eta^{j_{Wy}} - \eta^{j_{Wy}+1}\right]\left[\eta^{k_{Wy}} - \eta^{k_{Wy}+1}\right] =$$

$$= W_{j_W} W_{k_W} c_{jkx,m}\xi^{j_{Wx}+k_{Wx}+m-3} s_n \eta^{j_{Wy}+k_{Wy}+n-1} \qquad (m.n = 1,2,3) \qquad \text{(B.16a)}$$

where:

$$\begin{aligned} c_{ijx,1} &= i_{Wx} j_{Wx} & s_1 &= 1 \\ c_{ijx,2} &= -i_{Wx}(j_{Wx}+1) - (i_{Wx}+1)j_{Wx} & s_2 &= -2 \\ c_{ijx,3} &= (i_{Wx}+1)(j_{Wx}+1) & s_3 &= 1 \end{aligned} \qquad \text{(B.16b)}$$

and:

$$\left(\frac{\partial W}{\partial \eta}\right)^2 = W_{j_W} W_{k_W}\left[\xi^{j_{Wx}} - \xi^{j_{Wx}+1}\right]\left[\xi^{k_{Wx}} - \xi^{k_{Wx}+1}\right]\left[j_{Wy}\xi^{j_{Wy}-1} - (j_{Wy}+1)\xi^{j_{Wy}}\right]\left[k_{Wy}\xi^{k_{Wy}-1} - (k_{Wy}+1)\xi^{j_{Wy}}\right]$$

$$= W_{j_W} W_{k_W} s_n \xi^{j_{Wx}+k_{Wx}+n-1} c_{jky,m}\eta^{j_{Wy}+k_{Wy}+m-3} \qquad (m.n = 1,2,3) \qquad \text{(B.17a)}$$

where:

$$\begin{aligned} c_{ijy,1} &= i_{Wy} j_{Wy} \\ c_{ijy,2} &= -i_{Wy}(j_{Wy}+1) - (i_{Wy}+1)j_{Wy} \\ c_{ijy,3} &= (i_{Wy}+1)(j_{Wy}+1) \end{aligned} \qquad \text{(B.17b)}$$

Then the following integrals can be evaluated:

$$\int_\Sigma \left[\frac{\partial U}{\partial \xi}\left(\frac{\partial W}{\partial \xi}\right)^2\right] d\Sigma = U_{i_U} W_{j_W} W_{k_W} \mathcal{I}_{i_U j_W k_W}^{(UxWx2)} \qquad \text{(B.18a)}$$

$$\int_\Sigma \left[\frac{\partial U}{\partial \xi}\left(\frac{\partial W}{\partial \eta}\right)^2\right] d\Sigma = U_{i_U} W_{j_W} W_{k_W} \mathcal{I}_{i_U j_W k_W}^{(UxWy2)} \qquad \text{(B.19a)}$$

$$\int_\Sigma \left[\frac{\partial V}{\partial \eta}\left(\frac{\partial W}{\partial \xi}\right)^2\right] d\Sigma = V_{i_V} W_{j_W} W_{k_W} \mathcal{I}_{i_U j_W k_W}^{(VyWx2)} \qquad \text{(B.20a)}$$

$$\int_\Sigma \left[\frac{\partial V}{\partial \eta}\left(\frac{\partial W}{\partial \eta}\right)^2\right] d\Sigma = V_{i_V} W_{j_W} W_{k_W} \mathcal{I}_{i_U j_W k_W}^{(UyWy2)} \qquad \text{(B.21a)}$$

where:

$$\mathcal{I}_{i_U j_W k_W}^{(UxWx2)} = c_{jkx,m} s_n \left[\frac{i_{Ux} + i_p}{i_{Ux} + i_p + j_{Wx} + k_{Wx} - 3 + m} - i_p \frac{i_{Ux} + i_p + 1}{i_{Ux} + i_p + j_{Wx} + k_{Wx} - 2 + m} \right]$$

$$\times \left[\frac{1}{i_{Uy} + i_p + j_{Wy} + k_{Wy} + n} - i_p \frac{1}{i_{Uy} + i_p + j_{Wy} + k_{Wy} + n + 1} \right] \quad \text{(B.18b)}$$

$$\mathcal{I}_{i_U j_W k_W}^{(UxWy2)} = c_{jky,m} s_n \left[\frac{i_{Ux} + i_p}{i_{Ux} + j_{Wx} + k_{Wx} + i_p + n - 1} - i_p \frac{i_{Ux} + i_p + 1}{i_{Ux} + j_{Wx} + k_{Wx} + i_p + n} \right]$$

$$\times \left[\frac{1}{i_{Uy} + j_{Wy} + k_{Wy} + i_p + m - 2} - i_p \frac{1}{i_{Uy} + j_{Wy} + k_{Wy} + i_p + m - 1} \right] \quad \text{(B.19b)}$$

$$\mathcal{I}_{i_V j_W k_W}^{(VyWx2)} = c_{jkx,m} s_n \left[\frac{1}{i_{Vx} + j_{Wx} + k_{Wx} + i_p + m - 2} - i_p \frac{1}{i_{Vx} + j_{Wx} + k_{Wx} + i_p + m - 1} \right]$$

$$\times \left[\frac{i_{Vy} + i_p}{i_{Vy} + j_{Wy} + k_{Wy} + i_p + n - 1} - i_p \frac{i_{Vy} + i_p + 1}{i_{Vy} + j_{Wy} + k_{Wy} + i_p + n} \right] \quad \text{(B.20b)}$$

$$\mathcal{I}_{i_V j_W k_W}^{(VyWy2)} = c_{jky,m} s_n \left[\frac{1}{i_{Vx} + j_{Wx} + k_{Wx} + i_p + n} - i_p \frac{1}{i_{Vx} + j_{Wx} + k_{Wx} + i_p + n + 1} \right]$$

$$\times \left[\frac{i_{Vy} + i_p}{i_{Vy} + j_{Wy} + k_{Wy} + i_p + m - 3} - i_p \frac{i_{Vy} + i_p + 1}{i_{Vy} + j_{Wy} + k_{Wy} + i_p + m - 2} \right] \quad \text{(B.21b)}$$

The first terms between the first brackets in Eqs. (B.18b) and (B.19b) disappear if $i_p = 0$ and $i_{Ux} = 0$, while the first terms between the second beackets in Eqs. (B.20b) and (B.21b) disappear if $i_p = 0$ and $i_{Vy} = 0$.

Also the mixed product must be considered:

$$\frac{\partial W}{\partial \xi} \frac{\partial W}{\partial \eta} = W_{j_W} W_{k_W} \left[j_{Wx} \xi^{j_{Wx}-1} - (j_{Wx}+1) \xi^{j_{Wx}} \right] \left[\eta^{j_{Wy}} - \eta^{j_{Wy}+1} \right] \left[\xi^{k_{Wx}} - \xi^{k_{Wx}+1} \right] \left[k_{Wy} \xi^{k_{Wy}-1} - (k_{Wy}+1) \xi^{j_{Wy}} \right]$$

$$= W_{j_W} W_{k_W} t_{jx,m} \xi^{i_{Wx} + j_{Wx} + m - 1} t_{ky,n} \eta^{i_{Wy} + j_{Wy} + n - 1} \quad \text{(m,n=1,2,3)} \quad \text{(B.22a)}$$

where:

$$\begin{aligned} t_{ix,1} &= i_{Wx} & t_{iy,1} &= i_{Wy} \\ t_{ix,2} &= -2 i_{Wx} - 1 & t_{iy,2} &= -2 i_{Wy} - 1 \\ t_{ix,3} &= i_{Wx} + 1 & t_{iy,3} &= i_{Wy} + 1 \end{aligned} \quad \text{(B.22b)}$$

which allow to determine the integrals in the shear contribution to the in-plane strain energy expression of mixed forces, in Eq. (B.17), as:

$$\int_{\Sigma}\left(\frac{\partial U}{\partial \eta}\frac{\partial W}{\partial \xi}\frac{\partial W}{\partial \eta}\right)d\Sigma = U_{i_U} W_{j_W} W_{k_W} \mathcal{I}_{i_U j_W k_W}^{(UyWxWy)} \tag{B.23a}$$

$$\int_{\Sigma}\left(\frac{\partial V}{\partial \xi}\frac{\partial W}{\partial \xi}\frac{\partial W}{\partial \eta}\right)d\Sigma = U_{i_U} W_{j_W} W_{k_W} \mathcal{I}_{i_U j_W k_W}^{(VxWxWy)} \tag{B.24a}$$

where:

$$\begin{aligned}\mathcal{I}_{i_U j_W k_W}^{(UyWxWy)} &= t_{jx,m} t_{ky,n}\left[\frac{1}{i_{Ux}+j_{Wx}+k_{Wx}+i_p+n} - i_p\frac{1}{i_{Ux}+j_{Wx}+k_{Wx}+i_p+n+1}\right]\\ &\times\left[\frac{i_{u_Y}+i_p}{i_{Uy}+j_{Wy}+k_{Wy}+i_p+n-1} - i_p\frac{i_{u_y}+i_p+1}{i_{Uy}+j_{Wy}+k_{Wy}+i_p+n}\right]\end{aligned} \tag{B.23b}$$

$$\begin{aligned}\mathcal{I}_{i_U j_W k_W}^{(VxWxWy)} &= t_{jx,m} t_{ky,n}\left[\frac{i_{Vx}+i_p}{i_{Ux}+j_{Wx}+k_{Wx}+i_p+n} - i_p\frac{i_{Vx}+i_p+1}{i_{Ux}+j_{Wx}+k_{Wx}+i_p+n+1}\right]\\ &\times\left[\frac{1}{i_{Uy}+j_{Wy}+k_{Wy}+i_p+n} - i_p\frac{1}{i_{Uy}+j_{Wy}+k_{Wy}+i_p+n+1}\right]\end{aligned} \tag{B.24b}$$

Thus the tensor elements corresponding to the in-plane strain energy contribution of the mixed forces, introduced in Eq. (37), can be evaluated:

$$\begin{aligned} d_{i_1 j_3 k_3}^{(3)} &= A_x^* \gamma_a^4 \mathcal{I}_{i_U j_W k_W}^{(UxWx2)} + A_x^* \nu_{yx}\gamma_a^2\gamma_b^2 \mathcal{I}_{i_U j_W k_W}^{(UxWy2)} + 2G_{xy}^*\gamma_{ab}^4 \mathcal{I}_{i_U j_W k_W}^{(UxWxWy)}\\ d_{i_2 j_3 k_3}^{(3)} &= A_y^* \gamma_b^4 \mathcal{I}_{i_V j_W k_W}^{(VyWy2)} + A_x^* \nu_{yx}\gamma_a^2\gamma_b^2 \mathcal{I}_{i_V j_W k_W}^{(VyWx2)} + 2G_{xy}^*\gamma_{ab}^4 \mathcal{I}_{i_V j_W k_W}^{(VxWxWy)}\end{aligned} \tag{B.25}$$

At last the contribution of only non-linear structural forces has to be taken into account. Retaining only the terms containing the displacement W in the in-plane strain energy expression (B.5a), gives this contribution in non-dimensional form, as follows:

$$\mathcal{U}_{nl}^{(in)} = \frac{1}{2}\int_{\Sigma}\left\{A_x^*\gamma_a^4\frac{1}{4}\left(\frac{\partial W}{\partial \xi}\right)^4 + A_x^*\gamma_b^4\frac{1}{4}\left(\frac{\partial W}{\partial \eta}\right)^4 + A_x^*\nu_{yx}\gamma_a^2\gamma_b^2\frac{1}{2}\left(\frac{\partial W}{\partial \xi}\right)^2\left(\frac{\partial W}{\partial \eta}\right)^2 + G_{xy}^*\gamma_a^2\gamma_b^4\left(\frac{\partial W}{\partial \xi}\frac{\partial W}{\partial \eta}\right)^2\right\}d\Sigma \tag{B.26}$$

The fourth power of the derivatives has to be evaluated:

$$\left(\frac{\partial W}{\partial \xi}\right)^4 = W_{j_W} W_{k_W} W_{l_W} W_{m_W} c_{jkx,o} c_{lmx,p_r} \xi^{j_{Wx}+k_{Wx}+l_{Wx}+k_{Wx+}+o+p-6} s_q s_r \eta^{j_{Wy}+k_{Wy}+l_{Wx}+m_{Wy}+q+r-2} \tag{B.27}$$

$$(o,p,q,r=1,2,3)$$

where the meaning of the coefficients $c_{jkx,o}$, $c_{jkx,p}$, s_q and s_r, has been illustrated in Eq. (B.16b). Also the square power of the other derivative can be determined:

$$\left(\frac{\partial W}{\partial \eta}\right)^4 = W_{i_W} W_{j_W} W_{k_W} W_{l_W} s_q s_r \xi^{i_{Wx}+j_{Wx}+k_{Wx}+l_{Wx}+q+r-2} c_{ijy,o} c_{kly,p_r} \eta^{i_{Wy}+j_{Wy}+k_{Wy}+l_{Wy+}+o+p-6} \quad \text{(B.28)}$$

$$(o,p,q,r=1,2,3)$$

with the same meaning of the above mentioned coefficients, as in Eqs. (B.16b) and (B.17b), together with the mixed products:

$$\left(\frac{\partial W}{\partial \xi}\right)^2\left(\frac{\partial W}{\partial \eta}\right)^2 = W_{i_W} W_{j_W} W_{k_W} W_{l_W} c_{ijx,o} s_q \xi^{i_{Wx}+j_{Wx}+k_{Wx}+l_{Wx+}+o+q-4} c_{kly,p} s_r \eta^{i_{Wy}+j_{Wy}+k_{Wx}+l_{Wy}+p+r-4}$$

$$(o,p,q,r=1,2,3) \quad \text{(B.29)}$$

Thus the following integrals can be evaluated:

$$\int_\Sigma \left(\frac{\partial W}{\partial \xi}\right)^4 d\Sigma = W_{i_W} W_{j_W} W_{k_W} W_{l_W} \mathcal{I}_{i_W j_W k_W l_W}^{(WxWxWxWx)} \quad \text{(B.30a)}$$

$$\int_\Sigma \left(\frac{\partial W}{\partial \eta}\right)^4 d\Sigma = W_{i_W} W_{j_W} W_{k_W} W_{l_W} \mathcal{I}_{i_W j_W k_W l_W}^{(WyWyWyWy)} \quad \text{(B.31a)}$$

$$\int_\Sigma \left(\frac{\partial W}{\partial \xi}\frac{\partial W}{\partial \eta}\right)^2 d\Sigma = W_{i_W} W_{j_W} W_{k_W} W_{l_W} \mathcal{I}_{i_W j_W k_W l_W}^{(WxWxWyWy)} \quad \text{(B.32a)}$$

where:

$$\mathcal{I}_{i_W j_W k_W, l_W}^{(WxWxWxWx)} = c_{ijx,o} c_{klx,p} s_q s_r \times \frac{1}{i_{Wx}+j_{Wx}+k_{Wx}+l_{Wx}+o+p-5} \times \frac{1}{i_{Wy}+j_{Wy}+k_{Wy}+l_{Wy}+q+r-1} \quad \text{(B.30b)}$$

$$\mathcal{I}_{i_W j_W k_W, l_W}^{(WyWyWyWy)} = c_{ijy,o} c_{kly,p} s_q s_r \times \frac{1}{i_{Wx}+j_{Wx}+k_{Wx}+l_{Wx}+q+r-1} \times \frac{1}{i_{Wy}+j_{Wy}+k_{Wy}+l_{Wy}+o+p-5} \quad \text{(B.31b)}$$

$$\mathcal{I}_{i_W j_W k_W, l_W}^{(WxWxWyWy)} = c_{ijx,o} c_{kly,p} s_q s_r \times \frac{1}{i_{Wx}+j_{Wx}+k_{Wx}+l_{Wx}+o+q-3} \times \frac{1}{i_{Wy}+j_{Wy}+k_{Wy}+l_{Wy}+p+r-3} \quad \text{(B.32b)}$$

Thus the tensor elements $d_{i_3 j_3 k_3 l_3}^{(4)}$ can be determined:

$$d_{i_3 j_3 k_3 l_3}^{(4)} = \frac{1}{4} A_x^* \gamma_a^4 \mathcal{I}_{i_W j_W k_W l_W}^{(Wx4)} + \frac{1}{4} A_y^* \gamma_b^4 \mathcal{I}_{i_W j_W k_W l_W}^{(Wy4)} + \frac{1}{2} A_x^* \nu_{yx} \gamma_a^2 \gamma_b^2 \mathcal{I}_{i_W j_W k_W l_W}^{(Wx2y2)} + G_{xy}^* \gamma_a^2 \gamma_b^2 \mathcal{I}_{i_W j_W k_W l_W}^{(Wx2y2)} \quad \text{(B.33)}$$

