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STONELEY SIGNALS IN PERFECTLY BONDED DISSIMILAR THERMOELASTIC HALF-SPACES WITH AND WITHOUT THERMAL RELAXATION

LOUIS MILTON BROCK

The governing equations for each of two perfectly bonded, dissimilar thermoelastic half-spaces include as special cases the Fourier heat conduction model and models with either one or two thermal relaxation times. An exact solution in transform space for the problem of line loads applied in one half-space is obtained.

Study of the Stoneley function shows that conditions for existence of roots are more restrictive than in the isothermal case, and that both real and imaginary roots are possible. For the limit case of line loads applied to the interface, an analytical expression for the time transform of the corresponding residue contribution to interface temperature change is derived.

Asymptotic expressions for the inverses that are valid for either very long or very short times after loading occurs show that long-time behavior obeys Fourier heat conduction. Short-time results are sensitive to thermal relaxation effects. In particular, a time step load produces a propagating step in temperature for the Fourier and double-relaxation time models, but a propagating impulse for the single-relaxation time model.

1. Introduction

Joined dissimilar elastic materials occur in geological formations [Cagniard 1962] and as structural elements [Jones 1999]. Transient analyses [Stoneley 1924; Cagniard 1962] show that dynamic loading of these can produce, in addition to dilatational and rotational waves, interface (Stoneley) waves. Such waves are similar to Rayleigh surface waves [Lamb 1904] and so may be important in assessing interface integrity.

Studies such as [Stoneley 1924; Cagniard 1962] focus on isothermal materials. Studies such as [Brock 1997a; 1997b] consider both Stoneley and Rayleigh waves for materials that satisfy equations for coupled thermoelasticity [Chadwick 1960]. However, the equations are based on classical Fourier heat conduction [Carrier and Pearson 1988], and the Stoneley and Rayleigh signals are examined for times after the application of loading that greatly exceed the thermoelastic characteristic time.

Joseph and Preziosi [1989] have surveyed models that include the phenomenon of thermal relaxation in heat conduction. Lord and Shulman [1967], Green and Lindsay [1972] and Chandrasekharia [1986] have included thermal relaxation in formulations for coupled thermoelasticity. Sharma and Sharma [2002] have applied such formulations to homogeneous plates. Based on all this work, and on an effort in (nontransient) dynamic steady-state analysis of two joined half-spaces governed by the Fourier model [Brock and Georgiadis 1999], this article considers two perfectly bonded, dissimilar elastic half-spaces

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that are subject to thermal-mechanical line loads applied to the interface. Both half-spaces obey equations for coupled thermoelasticity that include the Fourier model [Chadwick 1960], and the single- and double-relaxation time models of Lord and Shulman [1967] and Green and Lindsay [1972], respectively, as special cases.

The study begins with construction of the exact solution in transform space for the general case of line loads applied in one of the half-spaces. The solution exhibits a Stoneley function that is more complicated in form than its isothermal counterpart [Cagniard 1962]. Conditions for the existence of Stoneley roots are determined, and found to be more restrictive than those for the isothermal case. Expressions for these roots, analytic to within a single integration, are developed, and found to give both real and imaginary values, again in contrast to the isothermal case. An exact formula for the time transform of the change in interface temperature when the line loads are applied to the interface is developed. Analytical expressions for the change itself, valid for either very long or very short times after loading is applied, are obtained for each of the three models. Consistent with previous observation [Brock 2004] the long-time results all have the character of the Fourier model, and describe a temperature change wave. The short-time results, on the other hand, are sensitive to the particular model but the Stoneley signals are again in the form of waves.

2. Statement of general problem and governing equations

In terms of Cartesian coordinates (x, y, z) two half-spaces of dissimilar isotropic, homogeneous, linear thermoelastic material are perfectly bonded along the plane $y = 0$. For time $t \leq 0$, both are at rest at the uniform ambient (absolute) temperature T_0 when, at $t = 0$, thermal-mechanical disturbances are introduced along the line $x = 0$, $y = L$. The disturbances may be time-dependent, but do not vary along the line, so that a state of plane strain is generated. For half-space $1 (y > 0)$ the field equations for $t > 0$ are

$$\left(\nabla^2 - s_{r1}^2 \frac{\partial^2}{\partial t^2}\right)(u_{x1}, u_{y1}) + \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)(m_1 \Delta_1 - \alpha_{v1} D_1^{II} \theta_1) = \frac{1}{\mu_1}(F_x, F_y)\delta(x)\delta(y - L), \tag{1a}$$

$$h_1 \nabla^2 \theta_1 - s_{r1} \frac{\partial}{\partial t}\left(\frac{\varepsilon_1}{\alpha_{v1}} D_1 \Delta_1 - D_1^I \theta_1\right) = F_T \delta(x)\delta(y - L), \tag{1b}$$

$$\frac{1}{\mu_1}(\sigma_{x1}, \sigma_{y1}, \sigma_{z1}) = (m_1 - 1)\Delta_1 - \alpha_{v1} D_1^{II} \theta_1 + 2\left(\frac{\partial u_{x1}}{\partial x}, \frac{\partial u_{y1}}{\partial y}, 0\right), \tag{1c}$$

$$\frac{1}{\mu_1}\sigma_{xy1} = \frac{\partial u_{x1}}{\partial y} + \frac{\partial u_{y1}}{\partial x}. \tag{1d}$$

In (1) $(u_{x1}, u_{y1}, \Delta_1, \theta_1)$ are, respectively, displacement components, dilatation and change in temperature from T_0, and $(\sigma_{x1}, \sigma_{y1}, \sigma_{z1}, \sigma_{xy1})$ are stress components. These vary with (x, y, t). In (1a), (1b) (F_x, F_y, F_T) are the t-dependent line loads, and δ is the Dirac function. For the Fourier model F [Chadwick 1960] and single- and double-relaxation time model I [Lord and Shulman 1967] and II [Green and Lindsay 1972], respectively,

$$\mathrm{F}: (D_1, D_1^I, D_1^{II}) = 1 \tag{2a}$$

$$\mathrm{I}: D_1^{II} = 1, \qquad (D_1, D_1^I) = 1 + \tau_1^I \frac{\partial}{\partial t} \tag{2b}$$

$$\mathrm{II}: (D_1, D_1^{II}) = 1 + \tau_1^{II} \frac{\partial}{\partial t}, \qquad D_1^I = 1 + \tau_1^I \frac{\partial}{\partial t}. \tag{2c}$$

Constants $\tau_1^I > \tau_1^{II} \geq 0$ are thermal relaxation times, and it is noted that model II serves to introduce thermal relaxation explicitly in constitutive Equation (1c), (1d). In (1)

$$m_1 = \frac{1}{1-2\nu_1}, \qquad a_1 = 2\frac{1-\nu_1}{1-2\nu_1}, \tag{3a}$$

$$\varepsilon_1 = \frac{\mu_1 T_0}{\rho_1 c_{v1}} \alpha_{1v}^2, \qquad h_1 = v_{r1}\tau_1^h, \qquad s_{r1} = \frac{1}{v_{r1}} \tag{3b}$$

$$\tau_1^h = \frac{k_1}{\mu_1 c_{v1}}, \qquad v_{r1} = \sqrt{\frac{\mu_1}{\rho_1}}. \tag{3c}$$

In (1) and (3) $(\nu_1, \mu_1, \rho_1, \alpha_{v1}, c_{v1}, k_1)$ are, respectively, Poisson's ratio, shear modulus, mass density, coefficient of volumetric thermal expansion, specific heat at constant volume and thermal conductivity. In turn $(\varepsilon_1, h_1, s_{r1}, v_{r1}, \tau_1^h)$ are, respectively, the thermal coupling constant, thermoelastic characteristic length, rotational wave slowness, rotational wave speed, and thermoelastic characteristic time. For half-space $2(y < 0)$ Equation (1)–(3) again hold, except that subscript 1 is replaced by 2 and (1a), (1b) are homogeneous. Data in a number of sources [Chadwick 1960; Achenbach 1973; Davis 1998; Sharma and Sharma 2002] suggests that in both half-spaces, that is, $n = (1, 2)$, we find

$$\begin{aligned} v_{rn} &\approx O(10^3)\ \mathrm{m/s}, & m_n &\geq 2, & \varepsilon_n &\approx O(10^{-2}), \\ h_n &\approx O(10^{-9})\ \mathrm{m}, & (\tau_n^I, \tau_n^{II}) &\approx O(10^{-13})\ \mathrm{s}. \end{aligned} \tag{4}$$

These values indicate in turn that $\tau_n^h \gg \tau_n^I > \tau_n^{II}$.

