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ON THE ELASTIC MODULI AND COMPLIANCES OF TRANSVERSELY ISOTROPIC AND ORTHOTROPIC MATERIALS

VLADO A. LUBARDA AND MICHELLE C. CHEN

The relationships between the elastic moduli and compliances of transversely isotropic and orthotropic materials, which correspond to different appealing sets of linearly independent fourth-order base tensors used to cast the elastic moduli and compliances tensors, are derived by performing explicit inversions of the involved fourth-order tensors. The deduced sets of elastic constants are related to each other and to common engineering constants expressed in the Voigt notation with respect to the coordinate axes aligned along the directions orthogonal to the planes of material symmetry. The results are applied to a transversely isotropic monocrystalline zinc and an orthotropic human femural bone.

1. Introduction

There has been a significant amount of research devoted to tensorial representation of elastic constants of anisotropic materials, which are the components of the fourth-order tensors in three-dimensional geometrical space, or the second-order tensors in six-dimensional stress or strain space. This was particularly important for the spectral decompositions of the fourth-order stiffness and compliance tensors and the determination of the corresponding eigenvalues and eigentensors for materials with different types of elastic anisotropy. The spectral decomposition, for example, allows an additive decomposition of the elastic strain energy into a sum of six or fewer uncoupled energy modes, which are given by trace products of the corresponding pairs of stress and strain eigentensors. The representative references include [Rychlewski 1984; Walpole 1984; Theocaris and Philippidis 1989; Mehrabadi and Cowin 1990; Sutcliffe 1992; Theocaris and Sokolis 2000]. A study of the fourth-order tensors of elastic constants, and their equivalent matrix representations, is also important in the analysis of the invariants of the fourth-order tensors and identification of the invariant combinations of anisotropic elastic constants [Srinivasan and Nigam 1969; Betten 1987; Zheng 1994; Ting 1987; 2000; Ting and He 2006]. Furthermore, a study of the representation of the fourth-order tensors with respect to different bases can facilitate tensor operations and the analytical determination of the inverse tensors [Kunin 1981; 1983; Walpole 1984; Lubarda and Krajcinovic 1994; Nadeau and Ferrari 1998; Gangi 2000].

The objective of the present paper is to derive the relationships between the elastic moduli and compliances of transversely isotropic and orthotropic materials, which correspond to different appealing sets of linearly independent fourth-order base tensors used to cast the elastic stiffness and compliance tensors. In the case of transversely isotropic materials we begin the analysis with a stress-strain relationship which follows from the representation theorem for transversely isotropic tensor functions of a symmetric second-order tensor and a unit vector. This leads to the representation of the stiffness tensor in terms of

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6 linearly independent tensors ($\mathbf{I}_r$) forming a basis of an algebra of fourth-order tensors that are made of the Kronecker delta and the unit vector. The corresponding elastic moduli are denoted by c_r. Since the multiplication table for $\mathbf{I}_r$ tensors is dense with nonzero tensors, and thus operationally less practical, we introduce a set of new base tensors ($\mathbf{J}_r$) following the procedure developed by Kunin [1981] and Walpole [1984]. The corresponding elastic moduli, denoted by $\bar{c}_r$, are related to the original moduli c_r. We then derive the explicit expressions for the elastic compliances tensor with respect to both bases, and the relationships between the corresponding parameters s_r and $\bar{s}_r$ and the original moduli c_r and $\bar{c}_r$. The two sets of elastic constants are also expressed in terms of the engineering constants, defined with respect to the coordinate system in which one of the axes is parallel to the axis of material symmetry.

Extending the analysis to orthotropic materials, three different representations of the elastic moduli tensor are constructed by using three different sets of 12 linearly independent fourth-order base tensors. This is achieved on the basis of the representation theorems for orthotropic tensor functions of a symmetric second-order tensor and the structural tensors associated with the principal axes of orthotropy [Spencer 1982; Boehler 1987]. The three sets of different elastic moduli, denoted by c_r, $\bar{c}_{rs}$ and $\hat{c}_{rs}$ (nine of which are independent in each case) are related to each other. The tensor of elastic compliances is then deduced by explicit inversion of the stiffness tensor expressed with respect to all three sets of base tensors. The corresponding compliances, denoted by s_r, $\bar{s}_{rs}$, and $\hat{s}_{rs}$, are related to each other and to the moduli c_r, $\bar{c}_{rs}$, and $\hat{c}_{rs}$. The three sets of elastic constants are then expressed in terms of engineering constants, appearing in the Voigt notation as the entries of 6×6 matrices [Voigt 1928], which do not constitute the components of the second-order tensor [Hearmon 1961; Nye 1964; Cowin and Mehrabadi 1995] and which are thus not amenable to easy tensor manipulations. The results are applied to calculate the different sets of elastic constants for a transversely isotropic monocrystalline zinc and an orthotropic human femur.