The out-of-plane strain energy expression [18] can be written as:

$$U^{(op)} = \frac{1}{2}\int_S \left\{ D_x \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + D_y \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2D_x \nu_{yx} \left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + 4D_t \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right\} dS \tag{B.34}$$

which, divided by D_r, can be also written in dimension-less form:

$$\mathcal{U}^{(op)} = \frac{1}{2}\int_\Sigma \left\{ D_x^* \left(\frac{\partial^2 W}{\partial \xi^2}\right)^2 + D_y^* \left(\frac{\partial^2 W}{\partial \eta^2}\right)^2 + 2B^* \left(\frac{\partial^2 W}{\partial \xi \partial \eta}\right)^2 \right\} d\Sigma \tag{B.35a}$$

where the in-plane coordinate, together the flexural displacement, have been reformulated in non-dimensional form, as in Eqs. (3b), and also the flexural-torsional rigidity parameters:

$$D_x^* = \frac{D_x L_w^2 b}{D_r a^3} \qquad D_y^* = \frac{D_y L_w^2 a}{D_r b^3} \qquad B^* = \frac{D_x \nu_{yx} L_w^2}{D_r a \ b} + 2\frac{D_t L_w^2}{D_r a \ b} \tag{B.35b}$$

and taking into account that for a simply supported plate it is true that [18]:

$$\int_\Sigma \left(\frac{\partial^2 W}{\partial \xi^2}\right)\left(\frac{\partial^2 W}{\partial \eta^2}\right) d\Sigma = \int_\Sigma \left(\frac{\partial^2 W}{\partial \xi \partial \eta}\right)^2 d\Sigma \tag{B.36}$$

The square power of the second derivative $\frac{\partial^2 W}{\partial \xi^2}$ has to be considered:

$$\left(\frac{\partial^2 W}{\partial \xi^2}\right)^2 = W_{i_W} W_{j_W} c_{ijx2,m} \xi^{i_{Wx} + j_{Wx} + m - 5} s_n \xi^{i_{Wy} + j_{Wy} + n - 1} \tag{B.37a}$$

where:

$$\begin{aligned} c_{ijx2,1} &= i_{Wx}(i_{Wx} - 1) j_{Wx}(j_{Wx} - 1) & s_1 &= 1 \\ c_{ijx2,2} &= -i_{Wx}(i_{Wx} - 1)(j_{Wx} + 1) j_{wx} - (i_{Wx} + 1) i_{Wx} j_{Wx}(j_{Wx} - 1) & s_2 &= -2 \\ c_{ijx2,3} &= (i_{Wx} + 1) i_{Wx}(j_{Wx} + 1) j_{Wx} & s_3 &= 1 \end{aligned} \tag{B.37b}$$

and also of the second derivative $\frac{\partial^2 W}{\partial \eta^2}$:

$$\left(\frac{\partial^2 W}{\partial \eta^2}\right)^2 = W_{i_W} W_{j_W} s_n \xi^{i_{Wx} + j_{Wx} + n - 1} c_{ijy2,m} \eta^{i_{Wy} + j_{Wy} + m - 5} \tag{B.38a}$$

$$\begin{aligned} c_{ijy2,1} &= i_{Wy}(i_{Wy} - 1) j_{Wy}(j_{Wy} - 1) \\ c_{ijy2,2} &= -i_{Wy}(i_{Wy} - 1)(j_{Wy} + 1) j_{wy} - (i_{Wy} + 1) i_{Wy} j_{Wy}(j_{Wy} - 1) \\ c_{ijy2,3} &= (i_{Wy} + 1) i_{Wy}(j_{Wy} + 1) j_{Wy} \end{aligned} \tag{B.38b}$$

It is necessary to know also the same power of the mixed derivative:

$$\left(\frac{\partial^2 W}{\partial\xi\partial\eta}\right)^2 = W_{i_W} W_{j_W} c_{ijx,m} c_{ijy,n} \xi^{i_{Wx}+j_{Wx}+m-3} \eta^{i_{Wy}+j_{Wy}+n-3} \tag{B.39}$$

where the coefficients $c_{ijx,m}$ and $c_{ijy,n}$ are the same of Eqs. (B.16b) and (B.17b).

Thus the following integrals in Eq. (B.36a) can be evaluated:

$$\int_\Sigma \left(\frac{\partial^2 W}{\partial\xi^2}\right)^2 d\Sigma = W_{i_W} W_{j_W} \mathcal{I}_{i_W j_W}^{(Wx2Wx2)} \tag{B.40a}$$

$$\int_\Sigma \left(\frac{\partial^2 W}{\partial\eta^2}\right)^2 d\Sigma = W_{i_W} W_{j_W} \mathcal{I}_{i_W j_W}^{(Wy2Wy2)} \tag{B.41a}$$

$$\int_\Sigma \left(\frac{\partial^2 W}{\partial\xi\partial\eta}\right)^2 d\Sigma = W_{i_W} W_{j_W} \mathcal{I}_{i_W j_W}^{(WxyWxy)} \tag{B.42a}$$

where:

$$\mathcal{I}_{i_W j_W}^{(Wx2Wx2)} = c_{ijx2,m} s_n \times \frac{1}{i_{Wx} + j_{Wx} + m - 4} \times \frac{1}{i_{Wy} + j_{Wy} + n} \tag{B.40b}$$

$$\mathcal{I}_{i_W j_W}^{(Wy2Wy2)} = c_{ijy2,m} s_n \times \frac{1}{i_{Wx} + j_{Wx} + n} \times \frac{1}{i_{Wy} + j_{Wy} + m - 4} \tag{B.41b}$$

$$\mathcal{I}_{i_W j_W}^{(WxyWxy)} = c_{ijx,m} c_{ijy,n} \times \frac{1}{i_{Wx} + j_{Wx} + m - 2} \times \frac{1}{i_{Wy} + j_{Wy} + n - 2} \tag{B.42b}$$

Hence the out-of-plane stiffness matrix elements can be evaluated:

$$k_{i_3 j_3}^{(op)} = D_x^* \mathcal{I}_{i_W j_W}^{(Wx2Wx2)} + D_y^* \mathcal{I}_{i_W j_W}^{(Wy2Wy2)} + 2B^* \mathcal{I}_{i_W j_W}^{(WxyWxy)} \tag{B.43}$$

Then the kinetic energy for the out-of-plane vibration can be considered:

$$T^{(op)} = \frac{1}{2}\mu \int_S \left(\frac{\partial w}{\partial t}\right)^2 dS \tag{B.44}$$

where μ is the mass density per unity surface, which, divided by D_r, can be also written in non-dimensional form:

$$\mathcal{T}^{(op)} = \frac{1}{2}\mathcal{M} \int_\Sigma \left(\frac{\partial W}{\partial\tau}\right)^2 d\Sigma \tag{B.45a}$$

where:

$$\tau = \sqrt{\frac{D_r \pi^4}{\mu L^4}}t \qquad \mathcal{M} = \pi^4 \frac{ab}{L^2}\gamma^2 \qquad \gamma = \frac{L_w}{L} \qquad W = \frac{w}{L_w} \tag{B.45b}$$

as in Eqs. (3c) and (11c).

The following integrals have to be evaluated:

$$\int_\Sigma \left(\frac{\partial W}{\partial \tau}\right)^2 d\Sigma = \dot{W}_{i_W} \dot{W}_{j_W} \mathcal{I}^{(WW)}_{i_W j_W} \tag{B.46a}$$

where:

$$\dot{W}_{i_W} = \frac{\partial W_{i_W}}{\partial \tau} \qquad \mathcal{I}^{(WW)}_{i_W j_W} = \left[\frac{1}{i_{Wx} + j_{Wx} + 1} - \frac{2}{i_{Wx} + j_{Wx} + 2} + \frac{1}{i_{Wx} + j_{Wx} + 3}\right]$$
$$\times \left[\frac{1}{i_{Wy} + j_{Wy} + 1} - \frac{2}{i_{Wy} + j_{Wy} + 2} + \frac{1}{i_{Wy} + j_{Wy} + 3}\right] \tag{B.46b}$$

Taking into account Eq. (38), which gives the classical expressions of the in-plane and out-of-plane kinetic energy, and Eqs. (B.3), which explains the meaning the Lagrangian degrees of freedom $q_i^{(op)}$, enables us to form the out-of-plane mass matrix, whose elements are:

$$m^{(op)}_{i_3 j_3} = \mathcal{M}\mathcal{I}^{(WW)}_{i_W j_W} \tag{B.47}$$

while the in-plane mass matrix is not considered, because the in-plane inertial forces are neglected.