For $y \neq 0$ the initial $(t \leq 0)$ conditions are

$$(u_{nx}, u_{ny}, \theta_n) \equiv 0, \quad n = (1, 2). \tag{5}$$

For $t > 0$ the interface $(y = 0)$ conditions are

$$\begin{aligned} u_{x1} - u_{x2} &= 0, & u_{y1} - u_{y2} &= 0, & \theta_1 - \theta_2 &= 0 \\ \sigma_{xy1} - \sigma_{xy2} &= 0, & \sigma_{y1} - \sigma_{y2} &= 0, & k_1\frac{\partial \theta_1}{\partial y} - k_2\frac{\partial \theta_2}{\partial y} &= 0. \end{aligned} \tag{6}$$

Equation (1a), (1b) imply for $(y = L, t > 0)$ that

$$\begin{gathered} [u_{x1}] = 0, \qquad [u_{y1}] = 0, \qquad [\theta_1] = 0 \\ \mu_1\left[\frac{\partial u_{x1}}{\partial y}\right] = F_x\delta(x), \qquad \mu_1 a_1\left[\frac{\partial u_{y1}}{\partial y}\right] = F_y\delta(x), \qquad h_1 c_{v1}\left[\frac{\partial \theta_1}{\partial y}\right] = F_T\delta(x). \end{gathered} \tag{7}$$

Here $[F]$ denotes the jump in function F for a given (x, t) as one moves from $y = L-0$ to $y = L+0$. For $t > 0(u_{x1}, u_{y1}, \theta_1)$ and $(u_{x2}, u_{y2}, \theta_2)$ should vanish as $y \to \infty$ and $y \to -\infty$, respectively, and singular behavior may occur at $(x = 0,\ y = L)$. By explicitly imposing (7), homogeneous forms of (1a), (1b) can be addressed in both half-space 1 and 2. Decomposition of these in view of (5) gives for $n = (1, 2)$, $y \neq (0, L)$

$$\nabla^2\big(a_n\Delta_n - \alpha_{vn}D_n^{II}\theta_n\big) - s_{rn}^2\frac{\partial^2\Delta_n}{\partial t^2} = 0, \qquad \Big(\nabla^2 - s_{rn}^2\frac{\partial^2}{\partial t^2}\Big)r_{xyn} = 0 \qquad (t > 0) \tag{8a}$$

$$(\Delta_n, \theta_n, r_{xyn}) \equiv 0 \quad (t \leq 0). \tag{8b}$$

In (8), Equation (2) holds, and r_{xyn} is rotation in plane strain.

3. Transform solution for general problem

Unilateral and bilateral [Sneddon 1972] Laplace transforms over (t, x) are

$$\hat{F}(x) = \int_0^\infty F(x,t)\exp(-pt)dt, \qquad \tilde{F} = \int_{-\infty}^\infty \hat{F}(x)\exp(-pqx)dq. \tag{9}$$

Here p is positive and real, and q is imaginary. Application of (9) to (8) gives eigenfunctions and eigenvalues

$$\exp(\pm pA_n^+ y), \qquad \exp(\pm pA_n^- y), \qquad \exp(\pm pB_n y) \tag{10a}$$

$$A_n^+(q^2) = \sqrt{s_n^{+2} - q^2}, \qquad A_n^-(q^2) = \sqrt{s_n^{-2} - q^2}, \qquad B_n(q^2) = \sqrt{s_{rn}^2 - q^2}. \tag{10b}$$

In (10) the branch points are defined by (3) and for $n = (1, 2)$

$$s_n^\pm = k_n^\pm s_{dn}, \qquad s_{dn} = \frac{s_{rn}}{\sqrt{a_n}} \tag{11a}$$

$$2k_n^\pm = \sqrt{\left(1 + \sqrt{\frac{a_n d_n^I}{\tau_n^h p}}\right)^2 + \frac{\varepsilon_n d_n}{\tau_n^h p}} \pm \sqrt{\left(1 - \sqrt{\frac{a_n d_n^I}{\tau_n^h p}}\right)^2 + \frac{\varepsilon_n d_n}{\tau_n^h p}}. \tag{11b}$$

Here s_{dn} is the isothermal dilatational wave slowness, and from (2), (5) and (9)

$$\begin{aligned} &\mathrm{F}: (d_n, d_n^I) = 1, \\ &\mathrm{I}: (d_n, d_n^I) = 1 + \tau_n^I p, \\ &\mathrm{II}: (d_n, d_n^{II}) = 1 + \tau_n^{II} p, \qquad d_n^I = 1 + \tau_n^I p. \end{aligned} \tag{12}$$

It can be shown in view of (4) for all three models that $k_n^+ > 1 > k_n^- > 0$ and thus $(s_n^+, s_{rn}) > s_n^-$ for positive real p. Inequality $s_n^+ > s_{rn}(k_n^+ > \sqrt{a_n})$ also holds when

$$\begin{aligned}
&\mathrm{F}: p < 1 + \frac{\varepsilon_n}{m_n},\\
&\mathrm{I}: p < \frac{m_n + \varepsilon_n}{m_n \tau_n^h - (m_n + \varepsilon_n)\tau_n^I},\\
&\mathrm{II}: p < \frac{m_n + \varepsilon_n}{m_n(\tau_n^h - \tau_n^I) - \varepsilon_n \tau_n^{II}}.
\end{aligned} \tag{13}$$

Application of (9) to the homogeneous versions of (1a), (1b) in light of (5) and using (10) and (12) gives transforms $(\tilde{u}_{1x}, \tilde{u}_{1y}, \tilde{\theta}_1)$ for $y > 0$, $y \neq L$ and $(\tilde{u}_{2x}, \tilde{u}_{2y}, \tilde{\theta}_2)$ for $y < 0$ as linear combinations of (10a). Operating on (1c), (1d), (6) and (7) with (9) then gives the equations required to find the coefficients of the linear combinations. For present purposes it is sufficient to display results for half-space 2:

$$\begin{bmatrix} \tilde{u}_{2x} \\ \tilde{u}_{2y} \\ \tilde{\theta}_2 \end{bmatrix} = \begin{bmatrix} q & q & 1 \\ A_2^+ & A_2^- & -q \\ \omega_2 \eta_2^+ & \omega_2 \eta_2^- & 0 \end{bmatrix} \begin{bmatrix} C_+ \exp(pA_2^+ y) \\ C_- \exp(pA_2^- y) \\ C_B \exp(pB_2 y) \end{bmatrix} \tag{14a}$$

$$\begin{bmatrix} C_+ \\ C_- \\ C_B \end{bmatrix} = \frac{1}{pS} \begin{bmatrix} M_+^+ & M_-^+ & \omega_1 q M_B^+ \\ M_+^- & M_-^- & \omega_1 q M_B^- \\ q M_+ & q M_- & M_B \end{bmatrix} \begin{bmatrix} F_+ \\ F_- \\ F_B \end{bmatrix}. \tag{14b}$$

For $n = (1, 2)$ in view of (11) and (12),

$$\omega_n = \frac{s_{rn}^2 p}{\alpha_{vn} d_n}, \qquad \eta_n^\pm = 1 - k_n^{\pm 2} \tag{15a}$$

$$\begin{aligned}
\eta_n^+ \eta_n^- &= -\frac{\varepsilon_n d_n}{\tau_n^h p},\\
\eta_n^- - \eta_n^+ = \eta_n &= \sqrt{\left[1 + \frac{1}{\tau_n^h p}(a_n d_n^I + \varepsilon_n d_n)\right]^2 - 4\sqrt{\frac{a_n d_n^I}{\tau_n^h p}}}.
\end{aligned} \tag{15b}$$

For ω_n parameter d_n is defined by

$$\mathrm{I, F}: d_n = 1, \qquad \mathrm{II}: (d_n, d_n^{II}) = 1 + \tau_n^{II} p. \tag{16}$$

In (15b), however, it is defined by (12). Introduction of branch cuts $\mathrm{Im}(q) = 0$, $|\mathrm{Re}(q)| > s_n^\pm$ and $\mathrm{Im}(q) = 0$, $|\mathrm{Re}(q)| > s_{rn}$ such that $\mathrm{Re}(A_n^\pm, B_n) \geq 0$ in the cut q-plane guarantees that (14a) is bounded as $y \to -\infty$ for positive real p. In (14b)

$$F_+ = \left[\omega_1 \eta_1^- (q\hat{F}_x + A_1^- \hat{F}_y) - \frac{\hat{F}_T}{h_1 c_{v1}}\right] \exp(-pA_1^+ L) \tag{17a}$$

$$F_- = \left[\omega_1 \eta_1^+ (q\hat{F}_x + A_1^+ \hat{F}_y) - \frac{\hat{F}_T}{h_1 c_{v1}}\right] \exp(-pA_1^- L) \tag{17b}$$

$$F_B = (q\hat{F}_y - B_1 \hat{F}_x) \exp(-pB_1 L). \tag{17c}$$

The matrix coefficients in (14b) are given by

$$M_+^+ = \frac{\omega_1 \eta_1^+ k_1}{\rho_1} S_{1-}^{2-} - \omega_2 \eta_2^- Q_B (K_1^- + K_2^-), \qquad M_-^+ = \omega_2 \eta_2^- Q_B (K_1^+ + K_2^-) - \frac{\omega_1 \eta_1^- k_1}{\rho_1} S_{1+}^{2-}, \tag{18a}$$

$$M_B^+ = \omega_2 \eta_2^- \left[\eta_1^- Q_1^+ (K_1^- + K_2^-) - \eta_1^+ Q_1^- (K_1^+ + K_2^-)\right] + \frac{\omega_1 \varepsilon_1 d_1}{\tau_1^h \rho_1 p} (K_1^- - K_1^+)\big(T_2 T_{12} - \mu_{12} T_1 K_2^- B_2\big), \tag{18b}$$

$$M_-^- = \frac{\omega_1 \eta_1^- k_1}{\rho_1} S_{1+}^{2+} - \omega_2 \eta_2^+ Q_B (K_1^+ + K_2^+), \qquad M_+^- = \omega_2 \eta_2^+ Q_B (K_1^- + K_2^+) - \frac{\omega_1 \eta_1^+ k_1}{\rho_1} S_{1-}^{2+}, \tag{19a}$$

$$M_B^- = \omega_2 \eta_2^+ \left[\eta_1^+ Q_1^- (K_1^+ + K_2^+) - \eta_1^- Q_1^+ (K_1^- + K_2^+)\right] + \frac{\omega_1 \varepsilon_1 d_1}{\tau_1^h \rho_1 p} (K_1^+ - K_1^-)\big(T_2 T_{12} - \mu_{12} T_1 K_2^+ B_2\big), \tag{19b}$$