2. Elastic moduli of transversely isotropic materials

The stress-strain relationship for a linearly elastic transversely isotropic material, based on the representation theorems for transversely isotropic tensor function of a strain tensor and a unit vector [Spencer 1982], can be written as

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} + \alpha(n_k n_l \epsilon_{kl}\delta_{ij} + n_i n_j \epsilon_{kk}) + 2(\mu_0 - \mu)(n_i n_k \epsilon_{kj} + n_j n_k \epsilon_{ki}) + \beta n_i n_j n_k n_l \epsilon_{kl}. \tag{1}$$

The rectangular components of the unit vector parallel to the axis of transverse isotropy are n_i. The elastic shear modulus within the plane of isotropy is μ, the other Lamé constant is λ, and the out-of-plane elastic shear modulus is μ_0. The remaining two moduli reflecting different elastic properties in the plane of isotropy and perpendicular to it are denoted by α and β.

The elastic stiffness tensor Λ corresponding to Equation (1), and defined such that $\sigma_{ij} = \Lambda_{ijkl}\epsilon_{kl}$, is

$$\mathbf{\Lambda} = \sum_{r=1}^{6} c_r \mathbf{I}_r, \tag{2}$$

where the fourth-order base tensors $\mathbf{I}_r$ are defined by

$$\mathbf{I}_1 = \tfrac{1}{2}\,\boldsymbol{\delta}\circ\boldsymbol{\delta}, \quad \mathbf{I}_2 = \boldsymbol{\delta}\,\boldsymbol{\delta}, \quad \mathbf{I}_3 = \mathbf{p}\,\boldsymbol{\delta}, \quad \mathbf{I}_4 = \boldsymbol{\delta}\,\mathbf{p}, \\ \mathbf{I}_5 = \tfrac{1}{2}\,\mathbf{p}\circ\boldsymbol{\delta}, \quad \mathbf{I}_6 = \mathbf{p}\,\mathbf{p}. \tag{3}$$

The second order tensor $\mathbf{p}$ has the components $p_{ij} = n_i n_j$. The tensor products between the second-order tensors $\mathbf{p}$ and $\boldsymbol{\delta}$ are such that

$$(\mathbf{p}\,\boldsymbol{\delta})_{ijkl} = p_{ij}\delta_{kl}, \quad (\mathbf{p}\circ\boldsymbol{\delta})_{ijkl} = \tfrac{1}{2}\,(p_{ik}\delta_{jl} + p_{il}\delta_{jk} + p_{jl}\delta_{ik} + p_{jk}\delta_{il}). \tag{4}$$

In the component form, the fourth-order tensors $\mathbf{I}_r$ are

$$\begin{aligned} I^{(1)}_{ijkl} &= \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), & I^{(2)}_{ijkl} &= \delta_{ij}\delta_{kl}, \\ I^{(3)}_{ijkl} &= n_i n_j \delta_{kl}, & I^{(4)}_{ijkl} &= \delta_{ij} n_k n_l, \\ I^{(5)}_{ijkl} &= \tfrac{1}{2}(\delta_{ik} n_j n_l + \delta_{il} n_j n_k + \delta_{jl} n_i n_k + \delta_{jk} n_i n_l), & I^{(6)}_{ijkl} &= n_i n_j n_k n_l. \end{aligned} \tag{5}$$

The material parameters (elastic moduli), appearing in Equation (2), are

$$c_1 = 2\mu, \quad c_2 = \lambda, \quad c_3 = c_4 = \alpha, \quad c_5 = 2(\mu_0 - \mu), \quad c_6 = \beta. \tag{6}$$

The set of linearly independent tensors $\mathbf{I}_r$ forms a basis of an algebra (of order 6) of the fourth-rank tensors made up of the Kronecker delta and the unit vector, symmetric with respect to first and second pair of indices, but not necessarily with respect to permutation of pairs. All such fourth-rank tensors can be represented as a linear combination of the base tensors.

Since the multiplication table of the tensors $\mathbf{I}_r$ is dense with nonzero tensors (see the upper left corner of Table 9 in the Appendix), and thus less convenient for tensor operations and the determination of the inverse tensors, we recast the elastic stiffness tensor (2) in terms of an alternative set of base tensors $\mathbf{J}_r$, which is characterized by a simple multiplication table, in which each product is equal to either zero tensor or to one of the tensors from the basis. Thus,

$$\boldsymbol{\Lambda} = \sum_{r=1}^{6} \bar{c}_r \mathbf{J}_r, \tag{7}$$

where the modified elastic moduli are

$$\begin{aligned} \bar{c}_1 &= c_1 + c_2 + 2c_3 + 2c_5 + c_6 = \lambda + 4\mu_0 - 2\mu + 2\alpha + \beta, \\ \bar{c}_2 &= c_1 + 2c_2 = 2(\lambda + \mu), \\ \bar{c}_3 &= \bar{c}_4 = \sqrt{2}\,(c_2 + c_3) = \sqrt{2}\,(\lambda + \alpha), \\ \bar{c}_5 &= c_1 = 2\mu, \quad \bar{c}_6 = c_1 + c_5 = 2\mu_0, \end{aligned} \tag{8}$$

with the inverse relationships

$$\begin{aligned} c_1 = \bar{c}_5, \quad c_2 = \tfrac{1}{2}(\bar{c}_2 - \bar{c}_5), \quad c_5 = \bar{c}_6 - \bar{c}_5, \\ c_3 = c_4 = \frac{1}{\sqrt{2}}\bar{c}_3 - \tfrac{1}{2}(\bar{c}_2 - \bar{c}_5), \\ c_6 = \bar{c}_1 + \tfrac{1}{2}\bar{c}_2 - \sqrt{2}\,\bar{c}_3 + \tfrac{1}{2}\bar{c}_5 - 2\bar{c}_6. \end{aligned} \tag{9}$$