Finally the expressions of the generalized aerodynamic forces in Eq. (47) have to be determined for the Ritz procedure. The aerodymic pressure p_z expression in Eq. (10b), has to be newly considered:

$$p_z = \frac{2q}{\beta}\left(\frac{\partial w}{\partial x} + \frac{1}{U_a}\frac{\beta^2 - 1}{\beta^2}\frac{\partial w}{\partial t}\right) \tag{B.48}$$

whose work, for the out-of-plane displacement w presence, can be evaluated:

$$L^{(aer)} = \frac{2q}{\beta}\int_S \left(\frac{\partial w}{\partial x} + \frac{1}{U_a}\frac{\beta^2 - 1}{\beta^2}\frac{\partial w}{\partial t}\right) w dS \tag{B.49}$$

which, divided by the reference plate flexural rigidity D_r, can be re-written in non-dimensional form:

$$\mathcal{L}^{(aer)} = \mathcal{A}^{(x)}\int_S \left(\frac{\partial W}{\partial \xi} W\right) d\Sigma + \mathcal{A}^{(t)}\int_s \left(\frac{\partial W}{\partial \tau} W\right) d\Sigma \tag{B.50a}$$

where:

$$\mathcal{A}^{(x)} = \frac{2q}{\beta}\frac{L_w^2 b}{aD_r} \qquad \mathcal{A}^{(t)} = \frac{2q}{\beta}\frac{\beta^2-1}{\beta^2}\frac{1}{U_a}\frac{L_w^2 ab}{D_r}\sqrt{\frac{D_r\pi^4}{\mu L^4}} \qquad \tau = \sqrt{\frac{D_r\pi^4}{\mu L^4}}t \tag{B.50b}$$

The series expansion for $W(\xi,\eta,\tau)$, as formulated in Eq. (B.2a), whose component elements are reported in Eq. (B.2b), can be substituted into Eq. (50a) and thence the work performed by the ith element of the series, which is equal to the ith generalized aerodynamic force $F_{i_3}^{(a)}$, can be evaluated:

$$F_{i_3}^{(a)} = W_{j_W}\mathcal{A}^{(x)}\int_S\left(\frac{\partial\chi_{j_W}}{\partial\xi}\chi_{i_W}\right)d\Sigma + \dot{W}_{j_W}\mathcal{A}^{(t)}\int_s\left(\chi_{i_W}\chi_{j_W}\right)d\Sigma \tag{B.51}$$

It is possible to point-out that it is formed by two components, as in Eq. (46), the first of which $F_{i_W}^{(a,x)}$ is the coupling term with spatial derivative, while the second one $F_{i_W}^{(a,t)}$ with temporal derivative is the damping component. Thus the coefficients $f_{ij}^{(x)}$ and $f_{ij}^{(t)}$ of the component aerodynamic forces can be evaluated, as in Eq. (47), from Eq. (B.51):

$$f_{i_3 j_3}^{(x)} = \mathcal{A}^{(x)}\int_S\left(\frac{\partial\chi_{j_W}}{\partial\xi}\chi_{i_W}\right)d\Sigma = \mathcal{A}^{(x)}\mathcal{I}_{i_W j_W}^{(WxW)} \tag{B.52a}$$

$$f_{i_3 j_3}^{(t)} = \mathcal{A}^{(t)}\int_s\left(\chi_{i_W}\chi_{j_W}\right)d\Sigma = \mathcal{A}^{(t)}\mathcal{I}_{i_W j_W}^{(WW)} \tag{B.53}$$

where $\mathcal{I}_{i_W j_W}^{(WW)}$ is the same of Eq. (B.46b), and:

$$\mathcal{I}_{i_W j_W}^{(WxW)} = \left[\frac{j_{Wx}}{i_{Wx}+j_{Wx}} - \frac{2j_{Wx}+1}{i_{Wx}+j_{Wx}+1} + \frac{j_{Wx}+1}{i_{Wx}+j_{Wx}+2}\right] \times\left[\frac{1}{i_{Wy}+j_{Wy}+1} - \frac{2}{i_{Wy}+j_{Wy}+2} + \frac{1}{i_{Wy}+j_{Wy}+3}\right] \tag{B.52b}$$

The coefficients $f_{i_3 j_3}^{(t)}$ are proportional the out-of-plane mass matrix elements by the relation [20]:

$$f_{i_3 j_3}^{(t)} = \gamma_d m_{i_3 j_3}^{(op)} \tag{B.54}$$

with $\gamma_d = \mathcal{A}^{(t)}/\mathcal{M}$, as in Eq. (53).

Appendix C

Now the FEM model utilized, as in Fig. 2, has to be illustrated.

The plate is divided into N_{ex} elements along the axis ξ and N_{ey} elements along the axis η, consequently the whole number of components elements of the fluttering panel is $N_{ex}N_{ey}$. A generic i_eth element with vertices ABCD is shown in Fig. 2, which lies in the coordinates range:

$$\xi_{i_{ex}-1} \le \xi \le \xi_{i_{ex}} \qquad \eta_{i_{ey}-1} \le \eta \le \eta_{i_{ey}} \qquad i_{ex} = 1,2....N_{ex} \qquad i_{ey} = 1,2....N_{ey} \tag{C.1}$$
$$i_e = (i_{ex} - 1)N_{ey} + i_{ey} \qquad N_e = N_{ex}N_{ey} \qquad i_e = 1,2....N_e$$

where $\xi_{i_{ex}-1}$ is the ascissa of the vertices A and D, $\xi_{i_{ex}}$ is the abscissa of the vertices B and C, $\eta_{i_{ey}-1}$ is the ordinate of the vertices A and B, $\eta_{i_{ey}}$ is the ordinate of the vertices C and D.

Normalized coordinates of the element are introduced:

$$\xi_{n,i_e} = (\xi - \xi_{i_{ex}-1})N_{ex} \qquad \eta_{n,i_e} = (\xi - \xi_{i_{ey}-1})N_{ey} \qquad 0 \le \xi_{n,i_e}, \eta_{n,i_e} \le 1 \tag{C.2}$$

Series expansion for non-dimensional in-plane and out-of-plane displacements functions can be introduced in the generic i_eth element:

$$Q^{(i_v)}(\xi,\eta,t) = q_{i_c i_d}^{(i_e,i_v)}(t)\varphi_{x,i_x}^{(i_e)}\varphi_{y,i_y}^{(i_e)} \tag{C.3a}$$

where $Q^{(i_v)}(\xi,\eta,t)$ corresponds to U,V,W for $i_v = 1,2,3$, respectively, the coefficients $q_{i_c i_d}^{(i_e,i_v)}(t)$ (which are the Lagrangian degrees of freedom in every grid point), are equal to:

$$Q^{(i_v)}, \frac{\partial Q^{(i_v)}}{\partial \xi}, \frac{\partial Q^{(i_v)}}{\partial \eta}, \frac{\partial^2 Q^{(i_v)}}{\partial \xi \partial \eta} \tag{C.3b}$$

for $i_d = 1,2,3,4$, respectively, evaluated on the vertices A,B,C,D, if $i_c = 1,2,3,4$, respectively, and with bicubic Hermitian polynomials used to interpolate each of the displacement components:

$$\begin{aligned}
\varphi_{x,1}^{(i_e)} &= 1 - 3\xi_{n,i_e}^2 + 2\xi_{n,i_e}^3 & \varphi_{y,1}^{(i_e)} &= 1 - 3\eta_{n,i_e}^2 + 2\eta_{n,i_e}^3 \\
\varphi_{x,2}^{(i_e)} &= \xi_{n,i_e} - 2\xi_{n,i_e}^2 + \xi_{n,i_e}^3 & \varphi_{y,2}^{(i_e)} &= \eta_{n,i_e} - 2\eta_{n,i_e}^2 + \eta_{n,i_e}^3 \\
\varphi_{x,3}^{(i_e)} &= 3\xi_{n,i_e}^2 - 2\xi_{n,i_e}^3 & \varphi_{y,3}^{(i_e)} &= 3\eta_{n,i_e}^2 - 2\eta_{n,i_e}^3 \\
\varphi_{x,4}^{(i_e)} &= -\xi_{n,i_e}^2 + \xi_{n,i_e}^3 & \varphi_{y,4}^{(i_e)} &= -\eta_{n,i_e}^2 + \eta_{n,i_e}^3
\end{aligned} \tag{C.3c}$$

and also:

$$\begin{aligned}
i_x &= i_d - 2\delta_{i_d 3} - 2\delta_{i_d 4} + 2\left(\frac{i_c}{2}\right) - 4\delta_{i_c 4} \\
i_y &= 2\left(\frac{i_c - 1}{2}\right) + \left(\frac{i_d - 1}{2}\right) + 1
\end{aligned} \tag{C.3d}$$

In Eq. (C.3d) the operations between round parentheses have to be performed with integer numers, and also the resulting number is integer, whereas δ_{ij} is the kronecker's delta. The values of i_x, i_y, connected with i_c, i_d as in Eqs. (C.3d), are summarized in Table 2.