$$M_+ = \omega_2 \left[\eta_2^- Q_2^+ (K_1^- + K_2^-) - \eta_2^+ Q_2^- (K_1^- + K_2^+)\right] + \frac{\omega_1 \eta_1^+}{\rho_1} (K_2^+ - K_2^-)\big(T_1 T_{12} - \mu_{12} T_2 K_1^- B_1\big), \tag{20a}$$

$$M_- = \omega_2 \left[\eta_2^+ Q_2^- (K_1^- + K_2^+) - \eta_2^- Q_2^+ (K_1^- + K_2^-)\right] + \frac{\omega_1 \eta_1^-}{\rho_1} (K_2^+ - K_2^-)\big(T_1 T_{12} - \mu_{12} T_2 K_1^+ B_1\big), \tag{20b}$$

$$\begin{aligned} M_B = {} & \omega_1 \omega_2 \left[\eta_1^+ \eta_2^+ (K_1^+ + K_2^+) Q_{1-}^{2-} + \eta_1^- \eta_2^- (K_1^- + K_2^-) Q_{1+}^{2+}\right] \\ & - \omega_1 \omega_2 \left[\eta_1^+ \eta_2^- (K_1^+ + K_2^-) Q_{1-}^{2+} + \eta_1^- \eta_2^+ (K_1^- + K_2^+) Q_{1+}^{2-}\right] \\ & - \left(\rho_1 \omega_2^2 \frac{k_2 \varepsilon_2 d_2}{\tau_2^h p} + \frac{\omega_1^2 k_1 \varepsilon_1 d_1}{\tau_1^h \rho_1 p} T_1 T_2\right)(K_1^+ - K_1^-)(K_2^+ - K_2^-). \end{aligned} \tag{20c}$$

Denominator term S is given by

$$\begin{aligned} S = {} & -Q_B \left(\rho_2 \frac{k_1 \omega_1^2 \varepsilon_1 d_1}{\tau_1^h p} + \rho_1 \frac{k_2 \omega_2^2 \varepsilon_2 d_2}{\tau_2^h p}\right)(A_1^+ - A_1^-)(A_2^+ - A_2^-) \\ & + \omega_1 \omega_2 \left[\eta_1^+ \eta_2^+ (K_1^+ + K_2^+) S_{1-}^{2-} + \eta_1^- \eta_2^- (K_1^- + K_2^-) S_{1+}^{2+}\right] \\ & - \omega_1 \omega_2 \left[\eta_1^+ \eta_2^- (K_1^+ + K_2^-) S_{1-}^{2+} + \eta_1^- \eta_2^+ (K_1^- + K_2^+) S_{1+}^{2-}\right]. \end{aligned} \tag{21}$$

Equation (12) defines d_n in (18)–(21) and in (18)–(20) functions

$$S_{1\pm}^{2\pm} = q^2 Q_1^{\pm} Q_2^{\pm} + Q_B Q_{1\pm}^{2\pm}, \qquad K_\eta^{\pm} = k_\eta A_\eta^{\pm}, \eta = (1, 2), \tag{22a}$$

$$Q_{1\pm}^{2\pm} = T_2 A_1^{\pm} + T_1 A_2^{\pm}, \qquad Q_B(q^2) = T_2 B_1 + T_1 B_2 \tag{22b}$$

$$Q_1^{\pm}(q^2) = T_{12} + \mu_{12} A_1^{\pm} B_2, \quad Q_2^{\pm}(q^2) = T_{12} + \mu_{12} A_2^{\pm} B_1, \tag{22c}$$

$$\mu_{12} = 2(\mu_2 - \mu_1) \tag{22d}$$

$$T_1 = \rho_1 + \mu_{12} q^2, \qquad T_2 = \rho_2 - \mu_{12} q^2, \tag{22e}$$

$$T_{12} = \rho_1 - \rho_2 + \mu_{12} q^2. \tag{22f}$$

If $s_n^{\pm}$ in $A_n^{\pm}$ is replaced by the isothermal dilatational wave slowness s_{dn}, then $(S_{1+}^{2+}, \dots)$ all assume the form of the Stoneley function S_i for isothermal half-spaces [Cagniard 1962]. Thus S is the Stoneley function for the present case, and is now discussed.

4. Stoneley function

For positive real p, S has branch cuts $\mathrm{Im}(q) = 0$, $|\mathrm{Re}(q)| > s_*$, where in view of (13),

$$s_* = \min(s_1^-, s_2^-), \qquad s^* = \max(s_1^+, s_2^+, s_{r1}, s_{r2}). \tag{23}$$

Study of (21) shows that

$$S(q) \approx -2(\omega_1 \eta_1 s_{d1}^2)(\omega_2 n_2 s_{d2}^2) M q^2 \sqrt{0 - q^2}, \quad |q| \to \infty, \tag{24a}$$

$$S(0) = (\rho_2 s_{r1} + \rho_1 s_{r2})\big(M_{12}\omega_1\omega_2 - M_1\omega_1^2 - M_2\omega_2^2\big). \tag{24b}$$

In (24), (M, M_1, M_2, M_{12}) are defined by

$$M = (k_1 + k_2)(\mu_1 + m_2\mu_2)(\mu_2 + m_1\mu_1), \tag{25a}$$

$$(M_1, M_2) = \left(\rho_2 \frac{k_1 \varepsilon_1 d_1}{\tau_1^h p}, \rho_1 \frac{k_2 \varepsilon_2 d_2}{\tau_2^h p}\right)(s_1^+ - s_1^-)(s_2^+ - s_2^-), \tag{25b}$$

$$\begin{aligned} M_{12} = {} & \eta_1^+ \eta_2^+ (k_1 s_1^+ + k_2 s_2^+)(\rho_2 s_1^- + \rho_1 s_2^-) + \eta_1^- \eta_2^- (k_1 s_1^- + k_2 s_2^-)(\rho_2 s_1^+ + \rho_1 s_2^+) \\ & - \eta_1^+ \eta_2^- (k_1 s_1^+ + k_2 s_2^-)(\rho_2 s_1^- + \rho_1 s_2^+) - \eta_1^- \eta_2^+ (k_1 s_1^- + k_2 s_2^+)(\rho_2 s_1^+ + \rho_1 s_2^-). \end{aligned} \tag{25c}$$

Equation (12) holds in (25b), and in view of (15) quantities $(M, M_1, M_2, M_{12}) > 0$ for positive real p. Study of (25a) shows for the isothermal case that $S_i(0) > 0$, and that this guarantees roots $q = \pm s_0^i$, $s_0^i > s_r^* = \max(s_{r1}, s_{r2})$ for S_i whenever $S_i(\pm s_r^*) < 0$. As noted in Appendix A, the sign of $S(0)$ depends on parameter P_- defined by (A3) and the dimensionless ratio ω_1/ω_2. In addition (22) and (25) show that S is real-valued at $q = \pm s_*$ but pure imaginary for $q = \pm s^*$ and $|q| \to \infty$, $\mathrm{Im}(q) = \pm 0$, respectively. The signs of the imaginary values depend on the side of the branch cut. Study of (21), (24), (25), these observations, and argument theory [Hille 1959] applied in the manner of [Brock 1997b] show that three cases arise.

$$\text{Case A}: \; S(0) > 0, \qquad \frac{S(s^* \pm i0)}{S(|q| \pm i0)}, \frac{S(-s^* \pm i0)}{S(-|q| \pm i0)} \longrightarrow -0, \quad |q| \to \infty, \tag{26a}$$

$$\text{Case B}: \; S(0) > 0, \qquad \frac{S(s^* \pm i0)}{S(|q| \pm i0)}, \frac{S(-s^* \pm i0)}{S(-|q| \pm i0)} \longrightarrow +0, \quad |q| \to \infty, \tag{26b}$$

$$\text{Case C}: \; S(0) < 0. \tag{26c}$$

For Case A, S exhibits roots $q = \pm s_0$, $s_0 > 0$. For Case B no roots arise in the cut q-plane. For Case C, S exhibits roots $q = \pm i\tau_0$, $\tau_0 > 0$.