The base tensors $\mathbf{J}_r$ are defined by Walpole [1984], [1]

$$\begin{aligned} \mathbf{J}_1 &= \mathbf{p}\,\mathbf{p}, & \mathbf{J}_2 &= \tfrac{1}{2}\,\mathbf{q}\,\mathbf{q}, \\ \mathbf{J}_3 &= \frac{1}{\sqrt{2}}\,\mathbf{p}\,\mathbf{q}, & \mathbf{J}_4 &= \frac{1}{\sqrt{2}}\,\mathbf{q}\,\mathbf{p}, \\ \mathbf{J}_5 &= \tfrac{1}{2}(\mathbf{q}\circ\mathbf{q} - \mathbf{q}\,\mathbf{q}), & \mathbf{J}_6 &= \mathbf{p}\circ\mathbf{q}. \end{aligned} \tag{10}$$

The second order tensor $\mathbf{q}$ is introduced such that

$$q_{ij} = \delta_{ij} - p_{ij}, \quad p_{ij} = n_i n_j. \tag{11}$$

When applied to an arbitrary vector v_j, the tensors $\mathbf{p}$ and $\mathbf{q}$ decompose that vector into its components parallel to n_i, $p_{ij}v_j = (v_j n_j)n_i$, and orthogonal to it, $q_{ij}n_j = v_i - p_{ij}v_j$. It readily follows that

$$\begin{aligned} p_{ik}p_{kj} = p_{ij}, \quad p_{ij}p_{ij} = 1, \quad p_{ii} = 1, \\ q_{ik}q_{kj} = q_{ij}, \quad q_{ij}q_{ij} = 2, \quad q_{ii} = 2, \\ p_{ik}q_{kj} = 0, \quad q_{ik}p_{kj} = 0. \end{aligned} \tag{12}$$

The two types of tensor products between the involved second-order tensors are defined by the type Equation (4), that is,

$$(\mathbf{p}\,\mathbf{q})_{ijkl} = p_{ij}q_{kl}, \quad (\mathbf{p}\circ\mathbf{q})_{ijkl} = \tfrac{1}{2}(p_{ik}q_{jl} + p_{il}q_{jk} + p_{jl}q_{ik} + p_{jk}q_{il}). \tag{13}$$

It can be easily verified that

$$\mathbf{p}\circ\mathbf{q} = \mathbf{q}\circ\mathbf{p}, \quad \mathbf{p}\circ\mathbf{p} = 2\,\mathbf{p}\,\mathbf{p}. \tag{14}$$

The trace products of the tensors $\mathbf{J}_r : \mathbf{J}_s$ of the type $J^{(r)}_{ijmn}J^{(s)}_{mnkl}$ can be readily evaluated by using Equation (12), with the results listed in Table 1. The symbol ↱ indicates that the products are in the order: *a tensor from the left column* traced with *a tensor from the top row*.

2.1. ***Elastic compliances of transversely isotropic materials.*** To derive the elastic compliances tensor, it is more convenient to invert the elastic stiffness tensor Λ in its representation (7) than (2). Thus, by writing

$$\mathbf{\Lambda}^{-1} = \sum_{r=1}^{6} \bar{s}_r \mathbf{J}_r, \tag{15}$$

and by using Table 1 to expand the trace products appearing in

$$\mathbf{\Lambda} : \mathbf{\Lambda}^{-1} = \mathbf{\Lambda}^{-1} : \mathbf{\Lambda} = \mathbf{I}_1, \tag{16}$$

[1] An alternative set of base tensors was used by Kunin[1981; 1983] and Lubarda and Krajcinovic [1994].

↱	$\mathbf{J}_1$	$\mathbf{J}_2$	$\mathbf{J}_3$	$\mathbf{J}_4$	$\mathbf{J}_5$	$\mathbf{J}_6$
$\mathbf{J}_1$	$\mathbf{J}_1$	0	$\mathbf{J}_3$	0	0	0
$\mathbf{J}_2$	0	$\mathbf{J}_2$	0	$\mathbf{J}_4$	0	0
$\mathbf{J}_3$	0	$\mathbf{J}_3$	0	$\mathbf{J}_1$	0	0
$\mathbf{J}_4$	$\mathbf{J}_4$	0	$\mathbf{J}_2$	0	0	0
$\mathbf{J}_5$	0	0	0	0	$\mathbf{J}_5$	0
$\mathbf{J}_6$	0	0	0	0	0	$\mathbf{J}_6$

Table 1. Table of products $\mathbf{J}_r : \mathbf{J}_s$.

there follows

$$\bar{s}_1 = \bar{c}_2/\bar{c}, \quad \bar{s}_2 = \bar{c}_1/\bar{c}, \quad \bar{s}_3 = \bar{s}_4 = -\bar{c}_3/\bar{c}, \quad \bar{s}_5 = 1/\bar{c}_5, \quad \bar{s}_6 = 1/\bar{c}_6, \tag{17}$$

where

$$\bar{c} = \bar{c}_1\bar{c}_2 - \bar{c}_3^2. \tag{18}$$

Note that the fourth-order identity tensor $\mathbf{I}_1$ can be expressed in terms of the base tensors $\mathbf{J}_r$ as

$$\mathbf{I}_1 = \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_5 + \mathbf{J}_6,$$

which was used in Equation (16).