The geometric boundary conditions have to be imposed at the rectangular edges, where in the case of in-plane clamped plate, all the three displacements vanish at the limit boundaries together with their first tangential derivative, for which in the i_Gth grid point, with:

$$i_G(i_e, i_c) = (i_{ex} - 1)(N_{ey} + 1) + i_{ey} + \delta_{i_c 4} + \delta_{i_c 3}(N_{ey} + 1) + \delta_{i_c 4}(N_{ey} + 2) \qquad \text{(C.4a)}$$

i_d can be only 3 and 4 if:

$$i_G = 1, N_{ey} + 1 + 1, 2(N_{ey} + 1) + 1, \ldots . N_{ex}(N_{ey} + 1) + 1 \qquad \text{(C.4b)}$$

corresponding to $\eta = 0$, and also if:

$$i_G = N_{ey} + 1, 2(N_{ey} + 1) \ldots . (N_{ex} + 1)(N_{ey} + 1) \qquad \text{(C.4c)}$$

corresponding to $\eta = 1$, whereas i_d can be only equal to 2 and 4 if:

$$i_e = 1, 2 \ldots . N_{ey} + 1 \qquad \text{(C.4d)}$$

corresponding to $\xi = 0$, and also if:

$$i_G = N_{ex}(N_{ey} + 1) + 1, N_{ex}(N_{ey} + 1) + 2 \ldots . (N_{ex} + 1)(N_{ey} + 1) \qquad \text{(C.4e)}$$

corresponding to $\xi = 1$.

There exist ever such limits for W, because the plate is simply-supported for the out-of-plane flutter vibration dynamics, apart from the in-plane boundary conditions. Consequently in the case of in-plane free edges there are such restrictions only for W, while for U and V both suffixes i_c and i_d can vary from 1 to 4 for every value of i_e.

Thus there are N_U degrees of freedom for both U and V, equal to:

$$N_U = 4(N_{ex} + 1)(N_{ey} + 1) - i_p[4(N_{ex} - 1) + 4(N_{ey} - 1) + 12] \qquad \text{(C.5)}$$

where, as for the Ritz procedure, $i_p = 0$ corresponds to the case with free in-plane edges of the plate, and $i_p = 1$ corresponds to the other with clamped boundaries. The number of the degrees of freedom N_W corresponding to W is obtained by Eq. (C.5) with $i_p = 1$, considering that in both cases with different in-plane boundary conditions, such number is limited by the same geometric boundary conditions. It means that in the case with free in-plane plate behaviour N_W is smaller than N_U.

The degrees of freedom in a generic i_Gth internal grid of the FEM model can be defined as follows:

$$q_i^{(in)} = q_{i_c i_d}^{(i_e,1)} \quad \text{with} \quad i = i_1 \quad \text{for} \quad i_{Gl} \le 4$$
$$q_i^{(in)} = q_{i_c i_d}^{(i_e,2)} \quad \text{with} \quad i = i_2 = i_1 + 4 \quad \text{for} \quad 4 < i_{Gl} \le 8 \tag{C.6a}$$
$$q_i^{(op)} = q_{i_c i_d}^{(i_e,3)} \quad \text{with} \quad i = i_3 = i_2 + 4 \quad \text{for} \quad 8 < i_{Gl} \le 12$$

where:

$$i_{Gl}(i_v, i_d) = 4(i_v - 1) + i_d \tag{C.6b}$$

In the external grids the number of degrees of freedom for U and V is reduced to 2, and for the same reason to 1 on the rectangular corners, if $i_p = 1$, and in both cases for W.

The index i_1 referring to the in-plane displacement U can be defined as:

$$\begin{aligned} i_1(i_G, i_d) = 12(i_G - 1) - (\delta_{i_{Gx}1} + \delta_{i_{Gx},N_{ex}+1})(2i_p + 1)[2(i_{Gy} - 1) + (1 - \delta_{i_{Gy}1})] \\ - (1 - \delta_{i_{Gx}1})(2i_p + 1)[4i_{Gx} - 4 + 2(1 - \delta_{i_{Gy}1}) + 2 + 2(N_{ey} - 1)] + i_d^* \end{aligned} \tag{C.7a}$$

where:

$$\begin{aligned} i_d^* = i_d [\delta_{i_p 0} + \delta_{i_p 1}(1 - \delta_{i_{Gx}1})(1 - \delta_{i_{Gx},N_{ex}+1})(1 - \delta_{i_{Gy}1})(1 - \delta_{i_{Gy},N_{ey}+1})] \\ + \delta_{i_p 1}(\delta_{i_{Gx}1} + \delta_{i_{Gx},N_{ex}+1})[(1 - \delta_{i_{Gy}1})(1 - \delta_{i_{Gy},N_{ey}+1})(\delta_{i_d 2} + 2\delta_{i_d 4}) + \delta_{i_{Gy}1} + \delta_{i_{Gy},N_{ey}+1}] \\ + \delta_{i_p 1}(\delta_{i_{Gy}1} + \delta_{i_{Gy},N_{ey}+1})[(1 - \delta_{i_{Gx}1})(1 - \delta_{i_{Gx},N_{ex}+1})(\delta_{i_d 3} + 2\delta_{i_d 4})] \end{aligned} \tag{C.7b}$$

and the index i_2, corresponding to the in-plane displacement V, can be defined as:

$$\begin{aligned} i_2(i_G, i_d) = i_1(i_G, i_d) + 4 [\delta_{i_p 0} + \delta_{i_p 1}(1 - \delta_{i_{Gx}1})(1 - \delta_{i_{Gx},N_{ex}+1})(1 - \delta_{i_{Gy}1})(1 - \delta_{i_{Gy},N_{ey}+1})] \\ + \delta_{i_p 1}(\delta_{i_{Gx}1} + \delta_{i_{Gx},N_{ex}+1})[2(1 - \delta_{i_{Gy}1})(1 - \delta_{i_{Gy},N_{ey}+1}) + \delta_{i_{Gy}1} + \delta_{i_{Gy},N_{ey}+1}] \\ + 2\delta_{i_p 1}(\delta_{i_{Gy}1} + \delta_{i_{Gy},N_{ey}+1})(1 - \delta_{i_{Gx}1})(1 - \delta_{i_{Gx},N_{ex}+1}) \end{aligned} \tag{C.7c}$$

and at last the index i_3, corresponding to W, as:

$$\begin{aligned} i_3(i_G, i_d) = i_2(i_G, i_d) + 4\, i_d [\delta_{i_p 0} + \delta_{i_p 1}(1 - \delta_{i_{Gx}1})(1 - \delta_{i_{Gx},N_{ex}+1})(1 - \delta_{i_{Gy}1})(1 - \delta_{i_{Gy},N_{ey}+1})] \\ + \delta_{i_p 1}(\delta_{i_{Gx}1} + \delta_{i_{Gx},N_{ex}+1})[(1 - \delta_{i_{Gy}1})(1 - \delta_{i_{Gy},N_{ey}+1})(2 + \delta_{i_p 0}\delta_{i_d 2}) + \delta_{i_{Gy}1} + \delta_{i_{Gy},N_{ey}+1}] \\ + 2\delta_{i_p 1}(\delta_{i_{Gy}1} + \delta_{i_{Gy},N_{ey}+1})(1 - \delta_{i_{Gx}1})(1 - \delta_{i_{Gx},N_{ex}+1}) \end{aligned} \tag{C.7d}$$

where:

$$i_{Gx} = \frac{i_G - 1}{N_{ey} + 1} + 1 \qquad i_{Gy} = i_G - (i_{Gx} - 1)(N_{ey} + 1) \qquad i_G = 1,2\mathbf{....}(N_{ex} + 1)(N_{ey} + 1) \tag{C.7e}$$

The operations in Eqs. (7e) are considered between integers numbers (g.e. $i_G - 1$ divided by $N_{ey} + 1$ is equal to 0 if $i_G - 1 < N_{ey} + 1$, etc.).