Following [Norris and Achenbach 1984] and [Brock 1998] an expression for s_0 that is analytic to within a single integration is obtained. We introduce function

$$G(q) = \frac{S(q)}{C^* \omega_1 \omega_2 M(\eta_1 s_{d1}^2)(\eta_2 s_{d2}^2)} \frac{1}{s_0^2 - q^2}, \qquad C^* = \sqrt{{s^*}^2 - q^2}. \tag{27}$$

It has branch cuts $\operatorname{Im}(q) = 0$, $s_* < |\operatorname{Re}(q)| < s^*$, approaches unity as $|q| \to \infty$, and has no roots or zeros in the cut q-plane. After [Noble 1958], it factors as the product of functions $G_\pm$ that are analytic in the overlapping strips $\operatorname{Re}(q) > -s_*$ and $\operatorname{Re}(q) < s_*$, respectively. These are given by

$$\ln G_\pm(q) = \frac{1}{\pi} \int_{s_*}^{s^*} \tan^{-1} \frac{\operatorname{Im} S(u + i0)}{\operatorname{Re} S(u + i0)} \frac{du}{u \pm q}. \tag{28}$$

Setting $G = G_+ G_-$ in (27) and evaluating it at $q = 0$ gives the formula

$$s_0 = \frac{1}{G_\pm(0)} \sqrt{\frac{\rho_2 s_{r1} + \rho_1 s_{r2}}{s^* M(\eta_1 s_{d1}^2)(\eta_2 s_{d2}^2)}} \sqrt{M_{12} - M_1 \frac{\omega_1}{\omega_2} - M_2 \frac{\omega_2}{\omega_1}}. \tag{29}$$

Replacing s_0^2 by the term $-\tau_0^2$ in (27) gives (28) again, but (29) is replaced by

$$\tau_0 = \frac{1}{G_\pm(0)} \sqrt{\frac{\rho_2 s_{r1} + \rho_1 s_{r2}}{s^* M(\eta_1 s_{d1}^2)(\eta_2 s_{d2}^2)}} \sqrt{M_1 \frac{\omega_1}{\omega_2} + M_2 \frac{\omega_2}{\omega_1} - M_{12}}. \tag{30}$$

Formula (28) shows that both G_+ and G_- are analytic at $q = \pm(s_* - 0)$ and $q = \pm(s^* + 0)$. Thus setting $G = G_+ G_-$ in (27) and evaluating at these locations shows by way of a check that $S(0)$ and $S(\pm s_*)$ have the same sign, and that the limit in (26b) is achieved whenever $S(0) < 0$. Because Case A and B are analogous to the isothermal problem, the results obtained so far are used to study Stoneley effects in interface temperatures for these cases. For simplicity, the limit problem of interface line loads ($L = 0$) is considered.

5. Interface temperature change when $L = 0$

When (F_x, F_y, F_T) act on the interface $y = 0$ itself $(L = 0)$, (14) and (16) give the transform of the temperature change on the interface

$$(\tilde{\theta}_1, \tilde{\theta}_2) = \tilde{\theta}_{12} = \frac{\omega_1\omega_2}{S}\left(qM_x\frac{\hat{F}_x}{p} + M_y\frac{\hat{F}_y}{p} + M_T\frac{\hat{F}_T}{\rho_1 p}\right). \tag{31}$$

In (49) the coefficients

$$M_x = \omega_2\eta_1^+\eta_1^-(K_1^+ - K_1^-)\big[\eta_2^-(Q_B - Q_2^+B_2) - \eta_2^+(Q_B - Q_2^-B_2)\big] \\ + \omega_1\eta_2^+\eta_2^-(K_2^+ - K_2^-)\big[\eta_1^-(Q_B + Q_1^+B_1) - \eta_1^+(Q_B + Q_1^-B_1)\big] \tag{32a}$$

$$M_y = \omega_2\eta_1^+\eta_1^-(K_1^+ - K_1^-)\big[\eta_2^+(q^2Q_2^- + Q_BA_2^-) - \eta_2^-(q^2Q_2^+ + Q_BA_2^+)\big] \\ + \omega_1\eta_2^+\eta_2^-(K_2^+ - K_2^-)\big[\eta_1^+(q^2Q_1^- - Q_BA_1^-) - \eta_1^-(q^2Q_1^+ - Q_BA_1^+)\big] \tag{32b}$$

$$M_T = \eta_1^+\eta_2^-S_{1-}^{2+} + \eta_1^-\eta_2^+S_{1+}^{2-} - \eta_1^+\eta_2^+S_{1-}^{2-} - \eta_1^-\eta_2^-S_{1+}^{2+}. \tag{32c}$$

The inverse of the bilateral Laplace transform [Sneddon 1972] in (9) can be written as

$$\hat{F}(x) = \frac{p}{2\pi i}\int \tilde{F}\exp(pqx)\,dq. \tag{33}$$

Integration is over a Bromwich contour which, for Case A, can be taken as the entire Im(q)-axis. However, (24a) and (32) show that

$$S \approx O(q^2\sqrt{-q^2}), \quad M_x \approx O(1), \tag{34}$$

$$M_y \approx O(\sqrt{-q^2}), \quad M_T \approx O(q^2), \quad |q| \to \infty. \tag{35}$$

Therefore, substitution of (31) in (33) gives integrands that vanish as $|q| \to \infty$ for all $x(M_x, M_y)$ and $x \neq 0(M_T)$. The (M_x, M_T)-contribution can then by Cauchy theory be obtained as principal value integrals about segment $\mathrm{Im}(q) = 0, \mathrm{Re}(q) < -s_*(x > 0)$ or $\mathrm{Im}(q) = 0, \mathrm{Re}(q) > s_*(x < 0)$. Similarly, the M_y-contribution becomes an integral about segment $\mathrm{Im}(q) = 0, -s^* < \mathrm{Re}(q) < -s_*(x > 0)$ or $\mathrm{Im}(q) = 0, s_* < \mathrm{Re}(q) < s^*(x < 0)$ and the pole residue

$$\hat{\theta}_{12}^S = \frac{\hat{F}_y}{2s_0p}\frac{v_{d1}^2v_{d2}^2N_y\exp(-ps_0|x|)}{\eta_1\eta_2MG_0\sqrt{s_0^2 - s^{*2}}} \tag{36a}$$

$$\ln G_0 = \frac{2}{\pi}\int_{s_*}^{s^*}\tan^{-1}\frac{\mathrm{Im}\,S(u+i0)}{\mathrm{Re}\,S(u+i0)}\frac{u\,du}{u^2 - s_0^2} \tag{36b}$$

$$N_y = \frac{\varepsilon_1 d_1}{\tau_1^h}(\kappa_1^+ - \kappa_1^-)\omega_2\big[\eta_2^-(s_0^2 T_2^+ - \alpha_2^+ T_\beta) - \eta_2^+(s_0^2 T_2^- - \alpha_2^- T_\beta)\big] + \frac{\varepsilon_2 d_2}{\tau_2^h}(\kappa_2^+ - \kappa_2^-)\omega_1\big[\eta_1^-(s_0^2 T_1^+ + \alpha_1^+ T_\beta) - \eta_1^+(s_0^2 T_1^- + \alpha_1^- T_\beta)\big]. \tag{36c}$$

Equation (12) governs d_n in (36c), and

$$(T_1^\pm, T_2^\pm) = \rho_1 - \rho_2 + \mu_{12}s_0^2 - \mu_{12}(\alpha_1^\pm \beta_2, \alpha_2^\pm \beta_1) \tag{37a}$$

$$T_\beta = (\rho_2 - \mu_{12}s_0^2)\beta_1 + (\rho_1 + \mu_{12}s_0^2)\beta_2 \tag{37b}$$

$$\alpha_n^\pm = \sqrt{s_0^2 - s_n^{\pm 2}}, \qquad \kappa_n^\pm = k_n \alpha_n^\pm, \qquad \beta_n = \sqrt{s_0^2 - s_{rn}^2}, \qquad n = (1, 2). \tag{37c}$$

Study of (36a) in view of (10a), (11), (12), (20)–(24) and (37) shows that $\hat{\theta}_{12}^S$ appropriately vanishes when the half-space materials are the same. For Case B a term such as (36a) does not arise. Inversion of (36a) is now sought for Case A for the three models. To allow more insight into behavior, analytical results are achieved with asymptotic versions of the transforms that are valid for very long or very short times after the line loads are applied.

6. Inversion for long times

A robust asymptotic result for long times, here defined for all three models as

$$t \gg \max(\tau_1^h, \tau_2^h) \tag{38}$$

is obtained by inverting an approximate transform valid for $\max(\tau_1^h p, \tau_2^h p) \ll 1$. It is noted that all D_n-operators (and thus corresponding d_n-factors) become unity, that is, all three models behave as Fourier model F. For $n = (1, 2)$ Equation (11)–(13) yield

$$k_n^+ \approx \sqrt{\frac{a_n^\varepsilon}{\tau_n^h p}}, \qquad \eta_n^+ \approx -\frac{a_n^\varepsilon}{\tau_n^h p}, \qquad k_n^- \approx \sqrt{\frac{a_n^\varepsilon}{a_n}}, \qquad \eta_n^- \approx \frac{\varepsilon_n}{a_n}, \qquad a_n^\varepsilon = a_n + \varepsilon_n \tag{39a}$$

$$s_n^+ \approx \frac{\lambda_n^\varepsilon}{\sqrt{p}}, \qquad s_n^- \approx \frac{s_{rn}}{\sqrt{a_n^\varepsilon}} = s_n^\varepsilon = \frac{1}{v_n^\varepsilon}, \qquad \omega_n \approx \frac{s_{rn}^2 p}{\alpha_{vn}}, \qquad \lambda_n^\varepsilon = \frac{a_n^\varepsilon s_{rn}}{a_n h_n}. \tag{39b}$$

In light of (11) and (39), $s_n^+ \gg s_{rn} > s_n^-$ and it is noted that $(v_n^\varepsilon, s_n^\varepsilon)$ are the thermoelastic dilatational wave speed and slowness [Brock and Georgiadis 1999]. For purposes of illustration we choose materials so that, in view of (39),

$$s_1^- < s_2^- < s_{r1} < s_{r2} \ll s_1^+ < s_2^+. \tag{40}$$

From Appendix A and (24) it can be shown that requirements for Case A are met if

$$s_{r2}^2(\rho_1 + \rho_2 - 2\mu_1 s_{r2}^2)^2 - \beta_1\big[(\rho_2 - 2\mu_1 s_{r2}^2)^2 \alpha_1 + \rho_1 \rho_2 \alpha_2\big] > 0 \tag{41a}$$

$$\alpha_n = \sqrt{s_{r2}^2 - s_n^{\varepsilon 2}}, \qquad n = (1, 2), \qquad \beta_1 = \sqrt{s_{r2}^2 - s_{r1}^2}. \tag{41b}$$