The elastic compliances tensor can be expressed in the original basis $\mathbf{I}_r$ by using the connections between the two bases, which are

$$\begin{aligned}
&\mathbf{I}_1 = \tfrac{1}{2}\,\boldsymbol{\delta}\circ\boldsymbol{\delta} = \mathbf{p}\,\mathbf{p} + \mathbf{p}\circ\mathbf{q} + \tfrac{1}{2}\,\mathbf{q}\circ\mathbf{q} = \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_5 + \mathbf{J}_6,\\
&\mathbf{I}_2 = \boldsymbol{\delta}\,\boldsymbol{\delta} = (\mathbf{p}+\mathbf{q})(\mathbf{p}+\mathbf{q}) = \mathbf{J}_1 + 2\mathbf{J}_2 + \sqrt{2}(\mathbf{J}_3 + \mathbf{J}_4),\\
&\mathbf{I}_3 = \mathbf{p}\,\boldsymbol{\delta} = \mathbf{p}\,\mathbf{p} + \mathbf{p}\,\mathbf{q} = \mathbf{J}_1 + \sqrt{2}\,\mathbf{J}_3, \qquad \mathbf{I}_4 = \boldsymbol{\delta}\,\mathbf{p} = \mathbf{p}\,\mathbf{p} + \mathbf{q}\,\mathbf{p} = \mathbf{J}_1 + \sqrt{2}\,\mathbf{J}_4,\\
&\mathbf{I}_5 = \mathbf{p}\circ\boldsymbol{\delta} = 2\mathbf{p}\,\mathbf{p} + \mathbf{p}\circ\mathbf{q} = 2\mathbf{J}_1 + \mathbf{J}_6, \qquad \mathbf{I}_6 = \mathbf{p}\,\mathbf{p} = \mathbf{J}_1.
\end{aligned} \tag{19}$$

Their inverse relationships are

$$\begin{aligned}
&\mathbf{J}_1 = \mathbf{I}_6, && \mathbf{J}_2 = \tfrac{1}{2}\,(\mathbf{I}_2 - \mathbf{I}_3 - \mathbf{I}_4 + \mathbf{I}_6),\\
&\mathbf{J}_3 = \frac{1}{\sqrt{2}}\,(\mathbf{I}_3 - \mathbf{I}_6), && \mathbf{J}_4 = \frac{1}{\sqrt{2}}\,(\mathbf{I}_4 - \mathbf{I}_6),\\
&\mathbf{J}_5 = \tfrac{1}{2}\,(2\mathbf{I}_1 - \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4 - 2\mathbf{I}_5 + \mathbf{I}_6), && \mathbf{J}_6 = \mathbf{I}_5 - 2\mathbf{I}_6.
\end{aligned} \tag{20}$$

The elastic compliance tensor is expressed in terms of the base tensors $\mathbf{I}_r$ by substituting (20) into (15). The result is

$$\boldsymbol{\Lambda}^{-1} = \sum_{r=1}^{6} s_r \mathbf{I}_r, \tag{21}$$

with the corresponding compliances

$$
\begin{aligned}
&s_1 = \bar{s}_5, && s_2 = \tfrac{1}{2}\,(\bar{s}_2 - \bar{s}_5),\\
&s_3 = s_4 = \frac{1}{\sqrt{2}}\,\bar{s}_3 - \tfrac{1}{2}\,(\bar{s}_2 - \bar{s}_5), && s_5 = \bar{s}_6 - \bar{s}_5,\\
&s_6 = \bar{s}_1 + \tfrac{1}{2}\,\bar{s}_2 - \sqrt{2}\,\bar{s}_3 + \tfrac{1}{2}\,\bar{s}_5 - 2\bar{s}_6. &&
\end{aligned}
\tag{22}
$$

The same type of relationships hold between the moduli $\bar{c}_r$ and c_r appearing in Equation (2) and (7). The inverse expressions to (22) are

$$
\begin{aligned}
&\bar{s}_1 = s_1 + s_2 + 2s_3 + 2s_5 + s_6, && \bar{s}_2 = s_1 + 2s_2,\\
&\bar{s}_3 = \bar{s}_4 = \sqrt{2}\,(s_2 + s_3), && \bar{s}_5 = s_1,\\
&\bar{s}_6 = s_1 + s_5. &&
\end{aligned}
\tag{23}
$$