From the expression (B.6a) of the linear structural forces contribution to the in-plane strain energy it is possible to form the in-plane stiffness matrix of the i_eth element, whose elements obtained in the Ritz procedure have been shown in Eq. (B.13), and in this FEM model can be written as:

$$k^{(in)}_{i_1 j_1} = A^*_x \gamma^4_a \mathcal{I}^{(Ux2)}_{i_1 j_1} + G^*_{xy} \gamma^4_{ab} \mathcal{I}^{(Uy2)}_{i_1 j_1}$$
$$k^{(in)}_{i_1 j_2} = A^*_x \nu_{yx} \gamma^2_a \gamma^2_b \mathcal{I}^{(UxVy)}_{i_1 j_2} + G^*_{xy} \gamma^2_a \gamma^2_b \mathcal{I}^{(UyVx)}_{i_1 j_2} \tag{C.8a}$$
$$k^{(in)}_{i_2 j_2} = A^*_y \gamma^4_b \mathcal{I}^{(Vy2)}_{i_2 j_2} + G^*_{xy} \gamma^4_{ab} \mathcal{I}^{(Vx2)}_{i_2 j_2}$$

where the same integrals in Eqs. (B.7a),(B.8a),(B.9a),(B.10a),(B.11a),(B.12a) are utilized:

$$\mathcal{I}^{(Ux2)}_{i_1 j_1} = \frac{N_{ex}}{N_{ey}} \int_{\Sigma^{(i_e)}} \left(\frac{\partial \varphi^{(i_e)}_{x,i_x}}{\partial \xi_{n,i_e}} \frac{\partial \varphi^{(i_e)}_{x,j_x}}{\partial \xi_{n,i_e}} \varphi^{(i_e)}_{y,i_y} \varphi^{(i_e)}_{y,j_y} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.8b}$$

$$\mathcal{I}^{(Vy2)}_{i_2 j_2} = \frac{N_{ey}}{N_{ex}} \int_{\Sigma^{(i_e)}} \left(\varphi^{(i_e)}_{x,i_x} \varphi^{(i_e)}_{x,j_x} \frac{\partial \varphi^{(i_e)}_{y,i_y}}{\partial \eta_{n,i_e}} \frac{\partial \varphi^{(i_e)}_{y,j_y}}{\partial \eta_{n,i_e}} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.8c}$$

$$\mathcal{I}^{(UxVy)}_{i_1 j_2} = \int_{\Sigma^{(i_e)}} \left(\frac{\partial \varphi^{(i_e)}_{x,i_x}}{\partial \xi_{n,i_e}} \varphi^{(i_e)}_{x,j_x} \varphi^{(i_e)}_{y,i_y} \frac{\partial \varphi^{(i_e)}_{y,j_y}}{\partial \eta_{n,i_e}} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.8d}$$

$$\mathcal{I}^{(UyVx)}_{i_1 j_2} = \int_{\Sigma^{(i_e)}} \left(\varphi^{(i_e)}_{x,i_x} \frac{\partial \varphi^{(i_e)}_{x,j_x}}{\partial \xi_{n,i_e}} \frac{\partial \varphi^{(i_e)}_{y,i_y}}{\partial \eta_{n,i_e}} \varphi^{(i_e)}_{y,j_y} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.8e}$$

$$\mathcal{I}^{(Uy2)}_{i_1 j_2} = \frac{N_{ey}}{N_{ex}} \int_{\Sigma^{(i_e)}} \left(\varphi^{(i_e)}_{x,i_x} \varphi^{(i_e)}_{x,j_x} \frac{\partial \varphi^{(i_e)}_{y,i_y}}{\partial \eta_{n,i_e}} \frac{\partial \varphi^{(i_e)}_{y,j_y}}{\partial \eta_{n,i_e}} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.8f}$$

$$\mathcal{I}^{(Vx2)}_{i_2 j_2} = \frac{N_{ex}}{N_{ey}} \int_{\Sigma^{(i_e)}} \left(\frac{\partial \varphi^{(i_e)}_{x,i_x}}{\partial \xi_{n,i_e}} \frac{\partial \varphi^{(i_e)}_{x,j_x}}{\partial \xi_{n,i_e}} \varphi^{(i_e)}_{y,i_y} \varphi^{(i_e)}_{y,j_y} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.8g}$$

because:

$$d\xi = \frac{d\xi_{n,i_e}}{N_{ex}} \qquad d\eta = \frac{d\eta_{n,i_e}}{N_{ey}} \qquad d\Sigma = d\xi d\eta = \frac{d\xi_{n,i_e} d\eta_{n,i_e}}{N_{ex} N_{ey}} \tag{C.8h}$$

The integrals extend over the area $\Sigma^{(i_e)}\left(0 \le \xi_{n,i_e}, \eta_{n,i_e} \le 1\right)$ of the i_eth element. The suffixes j_2 and j_1 are connected with j_x, j_y through i_G, i_e, i_c, i_d, like i_1 and i_2 with i_x, i_y, by the same Eqs. (C.7a), (C.7c), (C.4a) and (C.3d). The dual element $k^{(in)}_{i_2 j_1}$ is equal to $k^{(in)}_{i_1 j_2}$, for the symmetry of the stiffness matrix in each element, as in Eq. (B.14).

Likewise it is possible to form the mixed structural forces tensor of the i_eth element, whose contribution to the in-plane strain energy is shown in Eq. (15), which has been reported in Eq. (B.25) for the Ritz procedure, and in FEM model can be written as:

$$d^{(3)}_{i_1 j_3 k_3} = A^*_x \gamma^4_a \mathcal{I}^{(UxWxWx)}_{i_1 j_3 k_3} + A^*_x \nu_{yx} \gamma^2_a \gamma^2_b \mathcal{I}^{(UxWyWy)}_{i_1 j_3 k_3} + 2G^*_{xy} \gamma^2_a \gamma^2_b \mathcal{I}^{(UxWxWy)}_{i_1 j_3 k_3}$$

$$d^{(3)}_{i^*_2 j_3 k_3} = A^*_y \gamma^4_b \mathcal{I}^{(VyWyWy)}_{i_2 j_3 k_3} + A^*_x \nu_{yx} \gamma^2_a \gamma^2_b \mathcal{I}^{(VyWxWx)}_{i_2 j_3 k_3} + 2G^*_{xy} \gamma^2_a \gamma^2_b \mathcal{I}^{(VxWxWy)}_{i_2 j_3 k_3} \tag{C.9a}$$

where the integrals utilized are the same of Eqs. (B.18a),(B.19a),(B.20a),(B.21a),(B.23a) and (B.24a):

$$\mathcal{I}^{(UxWxWx)}_{i_1 j_3 k_3} = \frac{N^2_{ex}}{N_{ey}} \int_{\Sigma^{(i_e)}} \left(\frac{\partial \varphi^{(i_e)}_{x,i_x}}{\partial \xi_{n,i_e}} \frac{\partial \varphi^{(i_e)}_{x,j_x}}{\partial \xi_{n,i_e}} \frac{\partial \varphi^{(i_e)}_{x,k_x}}{\partial \xi_{n,i_e}} \varphi^{(i_e)}_{y,i_y} \varphi^{(i_e)}_{y,j_y} \varphi^{(i_e)}_{y,k_y} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.9b}$$

$$\mathcal{I}^{(UxWyWy)}_{i_1 j_2 k_3} = N_{ey} \int_{\Sigma^{(i_e)}} \left(\frac{\partial \varphi^{(i_e)}_{x,i_x}}{\partial \xi_{n,i_e}} \varphi^{(i_e)}_{x,j_x} \varphi^{(i_e)}_{x,k_x} \varphi^{(i_e)}_{y,i_y} \frac{\partial \varphi^{(i_e)}_{y,j_y}}{\partial \eta_{n,i_e}} \frac{\partial \varphi^{(i_e)}_{y,k_y}}{\partial \eta_{n,i_e}} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.9c}$$

$$\mathcal{I}^{(VyWxWx)}_{i_2 j_3 k_3} = N_{ex} \int_{\Sigma^{(i_e)}} \left(\varphi^{(i_e)}_{x,i_x} \frac{\partial \varphi^{(i_e)}_{x,j_x}}{\partial \xi_{n,i_e}} \frac{\partial \varphi^{(i_e)}_{x,k_x}}{\partial \xi_{n,i_e}} \frac{\partial \varphi^{(i_e)}_{y,i_y}}{\partial \eta_{n,i_e}} \varphi^{(i_e)}_{y,j_y} \varphi^{(i_e)}_{y,k_y} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.9d}$$