If (41b) does hold then (29), (36a) and a standard table [Sneddon 1972] give

$$s_0 \approx \sqrt{\rho_2 s_1^\varepsilon + \rho_1 s_2^\varepsilon}\sqrt{\rho_2 s_{r1} + \rho_1 s_{r2}}\,\frac{\sqrt{k_1\lambda_1^\epsilon + k_2\lambda_2^\epsilon}}{\sqrt{\lambda_1^\epsilon \lambda_2^\epsilon}}\,\frac{\exp(\Psi_F(0))}{s_2^\varepsilon\sqrt{M s_1^\varepsilon s_{r2}}} > s_{r2} \tag{42a}$$

$$\theta_{12}^S \approx \frac{N_F}{2M\sqrt{s_0^2 - s_{r2}^2}} \exp(-2\Psi_F(s_0)) F_y(t - s_0|x|) H(t - s_0|x|) \tag{42b}$$

$$N_F = \frac{k_1 \varepsilon_1 a_2 v_1^{\varepsilon 2}}{\tau_1^h \alpha_{v2}}\left(1 - \frac{\alpha_2^0}{s_0}\right)\left(s_0^2 T_1^0 + \alpha_1^0 T_\beta^0\right) + \frac{k_2 \varepsilon_2 a_1^\varepsilon v_2^{\varepsilon 2}}{\tau_2^h \alpha_{v1}}\left(1 - \frac{\alpha_1^0}{s_0}\right)\left(s_0^2 T_2^0 - \alpha_2^0 T_\beta^0\right). \tag{42c}$$

Here H is the Heaviside function, function Ψ_F is defined by (B1) in Appendix B and

$$(T_1^0, T_2^0) = \rho_1 - \rho_2 + \mu_{12} s_0^2 - \mu_{12}(\alpha_1^0 \beta_2^0, \alpha_2^0 \beta_1^0) \tag{43a}$$

$$T_\beta^0 = (\rho_2 - \mu_{12} s_0^2)\beta_1^0 + (\rho_1 + \mu_{12} s_0^2)\beta_2^0 \tag{43b}$$

$$\alpha_n^0 = \sqrt{s_0^2 - s_n^{\varepsilon 2}}, \qquad \beta_n^0 = \sqrt{s_0^2 - s_{rn}^2}, \quad n = (1, 2). \tag{43c}$$

7. Inversion for short times: model F

The short time range for Fourier model F is defined as

$$t \ll \min(\tau_1^h, \tau_2^h). \tag{44}$$

A robust asymptotic result can therefore be obtained from a transform approximation valid for

$$\min(\tau_1^h p, \tau_2^h p) \gg 1.$$

It can be shown that for $n = (1, 2)$

$$k_n^+ \approx 1, \qquad \eta_n^+ \approx \frac{-\varepsilon_n}{\tau_n^h p}, \qquad k_n^- \approx \sqrt{\frac{a_n}{\tau_n^h p}}, \qquad \eta_n^- \approx 1 \tag{45a}$$

$$s_n^+ \approx s_{dn}, \qquad s_n^- \approx \frac{\lambda_n}{\sqrt{p}}, \qquad \omega_n = \frac{s_{rn}^2 p}{\alpha_{vn}}, \qquad \lambda_n = \frac{s_{rn}}{h_n}. \tag{45b}$$

From (11) and (45) it follows that now $s_{rn} > s_n^+ \gg s_n^-$. For purposes of illustration the materials are chosen such that

$$s_1^- < s_2^- \ll s_1^+ < s_2^+ < s_{r1} < s_{r2}. \tag{46}$$

From Appendix A and (21) it can be shown that conditions for Case A are met if (38) is satisfied, but with (α_1, α_2) in (41b) replaced by

$$\alpha_n = \sqrt{s_{r2}^2 - s_{dn}^2}, \quad n = (1, 2). \tag{47}$$

If (41a) and (47) do hold then it can be shown that

$$s_0 \approx \sqrt{\rho_2 s_{d1} + \rho_1 s_{d2}}\sqrt{\rho_2 s_{r1} + \rho_1 s_{r2}}\,\frac{\sqrt{k_1\lambda_1 + k_2\lambda_2}}{\sqrt{k_1 k_2}}\,\frac{\exp(\Psi_F(0))}{s_{d2}\sqrt{s_{d1}s_{r2}M}} > s_{r2}, \tag{48a}$$

$$\theta_{12}^S \approx \frac{N_F}{2M\sqrt{s_0^2 - s_{r2}^2}}\exp(-2\Psi_F(s_0))F_y(t - s_0|x|)H(t - s_0|x|), \tag{48b}$$

$$N_F = \frac{k_1\varepsilon_1 a_2 v_{d1}^2}{\tau_1^h \alpha_{v2}}\left(1 - \frac{\alpha_2^0}{s_0}\right)(s_0^2 T_1^0 + \alpha_1^0 T_\beta^0) + \frac{k_2\varepsilon_2 a_1 v_{d2}^2}{\tau_2^h \alpha_{v1}}\left(1 - \frac{\alpha_1^0}{s_0}\right)(s_0^2 T_2^0 - \alpha_2^0 T_\beta^0). \tag{48c}$$

Function Ψ_F is now given by (B3) in Appendix B.

8. Inversion for short times: model I

For the single-relaxation time model, valid results are obtained for

$$t \ll \min(\tau_1^I, \tau_2^I) \tag{49}$$

with approximate transforms valid for $\max(\tau_1^I p, \tau_2^I p) \gg 1$. Then for $n = (1, 2)$

$$2k_n^{\pm} \approx \sqrt{\left(1 + \sqrt{a_n l_n^I}\right)^2 + \varepsilon_n l_n^I} \pm \sqrt{\left(1 - \sqrt{a_n l_n^I}\right)^2 + \varepsilon_n l_n^I}, \qquad l_n^I = \frac{\tau_n^I}{\tau_n^h} \ll 1 \tag{50a}$$

$$\omega_n \approx \frac{s_{rn}^2 p}{\alpha_{nv}}, \qquad \eta_n^+ \eta_n^- \approx -\varepsilon_n l_n^I. \tag{50b}$$

It is noted that l_n^I is a dimensionless ratio of characteristic times. In light of (13) inequality $s_{rn} > s_n^+$ holds, and one can again consider the situation (46). However, each s-parameter is now a constant, that is, wave slowness, so that a difference of scale between s_n^+ and s_n^- would be due to material mismatch. Use of Appendix A, (24) and (50) shows that Case A arises only if

$$z_- < \frac{s_{r1}^2 \alpha_{v2}}{s_{r2}^2 \alpha_{v1}} < z_+, \qquad M_I < 0. \tag{51}$$

Parameters $z_\pm$ are given by (A6) in Appendix A, with (50) understood and

$$(M_1, M_2) \approx (k_1\varepsilon_1 l_1^I \rho_2, k_2\varepsilon_2 l_2^I \rho_1)(s_1^+ - s_1^-)(s_2^+ - s_2^-). \tag{52}$$

Parameter M_I is defined as

$$M_I = \eta_1^+\eta_2^+(\kappa_1^+ + \kappa_2^+)M_{1-}^{2-} + \eta_1^-\eta_2^-(\kappa_1^- + \kappa_2^-)M_{1+}^{2+} - \eta_1^+\eta_2^-(\kappa_1^+ + \kappa_2^-)M_{1-}^{2+} - \eta_1^-\eta_2^+(\kappa_1^- + \kappa_2^+)M_{1+}^{2-}, \tag{53a}$$

$$M_{1\pm}^{2\pm} = s_{r2}^2 T_{12}^2 - (T_2^2 \alpha_1^{\pm} + \rho_1\rho_2\alpha_2^{\pm}), \tag{53b}$$

$$\alpha_n^{\pm} = \sqrt{s_{r2}^2 - s_n^{\pm 2}}, \qquad \kappa_\eta^{\pm} = \kappa_\eta \alpha_{\eta_1}^{\pm}, \qquad n = (1, 2), \qquad \beta_1 = \sqrt{s_{r2}^2 - s_{r1}^2}. \tag{53c}$$

Here (10b) and (22e) govern with argument u^2. For Case A (29) is valid, with

$$s^* = s_{r2}, \qquad G_\pm(0) \approx \exp \Psi_I(0). \tag{54}$$

Inversion of (36a) then produces in light of (37)

$$\theta_{12}^S \approx \frac{1}{2s_0} \frac{N_I \exp(-2\Psi_I(s_0))}{M\eta_1\eta_2\sqrt{s_0^2 - s_{r2}^2}} \dot{F}_y(t - s_0|x|)H(t - s_0|x|), \tag{55a}$$

$$N_I = \varepsilon_1 l_1^I(\kappa_1^+ - \kappa_1^-)\frac{a_2 v_{d1}^2}{\alpha_{v2}}\big[\eta_2^-(s_0^2 T_2^+ - \alpha_2^+ T_\beta) - \eta_2^+(s_0^2 T_2^- - \alpha_2^- T_\beta)\big]$$
$$+ \varepsilon_2 l_2^I(\kappa_2^+ - \kappa_2^-)\frac{a_1 v_{d2}^2}{\alpha_{v1}}\big[\eta_1^-(s_0^2 T_1^+ + \alpha_1^+ T_\beta) - \eta_1^+(s_0^2 T_1^- + \alpha_1^- T_\beta)\big]. \tag{55b}$$

The superposed dot signifies time differentiation; Ψ_I is defined by (C1) in Appendix C.