2.2. ***Relationships to engineering constants in Voigt notation.*** The parameters c_r are related to commonly used engineering moduli C_{ij} (in Voigt notation, with the axis of isotropy along the x_3 direction) by

$$
\begin{aligned}
&c_1 = C_{11} - C_{12} = 2C_{66}, \quad c_2 = C_{12}, \quad c_3 = c_4 = C_{13} - C_{12},\\
&c_5 = 2(C_{55} - C_{66}), \quad c_6 = C_{11} + C_{33} - 2C_{13} - 4C_{55},
\end{aligned}
\tag{24}
$$

which is in accord with [Boehler 1987]. The (symmetric) elastic moduli C_{ij} are defined such that

$$
\begin{bmatrix} \sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12} \end{bmatrix}
=
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\
C_{21} & C_{22} & C_{23} & 0 & 0 & 0\\
C_{31} & C_{32} & C_{33} & 0 & 0 & 0\\
0 & 0 & 0 & C_{44} & 0 & 0\\
0 & 0 & 0 & 0 & C_{55} & 0\\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\cdot
\begin{bmatrix} \epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ 2\epsilon_{23}\\ 2\epsilon_{31}\\ 2\epsilon_{12} \end{bmatrix},
\tag{25}
$$

where for transversely isotropic materials: $C_{11} = C_{22}$, $C_{23} = C_{31}$, $C_{44} = C_{55}$, and $C_{66} = (C_{11} - C_{12})/2$. The relationships between $\bar{c}_r$ and C_{ij} are deduced from Equation (8) and (24). They are

$$
\bar{c}_1 = C_{33}, \quad \bar{c}_2 = C_{11} + C_{12}, \quad \bar{c}_3 = \bar{c}_4 = \sqrt{2}C_{13}, \tag{26}
$$

$$
\bar{c}_5 = 2C_{66}, \quad \bar{c}_6 = 2C_{55}. \tag{27}
$$

The compliance constants s_i are related to the engineering compliances S_{ij} by

$$
s_1 = S_{11} - S_{12} = S_{66}/2, \quad s_2 = S_{12}, \quad s_3 = s_4 = S_{13} - S_{12}, \tag{28}
$$

$$
s_5 = (S_{55} - S_{66})/2, \quad s_6 = S_{11} + S_{33} - 2S_{13} - S_{55}. \tag{29}
$$

The compliances S_{ij} are also defined with respect to the coordinate system in which the axis of isotropy is along the x_3 direction, so that, in Voigt notation,

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}, \tag{30}$$

where $S_{11} = S_{22}$, $S_{23} = S_{31}$, $S_{44} = S_{55}$, and $S_{66} = 2(S_{11} - S_{12})$. The well known connections between S_{ij} and C_{ij} are

$$S_{11} = \frac{C_{11} - C_{13}^2/C_{33}}{C_{11} - C_{12}}\, S, \qquad S_{12} = -\frac{C_{12} - C_{13}^2/C_{33}}{C_{11} - C_{12}}\, S,$$
$$S_{13} = -\frac{C_{13}}{C_{33}}\, S, \qquad S_{33} = \frac{C_{11} + C_{12}}{C_{33}}\, S,$$
$$S_{44} = \frac{1}{C_{44}}, \qquad S_{66} = \frac{1}{C_{66}},$$
$$S = \frac{1}{C_{11} + C_{12} - 2C_{13}^2/C_{33}}. \tag{31}$$

The inverse relationships are obtained by reversing the role of C's and S's in Equation (31). The relationships between $\bar{s}_r$ and S_{ij} are obtained by substituting (28) into (23), with the result

$$\bar{s}_1 = S_{33}, \quad \bar{s}_2 = S_{11} + S_{12}, \quad \bar{s}_3 = \bar{s}_4 = \sqrt{2} S_{13}, \tag{32}$$
$$\bar{s}_5 = \tfrac{1}{2}\, S_{66}, \quad \bar{s}_6 = \tfrac{1}{2}\, S_{55}. \tag{33}$$

Let E and E_0 be the Young's moduli in the plane of isotropy and in the direction perpendicular to it; ν is the Poisson ratio characterizing transverse contraction in the plane of isotropy due to applied tension in the orthogonal direction within the plane of isotropy; ν_0 is the same due to applied tension perpendicular to the plane of isotropy; μ is the shear modulus in the plane of isotropy, and μ_0 in any plane perpendicular to the plane of isotropy (see [Lekhnitskii 1981]). Then, the elastic compliances S_{ij} in Equation (30) are

$$S_{11} = S_{22} = \frac{1}{E}, \qquad S_{33} = \frac{1}{E_0}, \qquad S_{12} = -\frac{\nu}{E},$$
$$S_{23} = S_{31} = -\frac{\nu_0}{E_0}, \qquad S_{44} = S_{55} = \frac{1}{\mu_0}, \qquad S_{66} = 2(S_{11} - S_{12}) = \frac{1}{\mu} = \frac{2(1+\nu)}{E}. \tag{34}$$

The matrices $[S_{ij}]$ and $[C_{ij}]$ are inverse to each other, although their components are not the components of the second-order tensors [Hearmon 1961; Nye 1964]. The substitution of (34) into (28) gives

$$s_1 = \frac{1+\nu}{E} = \frac{1}{2\mu}, \quad s_2 = -\frac{\nu}{E}, \quad s_3 = s_4 = \frac{\nu}{E} - \frac{\nu_0}{E_0}, \tag{35}$$

$$s_5 = \frac{1}{\mu_0} - \frac{1}{\mu}, \quad s_6 = \frac{1}{E} + \frac{1+2\nu_0}{E_0} - \frac{1}{\mu_0}. \tag{36}$$