$$\mathcal{I}^{(VyWyWy)}_{i_2 j_3 k_3} = \frac{N^2_{ex}}{N_{ey}} \int_{\Sigma^{(i_e)}} \left(\varphi^{(i_e)}_{x,i_x} \varphi^{(i_e)}_{x,j_x} \varphi^{(i_e)}_{x,k_x} \frac{\partial \varphi^{(i_e)}_{y,i_y}}{\partial \eta_{n,i_e}} \frac{\partial \varphi^{(i_e)}_{y,j_y}}{\partial \eta_{n,i_e}} \frac{\partial \varphi^{(i_e)}_{y,k_y}}{\partial \eta_{n,i_e}} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.9e}$$

$$\mathcal{I}^{(UyWxWy)}_{i_1 j_3 k_3} = N_{ey} \int_{\Sigma^{(i_e)}} \left(\varphi^{(i_e)}_{x,i_x} \frac{\partial \varphi^{(i_e)}_{x,j_x}}{\partial \xi_{n,i_e}} \varphi^{(i_e)}_{x,k_x} \frac{\partial \varphi^{(i_e)}_{y,i_y}}{\partial \eta_{n,i_e}} \varphi^{(i_e)}_{y,j_y} \frac{\partial \varphi^{(i_e)}_{y,k_y}}{\partial \eta_{n,i_e}} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.9f}$$

$$\mathcal{I}^{(VxWxWy)}_{i_2 j_3 k_3} = N_{ex} \int_{\Sigma^{(i_e)}} \left(\frac{\partial \varphi^{(i_e)}_{x,i_x}}{\partial \xi_{n,i_e}} \frac{\partial \varphi^{(i_e)}_{x,j_x}}{\partial \xi_{n,i_e}} \varphi^{(i_e)}_{x,k_x} \varphi^{(i_e)}_{y,i_y} \varphi^{(i_e)}_{y,j_y} \frac{\partial \varphi^{(i_e)}_{y,k_y}}{\partial \eta_{n,i_e}} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.9g}$$

The indices j_3, k_3 are connected with j_x, j_y, k_x, k_y through i_G, i_e, i_c, i_d, like i_3 with i_x, i_y, by the same relation in Eq. (C.7d), (C.4a) and (C.3d). It is possible also to determine the non-linear structural forces tensor of the i_eth element, like the corresponding ones in the Ritz procedure in Eq. (B.33), which can be written as:

$$d^{(4)}_{i_3 j_3 k_3 l_3} = \frac{1}{4} A^*_x \gamma^4_a \mathcal{I}^{(WxWxWxWx)}_{i_3 j_3 k_3 l_3} + \frac{1}{4} A^*_y \gamma^4_b \mathcal{I}^{(WyWyWyWy)}_{i_3 j_3 k_3 l_3} + \frac{1}{2} A^*_x \nu_{yx} \gamma^2_a \gamma^2_b \mathcal{I}^{(WxWxWyWy)}_{i_3 j_3 k_3 l_3} + G^*_{xy} \gamma^4_{ab} \mathcal{I}^{(WxWxWyWy)}_{i_3 j_3 k_3 l_3} \tag{C.10a}$$

where the reported integrals are the same of Eqs. (B.30a),(B.31a) and (B.32a):

$$\mathcal{I}_{i_3 j_3 k_3 l_3}^{(WxWxWxWx)} = \frac{N_{ex}^3}{N_{ey}} \int_{\Sigma^{(i_e)}} \left(\frac{\partial \varphi_{x,i_x}^{(i_e)}}{\partial \xi_{n,i_e}} \frac{\partial \varphi_{x,j_x}^{(i_e)}}{\partial \xi_{n,i_e}} \frac{\partial \varphi_{x,k_x}^{(i_e)}}{\partial \xi_{n,i_e}} \frac{\partial \varphi_{x,l_x}^{(i_e)}}{\partial \xi_{n,i_e}} \varphi_{y,i_y}^{(i_e)} \varphi_{y,j_y}^{(i_e)} \varphi_{y,k_y}^{(i_e)} \varphi_{y,l_y}^{(i_e)} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.10b}$$

$$\mathcal{I}_{i_3 j_3 k_3 l_3}^{(WyWyWyWy)} = \frac{N_{ey}^3}{N_{ex}} \int_{\Sigma^{(i_e)}} \left(\varphi_{x,i_x}^{(i_e)} \varphi_{x,j_x}^{(i_e)} \varphi_{x,k_x}^{(i_e)} \varphi_{x,l_x}^{(i_e)} \frac{\partial \varphi_{y,i_y}^{(i_e)}}{\partial \eta_{n,i_e}} \frac{\partial \varphi_{y,j_y}^{(i_e)}}{\partial \eta_{n,i_e}} \frac{\partial \varphi_{y,k_y}^{(i_e)}}{\partial \eta_{n,i_e}} \frac{\partial \varphi_{y,l_y}^{(i_e)}}{\partial \eta_{n,i_e}} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.10c}$$

$$\mathcal{I}_{i_3 j_3 k_3 l_3}^{(WxWxWyWy)} = N_{ex} N_{ey} \int_{\Sigma^{(i_e)}} \left(\frac{\partial \varphi_{x,i_x}^{(i_e)}}{\partial \xi_{n,i_e}} \frac{\partial \varphi_{x,j_x}^{(i_e)}}{\partial \xi_{n,i_e}} \varphi_{x,k_x}^{(i_e)} \varphi_{x,l_x}^{(i_e)} \varphi_{y,i_y}^{(i_e)} \varphi_{y,j_y}^{(i_e)} \frac{\partial \varphi_{y,k_y}^{(i_e)}}{\partial \eta_{n,i_e}} \frac{\partial \varphi_{y,l_y}^{(i_e)}}{\partial \eta_{n,i_e}} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.10d}$$

where l_3, like i_3, j_3, k_3, is connected with l_x, l_y, through i_G, i_e, i_c, i_d, by the same relations in Eq. (C.7d), (C.4a) and (C.3d).

Also the out-of-plane stiffness matrix can be formed, whose elements are:

$$k_{i_3 j_3}^{(op)} = D_x^* \mathcal{I}_{i_3 j_3}^{(Wx2Wx2)} + D_y^* \mathcal{I}_{i_3 j_3}^{(Wy2Wy2)} + 2B^* \mathcal{I}_{i_3 j_3}^{(WxyWxy)} \tag{C.11a}$$

where the integrals are the same in Eqs. (B.40a),(B.41a) and (B.42a):

$$\mathcal{I}_{i_3 j_3}^{(Wx2Wx2)} = \frac{N_{ex}^3}{N_{ey}} \int_{\Sigma^{(i_e)}} \left(\frac{\partial^2 \varphi_{x,i_x}^{(i_e)}}{\partial \xi_{n,i_e}^2} \frac{\partial^2 \varphi_{x,j_x}^{(i_e)}}{\partial \xi_{n,i_e}^2} \varphi_{y,i_y}^{(i_e)} \varphi_{y,j_y}^{(i_e)} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.11b}$$

$$\mathcal{I}_{i_3 j_3}^{(Wy2Wy2)} = \frac{N_{ey}^3}{N_{ex}} \int_{\Sigma^{(i_e)}} \left(\varphi_{x,i_x}^{(i_e)} \varphi_{x,j_x}^{(i_e)} \frac{\partial^2 \varphi_{y,i_y}^{(i_e)}}{\partial \eta_{n,i_e}^2} \frac{\partial^2 \varphi_{y,j_y}^{(i_e)}}{\partial \eta_{n,i_e}^2} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.11c}$$

$$\mathcal{I}_{i_3 j_3}^{(WxyWxy)} = N_{ex} N_{ey} \int_{\Sigma^{(i_e)}} \left(\frac{\partial \varphi_{x,i_x}^{(i_e)}}{\partial \xi_{n,i_e}} \frac{\partial \varphi_{x,j_x}^{(i_e)}}{\partial \xi_{n,i_e}} \frac{\partial \varphi_{y,i_y}^{(i_e)}}{\partial \eta_{n,i_e}} \frac{\partial \varphi_{y,j_y}^{(i_e)}}{\partial \eta_{n,i_e}} \right) d\xi_{n,i_e} d\eta_{n,i_e} \tag{C.11d}$$

Then it is necessary to form the mass matrix of the i_eth element, whose components are:

$$m_{i_3 j_3}^{(op)} = \mathcal{M} \mathcal{I}_{i_3 j_3}^{(WW)} \tag{C.12a}$$

where the parameter $\mathcal{M}$ is the same of Eq. (B.45b), and the integral $\mathcal{I}_{i_3 j_3}^{(WW)}$ is the same of Eq. (B.46a):

$$\mathcal{I}_{i_3 j_3}^{(WW)} = \int_{\Sigma^{(i_e)}} \varphi_{i_x}^{(i_e)} \varphi_{j_x}^{(i_e)} \varphi_{i_y}^{(i_e)} \varphi_{j_y}^{(i_e)} d\Sigma^{(i_e)} \tag{C.12b}$$

All the stiffness and mass matrices, together with the tensor elements, of the single component elements of the plate structure elements have to be assembled to obtain the overall stiffness, mass and resulting tensor elements, whose knowledge allows to solve the dynamic problem.