9. Inversion for short times: model II

For the double-relaxation time model, valid results for

$$t < \min(\tau_1^{II}, \tau_2^{II}) \tag{56}$$

are obtained by examining approximate transforms valid for $\min(\tau_1^{II} p, \tau_2^{II} p) \gg 1$. For $n = (1, 2)$ asymptotic results are

$$2k_n^\pm \approx \sqrt{\left(1 + \sqrt{a_n l_n^I}\right)^2 + \varepsilon_n l_n^{II}} \pm \sqrt{\left(1 - \sqrt{a_n l_n^I}\right)^2 + \varepsilon_n l_n^{II}},$$
$$l_n^{II} = \frac{\tau_n^{II}}{\tau_n^h} < l_n^I \ll 1, \tag{57a}$$

$$\omega_n \approx \frac{s_{rn}^2}{\alpha_{vn}\tau_n^{II}}, \qquad \eta_n^+\eta_n^- \approx -\varepsilon_n l_n^{II}. \tag{57b}$$

As with model I each s-parameter is wave slowness, and situation (46) can again be considered, with the understanding that any difference in scale is due to material mismatch. Use of Appendix A, (24) and (57) shows that Case A arises only when

$$z_- < \frac{s_{r1}^2\alpha_{v2}\tau_2^{II}}{s_{r2}^2\alpha_{v1}\tau_1^{II}} < z_+, \qquad M_{II} < 0. \tag{58}$$

Again (A6) in Appendix A holds, but now

$$(M_1, M_2) \approx \left(\kappa_1\varepsilon_1 l_1^{II}\rho_2, \kappa_2\varepsilon_2 l_2^{II}\rho_1\right)(s_1^+ - s_1^-)(s_2^+ - s_2^-), \tag{59a}$$

$$M_{II} = M_I - (\rho_2 - \mu_{12}s_{r2}^2)\beta_1(\kappa_1^+ - \kappa_1^-)(\kappa_2^+ - \kappa_2^-)\Omega_{II}, \tag{59b}$$

$$\Omega_{II} = \kappa_1\mu_1\varepsilon_2 s_{r2}^2 l_2^{II}\frac{\alpha_{v1}}{\alpha_{v2}} + \kappa_2\mu_2\varepsilon_1 s_{r1}^2 l_1^{II}\frac{\alpha_{v2}}{\alpha_{v1}}. \tag{59c}$$

It is understood that (57) now holds for all quantities, including M_I. If (58) is satisfied then (29) holds, with

$$s^* = s_{r2}, \qquad G_0 \approx \exp \Psi_{II}(0). \tag{60}$$

Inversion of (36a) then gives

$$\theta_{12}^S \approx \frac{1}{2s_0} \frac{N_{II} \exp(-2\Psi_{II}(s_0))}{\eta_1\eta_2\sqrt{s_0^2 - s_{r2}^2}} F_y(t - s_0|x|) H(t - s_0|x|), \tag{61a}$$

$$N_{II} = \varepsilon_1 l_1^{II}(\kappa_1^+ - \kappa_1^-) \frac{s_{r2}^2}{\alpha_{v2}\tau_2^{II}} \big[\eta_2^- \big(s_0^2 T_2^+ - \alpha_2^+ T_\beta\big) - \eta_2^+ \big(s_0^2 T_2^- - \alpha_2^- T_\beta\big)\big] \\ + \varepsilon_2 l_2^{II}(\kappa_2^+ - \kappa_2^-) \frac{s_{r1}^2}{\alpha_{v1}\tau_1^{II}} \big[\eta_1^- \big(s_0^2 T_1^+ + \alpha_1^+ T_\beta\big) - \eta_1^+ \big(s_0^2 T_1^- + \alpha_1^- T_\beta\big)\big]. \tag{61b}$$

Here (57) governs and function Ψ_{II} is defined in Appendix D.

10. Some observations

Equation (21) shows that a Stoneley function arises in transform space in a dynamic study of perfectly bonded thermoelastic half-spaces. The function includes a linear combination of four terms, each (22a) of which has the form of an isothermal Stoneley function. Condition (26) for existence of thermoelastic Stoneley roots is similar to those for the isothermal case, but more restrictive. Expressions (29) and (30) for the roots, analytic to within a single integration, may depend on the unilateral Laplace (time) transform variable p, that is, not correspond to, as in the isothermal case, a constant Stoneley wave slowness. Moreover, a root can for positive real p be real (29) or imaginary (30).

It is found that a line load force applied directly to the interface and acting normal to it produces, from the residue of the real root, contribution (36a) to the time transform of the interface temperature change. The contribution has an analytical form, and asymptotic versions of this, valid for long times or short times after the line load is applied, can be inverted analytically.

Inversion (42b) shows that the residue contribution behaves for long times as if the half-spaces obey classical Fourier theory [Chadwick 1960] even when thermal relaxation [Lord and Shulman 1967; Green and Lindsay 1972] is present. Conditions for existence of the Stoneley root (in asymptotic form) are always met, and the root (42a) is a constant. As a result, (42b) describes a temperature change wave.

For short times, a constant real root (48a), (29) and (54), and (29) and (60) arises for, respectively, the Fourier and single- and double-relaxation time models, and the contribution of the residue to the interface temperature change for each model again defines a wave. However, existence conditions (51) and (58) for the relaxation time models are more restrictive than condition (41a) and (47) for the Fourier model. Moreover, contribution (48b) and (61a) for the Fourier and double-relaxation time models are proportional to line load function F_y. Contribution (55a) for the single-relaxation time model is proportional to the time derivative of F_y.

The observation that $\tau_n^h \gg \tau_n^I > \tau_n^{II}$, $n = (1, 2)$ made in connection with (4) shows in view of (38), (44), (49) and (56) that asymptotic result (42a) and (42b) are the most robust. Nevertheless, work in fluids [Fan and Lu 2002] shows that behavior for very short times after a load is applied can be distinctive.

As noted just above, this is the case here. Specifically, if F_y is a step (Heaviside) function in time, the Stoneley contribution to interface temperature for long times is a propagating step function whose form is the same for all three models. For short times, the contribution for the Fourier (F) and double-relaxation (II) time models are propagating step functions that are not identical, while the single-relaxation (I) time model gives a propagating impulse.

In summary, the present analysis shows the sensitivity of Stoneley signals in perfectly bonded thermoelastic half-spaces to the nature of the heat conduction model that governs. It is hoped that the results given here may prove useful in the transient study of solids that consist of dissimilar thermoelastic materials.

Appendix A

The sign of $S(0)$ in (24b) is determined by the second factor on its right-hand side. Equation (15) indicates that (ω_1, ω_2) for positive real p is positive, so that this factor can be studied in terms of the quadratic

$$M_{12}z - M_1 z^2 - M_2, \qquad z = \frac{\omega_1}{\omega_2} > 0. \tag{A1}$$

Its discriminant and the location of its maximum value are

$$M_{12}^2 - 4M_1 M_2, \qquad z = \frac{M_{12}}{2M_1}, \qquad (M_1, M_2, M_{12}) > 0. \tag{A2}$$

The former can be factored as

$$(k_1^+ - k_1^-)^2 (k_2^+ - k_2^-)^2 P_+ P_-, \qquad P_\pm = C_1 \rho_1 s_{d2} + C_2 \rho_2 s_{d1} \pm 2C_3 \sqrt{\rho_1 s_{d2} \rho_2 s_{d1}}. \tag{A3}$$

Term $P_\pm$ is quadratic in $(\sqrt{\rho_1 s_{d2}}, \sqrt{\rho_2 s_{d1}})$ and (C_1, C_2, C_3) are quadratic in $(\sqrt{k_1 s_{d1}}, \sqrt{k_2 s_{d2}})$:

$$\begin{aligned} C_1 &= C_{11} k_1 s_{d1} + C_{12} k_2 s_{d2}, \\ C_2 &= C_{21} k_1 s_{d1} + C_{22} k_2 s_{d2}, \end{aligned} \tag{A4a}$$

$$\begin{aligned} C_{11} &= (k_1^{+2} + k_1^+ k_1^- + k_1^{-2})(1 + k_2^+ k_2^-), \\ C_{12} &= k_2^+ k_2^- (k_1^+ + k_1^-)(k_2^+ + k_2^-), \end{aligned} \tag{A4b}$$

$$\begin{aligned} C_{22} &= (k_2^{+2} + k_2^+ k_2^- + k_2^{-2})(1 + k_1^+ k_1^-), \\ C_{21} &= k_1^+ k_2^- (k_1^+ + k_1^-)(k_2^+ + k_2^-), \end{aligned} \tag{A4c}$$

$$C_3 = \frac{1}{p} \sqrt{d_1 \epsilon_1 d_2 \epsilon_2} \sqrt{\frac{k_1 \varepsilon_1 s_{d1}}{\tau_1^h}} \sqrt{\frac{k_2 \varepsilon_2 s_{d2}}{\tau_2^h}}. \tag{A4d}$$

Equation (12) holds in (A4d) and because $k_n^+ > 1 > k_n^- > 0$, $n = (1, 2)$, terms $(C_1, C_2, C_3, P_+) > 0$. Therefore if $P_- > 0$ the quadratic in (A2) has a positive maximum and two positive real roots. If $P_- < 0$ the quadratic in (A2) is itself negative for all $\omega_1/\omega_2 > 0$. It follows that

$$P_- > 0 : S(0) > 0\big(z_- < \frac{\omega_1}{\omega_2} < z_+\big), \qquad S(0) < 0\Big(0 < \frac{\omega_1}{\omega_2} < z_-, \frac{\omega_1}{\omega_2} > z_+\Big), \tag{A5a}$$

$$P_- < 0 : S(0) < 0\Big(\frac{\omega_1}{\omega_2} > 0\Big). \tag{A5b}$$

In (A5a) the terms $z_\pm$ are given by

$$z_\pm = \frac{1}{2M_1}\Big(M_{12} \pm \sqrt{M_{12}^2 - 4M_1M_2}\Big). \tag{A6}$$