The inverse relations are

$$\begin{gathered} E = \frac{1}{s_1 + s_2}, \quad \nu = -E s_2, \quad \mu = \frac{1}{2s_1}, \\ E_0 = \frac{1}{s_1 + s_2 + 2s_3 + s_5 + s_6}, \quad \nu_0 = -E_0(s_2 + s_3), \quad \mu_0 = \frac{1}{2s_1 + s_5}. \end{gathered} \tag{37}$$

These expressions reveal the relationships between the original moduli appearing in Equation (6) and the engineering moduli. For example, the Young's modulus in the plane of isotropy is

$$E = 4\left[\frac{\lambda + 4\mu_0 - 2\mu + 2\alpha + \beta}{(\lambda+\mu)(\lambda + 4\mu_0 - 2\mu + 2\alpha + \beta) - (\lambda+\alpha)^2} + \frac{1}{\mu}\right]^{-1}. \tag{38}$$

If $\alpha = \beta = 0$ and $\mu_0 = \mu$ this reduces to the isotropic elasticity result $E = \mu(3\lambda + 2\mu)/(\lambda+\mu)$.

2.3. ***Elastic constants of a monocrystalline zinc.*** The elastic moduli C_{ij} of a transversely isotropic monocrystalline zinc, in units of GPa and with x_3 axis as the axis of isotropy, are [Landolt-Börnstein 1979]

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 160.28 & 29.04 & 47.84 \\ 29.04 & 160.28 & 47.84 \\ 47.84 & 47.84 & 60.28 \end{bmatrix}, \quad \begin{bmatrix} C_{44} \\ C_{55} \\ C_{66} \end{bmatrix} = \begin{bmatrix} 39.53 \\ 39.53 \\ 65.62 \end{bmatrix}.$$

The moduli c_i and $\bar{c}_i$ follow from Equation (24) and Equation (26) and are listed in Table 2. The moduli μ, λ, μ_0, α, and β, calculated from Equation (6), are listed in Table 3. The elastic compliances S_{ij} (in units of TPa^{-1}), corresponding to elastic moduli C_{ij} given above, are

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \begin{bmatrix} 8.22 & 0.6 & -7 \\ 0.6 & 8.22 & -7 \\ -7 & -7 & 27.7 \end{bmatrix}, \quad \begin{bmatrix} S_{44} \\ S_{55} \\ S_{66} \end{bmatrix} = \begin{bmatrix} 25.3 \\ 25.3 \\ 15.24 \end{bmatrix}.$$

c_1	c_2	$c_3 = c_4$	c_5	c_6
131.24	29.04	18.80	−52.18	−33.24
$\bar{c}_1$	$\bar{c}_2$	$\bar{c}_3 = \bar{c}_4$	$\bar{c}_5$	$\bar{c}_6$
60.28	189.32	67.66	131.24	79.06

Table 2. Elastic moduli c_i and $\bar{c}_i$ (in units of GPa).

$\mu = c_1/2$	$\lambda = c_2$	$\alpha = c_3$	$\mu_0 = \mu + c_5/2$	$\beta = c_6$
65.62	29.04	18.80	39.53	−33.24

Table 3. Table of elastic moduli μ, λ, μ_0, α, and β (in units of GPa).

s_1	s_2	$s_3 = s_4$	s_5	s_6
7.62	0.6	−7.6	5.03	24.62
$\bar{s}_1$	$\bar{s}_2$	$\bar{s}_3 = \bar{s}_4$	$\bar{s}_5$	$\bar{s}_6$
27.7	8.82	−9.9	7.62	12.65

Table 4. Elastic compliances s_i and $\bar{s}_i$ (in units of TPa^{-1}).

The compliance coefficients s_i and $\bar{s}_i$ are readily calculated from Equation (28) and Equation (32), and are listed in Table 4. Finally, the elastic constants E, E_0, μ, μ_0, ν, and ν_0 are determined from Equation (37), and are given in Table 5. Since $s_2 = 0.6 > 0$ and $\nu = -s_2 E$, the Poisson ratio in the (basal) plane of isotropy is negative ($\nu = -0.073$). This was first noted by Lubarda and Meyers [1999], who also determined the range of the directions associated with negative values of the Poisson ratio.

3. Elastic moduli of orthotropic materials

Consider an orthotropic elastic material whose principal axes of orthotropy are along the directions **a**, **b**, and **c**, arbitrarily oriented with respect to the reference coordinate system. From the representation theorems for orthotropic tensor functions [Spencer 1982], it follows that the components of the stress tensor σ_{ij} can be expressed in terms of the strain components ϵ_{ij}, and the vector components a_i and b_i, as

$$\begin{aligned}\sigma^{\text{e}}_{ij} = (\lambda\epsilon_{kk} + \alpha_1\epsilon_a + \alpha_2\epsilon_b)\delta_{ij} + 2\mu\epsilon_{ij} \\ + (\alpha_1\epsilon_{kk} + \beta_1\epsilon_a + \beta_3\epsilon_b)a_i a_j + (\alpha_2\epsilon_{kk} + \beta_3\epsilon_a + \beta_2\epsilon_b)b_i b_j \\ + 2\mu_1(a_i a_k\epsilon_{kj} + a_j a_k\epsilon_{ki}) + 2\mu_2(b_i b_k\epsilon_{kj} + b_j b_k\epsilon_{ki}). \quad (39)\end{aligned}$$