At last the presence of the generalized aerodynamic forces, obtained by the quasi-steady linearized aerodynamic "Piston Theory", as in Eq. (10b), has to be considered also in the FEM model. If the series expansion in Eq. (C.3a) is considered and substituted into Eq. (B.52a), it is possible to derive the generalized force corresponding to the i_3th degree of freedom, as in Eq. (B.52):

$$F_{i_3}^{(a)} = q_{i_c i_d}^{(j_e,3)} \mathcal{A}^{(x)} \int_{\Sigma^{(i_e)}} \left(\frac{\partial \varphi_{x,j_x}}{\partial \xi_{n,i_e}} \varphi_{x,i_x} \right) d\xi_{n,i_e} d\eta_{n,i_e} + \dot{q}_{i_c i_d}^{(j_e,3)} \mathcal{A}^{(t)} \int_{\Sigma^{(i_e)}} \left(\varphi_{x,i_x} \varphi_{x,j_x} \right) d\Sigma \tag{C.13}$$

and the coefficients $f_{ij}^{(x)}$ and $f_{ij}^{(t)}$ as in Eqs. (47), (B.53) and (B.54) can be obtained:

$$f_{i_3 j_3}^{(x)} = \mathcal{A}^{(x)} \int_{\Sigma^{(i_e)}} \left(\frac{\partial \varphi_{x,j_x}}{\partial \xi_{n,i_e}} \varphi_{x,i_x} \right) d\xi_{n,i_e} d\eta_{n,i_e} \qquad f_{i_3 j_3}^{(t)} = \mathcal{A}^{(t)} \int_{\Sigma^{(i_e)}} \left(\varphi_{x,i_x} \varphi_{x,j_x} \right) d\Sigma \tag{C.14}$$

and the non-dimensional parameter $\gamma_d = \mathcal{A}^{(t)} / \mathcal{M}$ of Eq. (53) is the same as in the Ritz method.

Appendix D. Nomenclature

A_x, A_y	in-plane rigidity parameters of the orthotropic plate
A_r	in-plane rigidità parameter of the reference isotropic plate
A_x^*, A_y^*	non-dimensional in-plane rigidity parameters of the orthotropic plate
a, b	rectangulare plate dimensions
a_1, b_1, c_1	non-dimensional parameters
a_{ij}	coefficients of the non-dimensional Airy function series expansion
D_r	flexural rigidity modulus of the reference isotropic plate
D_t	torsional rigidity modulus of the orthotropic plate
D_t^*	non-dimensional torsional rigidity modulus of the orthotropic plate
D_x, D_y	flexural rigidity moduli of the orthotropic plate

D_x^*, D_y^*, B^*	non-dimensional flexural and torsional rigidity parameters of the orthotropic plate
$E_{\Downarrow}, E_{\lrcorner}$	Young's moduli of the orthotropic plate along the fibers direction and the perpendicular one, respectively
E_x, E_y	Young's moduli of the orthotropic plate along the axes x and y, respectively
E_r	Young's modulus of the isotropic reference plate
$g_{xy} = G_{\Downarrow\lrcorner}$	in-plane shear rigidity modulus of the orthotropic plate
$G_{xy} = g_{xy}h$	resultant in-plane shear rigidity modulus
G_{Ax}, G_{Ay}, G_{xy}^*	nondimensional resultant in-plane shear rigidity parameters
h	plate thickness
k_{ij}	stiffness matrix elements
k_{ij}^*	elements of the linear structural and aerodynamic forces resultant matrix
L_a	nondimensional parameter L/a
L, L_w	in-plane and out-of-plane reference lengths
M_{ach}	Mach number
m_{ij}	mass matrix elements
N_x, N_y, N_{xy}	in-plane membrane stresses
r_{ab}	non-dimensional ratio a/b between the plate dimensions
t	time symbol
U_a	supersonic flow speed
U, V, W	non-dimensional displacements along the axes x, y, z
$U_{i_U}, V_{i_V}, W_{i_W}$	coefficients of the non-dimensional displacements series expansions
u, v, w	displacements along the axes x, y, z
x, y, z	plate reference system axes

Greek symbols

$\alpha_1, \beta_1, \gamma_1$	flexural and torsional rigidity non-dimensional parameters
$\alpha_a, \beta_a, \gamma_{as}$	in-plane extensional rigidity non-dimensional parameters
β	non-dimensional parameter equal to $\sqrt{M_{ach}^2 - 1}$
$\gamma, \gamma_a, \gamma_b$	non-dimensional parameters
δ_{ij}	Kronecker's delta
ζ_{ij}	utilized matrix elements
ϑ	non-dimensional parameter in the flutter vibration equation
$\nu_{xy}, \nu_{yx}, \nu_{\Downarrow\lrcorner}, \nu_{\lrcorner\Downarrow}$	Poisson's moduli
ξ, η	non-dimensional coordinates of the in-plane reference system
σ	non-dimensional dynamic pressare
τ	non-dimensional time
φ_{i_ψ}	generic element in the non-dimensional Airy function series expansion
$\chi_{i_U}\ \chi_{i_V}\ \chi_{i_W}$	generic elements of the displacements series expansion
ψ	non-dimensional Airy function

Special symbols

∂	partial differentiation
$d_{ijk}^{(3)}, d_{ijkl}^{(4)}, e_{ijkl}, h_{ijk}$	tensor elements
$\mathcal{I}_{\dots}^{(\ )}$	generic integral
$[\mathbf{A}][\mathbf{D}^{(\ )}][\mathbf{E}^{(\ )}][\mathbf{H}][\mathbf{I}^{(\ )}]$ $[\mathbf{P}^{(\ )}][\mathbf{Q}^{(\ )}][\mathbf{R}^{(\ )}][\mathbf{T}]\ [\mathbf{Z}][\Lambda]$	utilized matrices
$[\mathbf{K}^{(in)}][\mathbf{M}^{(in)}]$ $[\mathbf{K}^{(op)}][\mathbf{M}^{(op)}]$	in-plane and out-of-plane stiffness and mass matrices
$[\mathbf{K}^{(op)}]^*$	out-of-plane structural and aerodynamic forces matrix
$[\mathbf{Q}^{(in)}]\ \ [\mathbf{Q}^{(op)}]$	in-plane and out-of-plane Lagrangian degrees of freedom
$[\mathbf{U}][\mathbf{V}][\mathbf{W}]$	utilized column vectors of the displacements U , V , W series expansions coefficients
$[\mathbf{W}^{(2)}][\mathbf{W}^{(3)}]$	utilized column vectors containing double and triple products

	between coefficients of W series expansions
$sin(\ )cos(\ )tan(\ )$	trigonometric functions
$\mathcal{T}^{(\ )}$	non-dimensional kinetic energy expression
$\mathcal{U}_l^{(\ )}\mathcal{U}_m^{(\ )}\mathcal{U}_{nl}^{(\ )}$	non-dimensional strani energy expressions due to linear, mixed and non-linear structural forces

Subscripts

i, j	subscripts with generic meaning
$i_U j_U k_U, i_V j_V k_V$ $i_W j_W k_W$	subscripts referring to U,V,W, respectively, in the series expansions
$i_1 j_1 k_1, i_2 j_2 k_2$ $i_3 j_3 k_3$	subscripts referring to the Lagrangian degrees of freedom corresponding to U,V,W, respectively
i_c	subscript referring to a grid point of an element of FEM model
i_d	subscript referring to one of four degrees of freedom of an unknown variable
$i_{ex} j_{ex}, i_e$	subscripts referring to the generic i_e th element in FEM model
l, m, nl	subscripts referring to the linear,mixed and non-linear structural forces
⇓,↵	subscripts referring to the fibers direction and its perpendicular one

Superscripts

(i_e)	superscript referring to the generic i_e th element
(i_v)	superscript referring of one of the three unknown variables U,V,W
$(in)(op)$	supescripts referring to the in-plane and out-of-plane situation
$(U..V..W..)$	superscipts referring to operations on U,V,W

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