Study of P_- is aided by several observations: its discriminant is

$$-C_{11}C_{21}k_1^2s_{d1}^2 - C_{22}C_{12}k_2^2s_{d2}^2 + \bigg[\frac{2}{p}\sqrt{\frac{\varepsilon_1 d_1 \varepsilon_2 d_2}{\tau_1^h\tau_2^h}} - C_{11}C_{22} - C_{12}C_{21}\bigg]k_1s_{d1}k_2s_{d2}. \tag{A7}$$

This quadratic in turn has discriminant

$$D_+D_-, \qquad D_\pm = \frac{\varepsilon_1 d_1 \varepsilon_2 d_2}{\tau_1^h\tau_2^h p^2} - \frac{1}{2}\Big(\sqrt{C_{11}C_{22}} \pm \sqrt{C_{12}C_{22}}\Big)^2. \tag{A8}$$

The first term in $D_\pm$ can be written in light of (15) as

$$(1-k_1^{+2})(1-k_1^{-2})(1-k_2^{+2})(1-k_2^{-2}). \tag{A9}$$

Thus if $(k_1^\pm, k_2^\pm)$ have values for positive real p such that $D_+D_- < 0$, then (A7) is negative in (k_1s_{d1}, k_2s_{d2}), and $P_- > 0$ in $(\sqrt{\rho_1 s_{d2}}, \sqrt{\rho_2 s_{d1}})$. If $D_+D_- > 0$ however, (A7) exhibits $(k_1^\pm, k_2^\pm)$-dependent roots in on the s_{d1}/s_{d2}-axis and its sign depends on (k_1s_{d1}, k_2s_{d2}). Then, when it is positive the sign of P_- depends on $(\sqrt{\rho_1 s_{d2}}, \sqrt{\rho_2 s_{d1}})$.

Appendix B

Function Ψ_F that appears in (47) is defined as

$$\ln \Psi_F(q) = \frac{1}{\pi}\bigg(\int_{s_1^\varepsilon}^{s_2^\varepsilon} \frac{\psi_1 u\,du}{u^2-q^2} + \int_{s_2^\varepsilon}^{s_{r1}} \frac{\psi_2 u\,du}{u^2-q^2} + \int_{s_{r1}}^{s_{r2}} \frac{\psi_3 u\,du}{u^2-q^2}\bigg), \tag{B1a}$$

$$\psi_1 = \tan^{-1}\frac{1}{\alpha_1}\frac{(\rho_1\rho_2B_1 + T_1^2B_2)A_2 + u^2T_{12}^2}{\rho_1\rho_2B_2 + (T_2^2 + \mu_{12}^2A_2B_2)B_1}, \tag{B1b}$$

$$\psi_2 = \tan^{-1} u^2 \frac{T_{12}^2 - \mu_{12}^2\alpha_1B_1\alpha_2B_2}{T_1^2\alpha_2B_2 + T_2^2\alpha_1B_1 + \rho_1\rho_2(\alpha_1B_2 + \alpha_2B_1)}, \tag{B1c}$$

$$\psi_3 = \tan^{-1}\frac{1}{B_2}\frac{u^2T_{12}^2 - (T_2^2\alpha_1 + \rho_1\rho_2\alpha_2)\beta_1}{(T_1^2 - \mu_{12}^2u^2\alpha_1\beta_1)\alpha_2 + \rho_1\rho_2\alpha_1}. \tag{B1d}$$

Here (10b), (22e) and (45) hold, with argument u^2, and

$$\alpha_n = \sqrt{u^2 - s_n^{\varepsilon 2}}, \quad n = (1,2), \qquad \beta_1 = \sqrt{u^2 - s_{r1}^2}. \tag{B2}$$

In Equation (48a) and (48b) function Ψ_F is given by

$$\Psi_F(q) = \frac{1}{\pi}\left(\int_{s_{d1}}^{s_{d2}} \frac{\psi_1 u du}{u^2 - q^2} + \int_{s_{d2}}^{s_{r1}} \frac{\psi_2 u du}{u^2 - q^2} + \int_{s_{r1}}^{s_{r2}} \frac{\psi_3 u du}{u^2 - q^2}\right). \tag{B3}$$

Equation (B1) and (B2) again hold but with modification

$$\alpha_n = \sqrt{u^2 - s_{dn}^2}, \quad n = (1, 2). \tag{B4}$$

Appendix C

Function Ψ_I that appears in (55) is defined by

$$\Psi_I(q) = \frac{1}{\pi}\left(\int_{s_1^-}^{s_2^-} \frac{\psi_1 u du}{u^2 - q^2} + \int_{s_2^-}^{s_1^+} \frac{\psi_2 u du}{u^2 - q^2} + \int_{s_1^+}^{s_2^+} \frac{\psi_3 u du}{u^2 - q^2} + \int_{s_2^+}^{s_{r1}} \frac{\psi_4 u du}{u^2 - q^2} + \int_{s_{r1}}^{s_{r2}} \frac{\psi_5 u du}{u^2 - q^2}\right), \tag{C1a}$$

$$\psi_1 = \tan^{-1} \alpha_1^- \frac{N_1}{D_1}, \quad \psi_2 = \tan^{-1} \frac{1}{D_2}(\alpha_1^- N_{21} + \alpha_2^- N_{22}), \quad \psi_3 = \tan^{-1} \frac{N_3}{D_3}, \tag{C1b}$$

$$\psi_4 = \tan^{-1} u^2 \frac{N_4}{D_4}, \quad \psi_5 = \tan^{-1} \frac{N_5}{B_2 D_5}. \tag{C1c}$$

Equation (C1b) and (C1c) employ the quantities

$$N_1 = \eta_1^-\left(\eta_2^- S_{1+}^{2+} - \eta_2^+ S_{1+}^{2-}\right) + \eta_1^+\left[\eta_2^+(K_1^+ + K_2^+)U_{2-} - \eta_2^-(K_1^+ + K_2^-)U_{2+}\right], \tag{C2a}$$

$$D_1 = \eta_1^-\left(\eta_2^+ A_2^+ S_{1+}^{2-} - \eta_2^- A_2^- S_{1+}^{2+}\right) + \eta_1^+\left[\eta_2^-(K_1^+ + K_2^-)V_{2+} - \eta_2^+(K_1^+ + K_2^+)V_{2-}\right], \tag{C2b}$$

$$N_{21} = \eta_1^- \eta_2^- S_{1+}^{2+} + \eta_1^+ \eta_2^+ (K_1^+ + K_2^+)(\rho_1 \rho_2 B_2 + T_2^2 B_1) - k_1 \eta_1^+ \eta_2^- A_1^+ U_{2+} - k_1 \eta_1^- \eta_2^+ V_{1+}, \tag{C3a}$$

$$N_{22} = \eta_2^- \eta_1^- S_{1+}^{2+} + \eta_2^+ \eta_1^+ (K_1^+ + K_2^+)(\rho_1 \rho_2 B_1 + T_1^2 B_2) - k_2 \eta_2^+ \eta_1^- A_2^+ U_{1+} - k_2 \eta_2^- \eta_1^+ V_{2+}, \tag{C3b}$$

$$D_2 = \eta_1^+ \eta_2^- (A_1^+ V_{2+} - \alpha_1^- k_2^- U_{2+}) + \eta_1^- \eta_2^+ (A_2^+ V_{1+} - k_1^- \alpha_2^- U_{1+}), \tag{C3c}$$

$$\begin{aligned} N_3 = \eta_2^-\left[\eta_1^-(\kappa_1^- + \kappa_2^-) - \eta_1^+(\kappa_1^+ + \kappa_2^-)\right] \\ + \eta_2^+\left[\eta_1^+\left(u^2 \alpha_1^+ V_{1-}^{2-} + A_2^+ U_{1-}^{2-}\right) + \eta_1^-\left(u^2 k_1^- V_{1+}^{2-} - k_2^+ U_{1+}^{2-}\right)\right], \end{aligned} \tag{C4a}$$

$$\begin{aligned} D_3 = \eta_2^-\left[\eta_1^- \alpha_1^+(\eta_1^- + \kappa_2^+) + V_{2+} - \eta_1^+ \alpha_1^-\left(\eta_1^- \kappa_1^+ - \kappa_2^-\right)U_{2+}\right] \\ + \eta_2^+\left[\eta_1^-\left(u^2 K_2^+ V_{1+}^{2-} - \kappa_1^- U_{1+}^{2-}\right) - \eta_1^+\left(u^2 K_2^+ V_{1-}^{2-} - \kappa_1^+ U_{1-}^{2-}\right)\right], \end{aligned} \tag{C4b}$$

$$N_4=\eta_1^+\eta_2^-(\kappa_1^++\kappa_2^-)V_{1-}^{2+}+\eta_1^-\eta_2^+(\kappa_1^-+\kappa_2^+)V_{1+}^{2-}-\eta_1^+\eta_2^+(\kappa_1^++\kappa_2^+)V_{1-}^{2-}-\eta_1^-\eta_2^-(\kappa_1^-+\kappa_2^-)V_{1+}^{2+}, \quad \text{(C5a)}$$

$$D_4=\eta_1^+\eta_2^+(\kappa_1^++\kappa_2^+)U_{1-}^{2-}+\eta_1^-\eta_2^-(\kappa_1^-+\kappa_2^-)U_{1+}^{2+}+\eta_1^+\eta_2^-(\kappa_1^++\kappa_2^-)U_{1-}^{2+}+\eta_1^-\eta_2^+(\kappa_1^-+\kappa_2^+)U_{1+}^{2-}, \quad \text{(C5b)}$$

$$\begin{aligned}(N_5,D_5)=\eta_1^+\eta_2^+(\kappa_1^++\kappa_2^+)(X_{1-}^{2-},Y_{1-}^{2-})+\eta_1^-\eta_2^-(\kappa_1^-+\kappa_2^-)(X_{1+}^{2+},Y_{1+}^{2+})&\\ +\eta_1^+\eta_2^-(\kappa_1^++\kappa_2^-)(-X_{1-}^{2+},Y_{1-}^{2+})-\eta_1^-\eta_2^+(\kappa_1^-+\kappa_2^+)(-X_{1+}^{2-},Y_{1+}^{2-}).& \quad \text{(C6)}\end{aligned}$$