The longitudinal strains in the directions of two of the principal axes of orthotropy are $\epsilon_a = a_k a_l \epsilon_{kl}$ and $\epsilon_b = b_k b_l \epsilon_{kl}$. The nine elastic moduli are λ, μ, μ_1, μ_2, α_1, α_2, and β_1, β_2, β_3. The elastic stiffness tensor

E	E_0	μ	μ_0	ν	ν_0
121.66	44.11	65.62	49.34	−0.073	0.309

Table 5. Elastic moduli E, E_0, μ, μ_0 (in units of GPa) and Poisson's ratios ν and ν_0.

corresponding to Equation (39) is

$$\boldsymbol{\Lambda} = \sum_{r=1}^{12} c_r \mathbf{I}_r, \tag{40}$$

where the material parameters c_r are

$$\begin{gathered} c_1 = 2\mu, \quad c_2 = \lambda, \quad c_3 = c_4 = \alpha_1, \quad c_5 = c_6 = \alpha_2, \\ c_7 = 2\mu_1, \quad c_8 = 2\mu_2, \quad c_9 = \beta_1, \quad c_{10} = \beta_2, \quad c_{11} = c_{12} = \beta_3. \end{gathered} \tag{41}$$

The fourth-order tensors $\mathbf{I}_r$ are defined by

$$\begin{aligned} &\mathbf{I}_1 = \tfrac{1}{2}\,\boldsymbol{\delta} \circ \boldsymbol{\delta}, && \mathbf{I}_2 = \boldsymbol{\delta}\,\boldsymbol{\delta}, \\ &\mathbf{I}_3 = \mathbf{A}\,\boldsymbol{\delta}, && \mathbf{I}_4 = \boldsymbol{\delta}\,\mathbf{A}, \\ &\mathbf{I}_5 = \mathbf{B}\,\boldsymbol{\delta}, && \mathbf{I}_6 = \boldsymbol{\delta}\,\mathbf{B}, \\ &\mathbf{I}_7 = \mathbf{A} \circ \boldsymbol{\delta}, && \mathbf{I}_8 = \mathbf{B} \circ \boldsymbol{\delta}, \\ &\mathbf{I}_9 = \mathbf{A}\,\mathbf{A}, && \mathbf{I}_{10} = \mathbf{B}\,\mathbf{B}, \\ &\mathbf{I}_{11} = \mathbf{A}\,\mathbf{B}, && \mathbf{I}_{12} = \mathbf{B}\,\mathbf{A}. \end{aligned} \tag{42}$$

In these expressions the (idempotent) tensors **A**, **B**, and **C** have the components

$$A_{ij} = a_i a_j, \quad B_{ij} = b_i b_j, \quad C_{ij} = c_i c_j, \tag{43}$$

which are related by the identity $A_{ij} + B_{ij} + C_{ij} = \delta_{ij}$. Two types of tensor products are defined by the formulas of the following type:

$$\begin{aligned} (\mathbf{A}\,\mathbf{B})_{ijkl} &= A_{ij} B_{kl}, \\ (\mathbf{A} \circ \mathbf{B})_{ijkl} &= \tfrac{1}{2}\,(A_{ik} B_{jl} + A_{il} B_{jk} + A_{jl} B_{ik} + A_{jk} B_{il}). \end{aligned} \tag{44}$$

It can be easily verified that

$$\mathbf{A} \circ \mathbf{B} = \mathbf{B} \circ \mathbf{A}, \quad \mathbf{A} \circ \mathbf{A} = 2\mathbf{A}\,\mathbf{A}. \tag{45}$$

The symmetric tensors such as $\mathbf{A}\,\mathbf{A}$, and $\mathbf{A} \circ \mathbf{B}$ are also idempotent.

The multiplication table for tensors $\mathbf{I}_r$ (Table 9 of the Appendix) is dense with nonzero entries, although these are simple multiples of one of the $\mathbf{I}_r$ tensors, except for

$$\mathbf{I}_7 : \mathbf{I}_7 = \mathbf{I}_7 + 2\mathbf{I}_9, \quad \mathbf{I}_8 : \mathbf{I}_8 = \mathbf{I}_8 + 2\mathbf{I}_{10}, \tag{46}$$

and

$$\mathbf{I}_7 : \mathbf{I}_8 = \mathbf{I}_8 : \mathbf{I}_7 = \mathbf{A} \circ \mathbf{B}. \tag{47}$$

The latter can be expressed as a linear combination of $\mathbf{I}_r$ tensors, given by the tensor $\mathbf{E}_{66}$ introduced in the sequel.