In (C2)–(C6) Equation (10b), (22a), (22e) and (49) hold, with argument u^2, and

$$\begin{aligned}U_{2\pm}&=T_2^2B_1+(\rho_1\rho_2+\mu_{12}^2u^2A_2^\pm B_1)B_2,\\ V_{2\pm}&=u^2T_{12}^2+(\rho_1\rho_2B_1+T_1^2B_2)A_2^\pm,\end{aligned} \quad \text{(C7a)}$$

$$\begin{aligned}U_{1+}&=T_1^2B_2+(\rho_1\rho_2+\mu_{12}^2u^2A_1^+B_2)B_1,\\ V_{1+}&=u^2T_{12}^2+(\rho_1\rho_2B_2+T_2^2B_1)A_1^+,\end{aligned} \quad \text{(C7b)}$$

$$\begin{aligned}U_{1\pm}^{2\pm}&=T_2^2\alpha_1^\pm B_1+T_1^2\alpha_2^\pm B_2+\rho_1\rho_2(\alpha_1^\pm B_2+\alpha_2^\pm B_1),\\ V_{1\pm}^{2\pm}&=T_{12}^2-\mu_{12}^2\alpha_1^\pm B_1\alpha_2^\pm B_2,\end{aligned} \quad \text{(C7c)}$$

$$\begin{aligned}X_{1\pm}^{2\pm}&=u^2T_{12}^2-(T_2^2\alpha_1^\pm+\rho_1\rho_2\alpha_2^\pm)\beta_1,\\ Y_{1\pm}^{2\pm}&=T_1^2\alpha_2^\pm+(\rho_1\rho_2-\mu_{12}^2u^2\alpha_2^\pm\beta_1)\alpha_1^\pm.\end{aligned} \quad \text{(C7d)}$$

Appendix D

Function Ψ_{II} that appears in (61) has the same form as that given for Ψ_I by (C1a). However, (C1b) and (C1c) are modified:

$$\psi_1=\tan^{-1}\alpha_1^-\frac{N_1+\Omega_{12}Q_B(A_2^+-A_2^-)}{D_1+\Omega_{12}Q_BA_1^+(A_2^+-A_2^-)}, \quad \text{(D1a)}$$

$$\psi_2=\tan^{-1}\frac{\alpha_1^-N_{21}+\alpha_2^-N_{22}+\Omega_{12}Q_B(\alpha_1^-A_2^++\alpha_2^-A_1^+)}{D_2+\Omega_{12}Q_B(A_1^+A_2^+-\alpha_1^-\alpha_2^-)}, \quad \text{(D1b)}$$

$$\psi_3=\tan^{-1}\frac{N_3+\Omega_{12}Q_B\alpha_2^-(\alpha_1^+-\alpha_1^-)}{D_3-\Omega_{12}Q_BA_2^+(\alpha_1^+-\alpha_1^-)}, \quad \text{(D1c)}$$

$$\psi_4=\tan^{-1}\frac{u^2N_4}{D_4+\Omega_{12}Q_B(\alpha_1^+-\alpha_1^-)(\alpha_2^+-\alpha_2^-)}, \quad \text{(D1d)}$$

$$\psi_5=\tan^{-1}\frac{1}{B_2}\frac{N_5-\Omega_{12}T_2\beta_1(\alpha_1^+-\alpha_1^-)(\alpha_2^+-\alpha_2^-)}{D_5+\Omega_{12}T_1(\alpha_1^+-\alpha_1^-)(\alpha_2^+-\alpha_2^-)}. \quad \text{(D1e)}$$

In Equation (C2)–(C6) in Appendix C, (D1), (10b), (22a), (22e), (C2), (56) and (58) now hold, with argument u^2.

References

[Achenbach 1973] J. D. Achenbach, *Wave propagation in elastic solids*, North-Holland, Amsterdam, 1973.

[Brock 1997a] L. M. Brock, "Some results for Rayleigh and Stoneley signals in thermoelastic solids", *Indian J. Pure Ap. Mat.* **28**:6 (1997), 835–850. MR 1461194 (98e:73014)

[Brock 1997b] L. M. Brock, "Transient three-dimensional rayleigh and stoneley signal effects in thermoelastic solids", *Int. J. Solids Struct.* **34**:12 (1997), 1463–1478. MR 1442859 (98b:73015)

[Brock 1998] L. M. Brock, "Analytic results for roots of two irrational functions in elastic wave propagation", *J. Aust. Math. Soc. B* **40**:1 (1998), 72–79. MR 1707766 (2000d:74037)

[Brock 2004] L. M. Brock, "Dynamic contact and fracture: some results for transversely isotropic solids", pp. 50–60 in *Proceedings of the 7th National Congress on Mechanics* (Chania, Crete), edited by A. Kounadis et al., Hellenic Society for Theoretical and Applied Mechanics, 2004.

[Brock and Georgiadis 1999] L. M. Brock and H. G. Georgiadis, "Response of welded thermoelastic solids to the rapid motion of thermomechanical sources parallel to the interface", *Int. J. Solids Struct.* **36**:10 (1999), 1503–1521.

[Cagniard 1962] L. Cagniard, *Reflection and refraction of progressive seismic waves*, McGraw-Hill, New York, 1962. (E. A. Flinn and C. H. Dix, translators).

[Carrier and Pearson 1988] G. F. Carrier and C. E. Pearson, *Partial differential equations: theory and technique*, Academic Press, Boston, 1988. MR 952148 (89j:35001)

[Chadwick 1960] P. Chadwick, "Thermoelasticity. The dynamical theory", pp. 263–328 in *Progress in solid mechanics*, vol. 1, edited by I. N. Sneddon and R. Hill, North-Holland Publishing, Amsterdam, 1960. MR 0113406 (22 #4244)

[Chandrasekharia 1986] D. S. Chandrasekharia, "Thermoelasticity with second sound", *Appl. Mech. Rev.* **39** (1986), 355–376.

[Davis 1998] J. R. Davis (editor), *Metals handbook: desk edition*, edited by J. R. Davis, ASM International, Metals Park, OH, 1998.

[Fan and Lu 2002] Q.-M. Fan and W.-Q. Lu, "A new numerical method to simulate the non-fourier heat conduction in a single-phase medium", *Int. J. Heat Mass Tran.* **45**:13 (2002), 2815–2821.

[Green and Lindsay 1972] A. E. Green and K. A. Lindsay, "Thermoelasticity", *J. Elasticity* **2**:1 (1972), 1–7.

[Hille 1959] E. Hille, *Analytic function theory*, vol. 1, Introduction to higher mathematics, Ginn, Boston, 1959. MR 0107692 (21 #6415)

[Jones 1999] R. M. Jones, *Mechanics of composite materials*, Taylor and Francis, New York, 1999.

[Joseph and Preziosi 1989] D. D. Joseph and L. Preziosi, "Heat waves", *Rev. Mod. Phys.* **61**:1 (1989), 41–73. MR 977943 (89k:80001)

[Lamb 1904] H. Lamb, "On the propagation of tremors over the surface of an elastic solid", *Philos. T. Roy. Soc. A* **203** (1904), 1–42.

[Lord and Shulman 1967] H. W. Lord and Y. Shulman, "A generalized dynamical theory of thermoelasticity", *J. Mech. Phys. Solids* **15**:5 (1967), 299–309.

[Noble 1958] B. Noble, *Methods based on the Wiener-Hopf technique for the solution of partial differential equations*, vol. 7, International series of monographs on pure and applied mathematics, Pergamon Press, New York, 1958. MR 0102719 (21 #1505)

[Norris and Achenbach 1984] A. N. Norris and J. D. Achenbach, "Elastic wave diffraction by a semi-infinite crack in a transversely isotropic material", *Q. J. Mech. Appl. Math.* **37**:4 (1984), 565–580. MR 774907 (86a:73014)

[Sharma and Sharma 2002] J. N. Sharma and P. K. Sharma, "Free vibration analysis of homogeneous transversely isotropic thermoelastic cylindrical panel", *J. Therm. Stresses* **25**:2 (2002), 169–182.

[Sneddon 1972] I. N. Sneddon, *The use of integral transforms*, McGraw-Hill, New York, 1972.

[Stoneley 1924] R. Stoneley, "Elastic waves at the surface of separation of two solids", *P. Roy. Soc. Lond. A Mat.* **106**:738 (1924), 416–428.

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LOUIS MILTON BROCK: brock@engr.uky.edu
Mechanical Engineering, University of Kentucky, 265 RGAN, Lexington, KY 40506-0503, United States