The elastic stiffness Equation (40) can be recast in terms of a more convenient set of base tensors $\mathbf{E}_{rs}$, such that

$$\boldsymbol{\Lambda} = \sum_{r,s=1}^{3} \bar{c}_{rs} \mathbf{E}_{rs} + \bar{c}_{44} \mathbf{E}_{44} + \bar{c}_{55} \mathbf{E}_{55} + \bar{c}_{66} \mathbf{E}_{66}. \tag{48}$$

The corresponding moduli are

$$
\begin{aligned}
\bar{c}_{11} &= c_1 + c_2 + 2c_3 + 2c_7 + c_9 = \lambda + 2\mu + 4\mu_1 + 2\alpha_1 + \beta_1, \\
\bar{c}_{22} &= c_1 + c_2 + 2c_5 + 2c_8 + c_{10} = \lambda + 2\mu + 4\mu_2 + 2\alpha_2 + \beta_2, \\
\bar{c}_{33} &= c_1 + c_2 = \lambda + 2\mu, \\
\bar{c}_{12} &= \bar{c}_{21} = c_2 + c_3 + c_5 + c_{11} = \lambda + \alpha_1 + \alpha_2 + \beta_3, \\
\bar{c}_{23} &= \bar{c}_{32} = c_2 + c_5 = \lambda + \alpha_2, \\
\bar{c}_{31} &= \bar{c}_{13} = c_2 + c_3 = \lambda + \alpha_1, \\
\bar{c}_{44} &= c_1 + c_8 = 2(\mu + \mu_2), \\
\bar{c}_{55} &= c_1 + c_7 = 2(\mu + \mu_1), \\
\bar{c}_{66} &= c_1 + c_7 + c_8 = 2(\mu + \mu_1 + \mu_2).
\end{aligned}
\tag{49}
$$

The base $\mathbf{E}_{rs}$ are defined by

$$
\begin{aligned}
&\mathbf{E}_{11} = \mathbf{A}\,\mathbf{A}, \quad \mathbf{E}_{12} = \mathbf{A}\,\mathbf{B}, \quad \mathbf{E}_{13} = \mathbf{A}\,\mathbf{C}, \\
&\mathbf{E}_{21} = \mathbf{B}\,\mathbf{A}, \quad \mathbf{E}_{22} = \mathbf{B}\,\mathbf{B}, \quad \mathbf{E}_{23} = \mathbf{B}\,\mathbf{C}, \\
&\mathbf{E}_{31} = \mathbf{C}\,\mathbf{A}, \quad \mathbf{E}_{32} = \mathbf{C}\,\mathbf{B}, \quad \mathbf{E}_{33} = \mathbf{C}\,\mathbf{C},
\end{aligned}
\tag{50}
$$

and

$$
\mathbf{E}_{44} = \mathbf{B} \circ \mathbf{C}, \quad \mathbf{E}_{55} = \mathbf{C} \circ \mathbf{A} \quad \mathbf{E}_{66} = \mathbf{A} \circ \mathbf{B}. \tag{51}
$$

The ordering of the last three tensors is made to facilitate the later comparison with the classical moduli given in the Voigt notation.

The trace products of the tensors $\mathbf{E}_{rs}$ are specified by the following rules [Walpole 1984]

$$
\begin{aligned}
\mathbf{E}_{pq} : \mathbf{E}_{qr} &= \mathbf{E}_{pr}, \quad \text{(no sum on } q\text{)}, \\
\mathbf{E}_{pq} : \mathbf{E}_{rs} &= \mathbf{0}, \qquad (q \neq r).
\end{aligned}
\tag{52}
$$

Furthermore, a zero trace product is obtained when any one of $\mathbf{E}_{44}$, $\mathbf{E}_{55}$, and $\mathbf{E}_{66}$ is traced with the other two or any one of $\mathbf{E}_{rs}$ (see Table 10 of the Appendix).

An alternative representation of the elastic stiffness tensor is obtained from the representation theorem for orthotropic tensor functions, if the stress tensor $\boldsymbol{\sigma}$ is expressed in terms of the strain tensor $\boldsymbol{\epsilon}$ and the structural tensors **A**, **B**, and **C** [Boehler 1987]. This is

$$
\boldsymbol{\sigma} = \hat{c}_1\mathbf{A} + \hat{c}_2\mathbf{B} + \hat{c}_3\mathbf{C} + \hat{c}_4(\mathbf{A}\cdot\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\cdot\mathbf{A}) + \hat{c}_5(\mathbf{B}\cdot\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\cdot\mathbf{B}) + \hat{c}_6(\mathbf{C}\cdot\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\cdot\mathbf{C}). \tag{53}
$$

To preserve the linear dependence on strain, the functions $\hat{c}_r$ are

$$
\hat{c}_r = \hat{c}_{r1}\,\mathbf{A} : \boldsymbol{\epsilon} + \hat{c}_{r2}\,\mathbf{B} : \boldsymbol{\epsilon} + \hat{c}_{r3}\,\mathbf{C} : \boldsymbol{\epsilon}, \quad r = 1, 2, 3, \tag{54}
$$

where the parameters $\hat{c}_{rs}$, $\hat{c}_4 = \hat{c}_{44}$, $\hat{c}_5 = \hat{c}_{55}$, and $\hat{c}_6 = \hat{c}_{66}$ are the material constants. The dot $(\cdot)$ denotes the inner tensor product, and : is used for the trace product. The elastic stiffness tensor associated with

Equation (53) and (54) is

$$\mathbf{\Lambda} = \sum_{r,s=1}^{3} \hat{c}_{rs}\mathbf{E}_{rs} + \hat{c}_{44}\mathbf{A}\circ\boldsymbol{\delta} + \hat{c}_{55}\mathbf{B}\circ\boldsymbol{\delta} + \hat{c}_{66}\mathbf{C}\circ\boldsymbol{\delta}. \tag{55}$$

3.1. ***Elastic compliances of orthotropic materials.*** If the elastic moduli tensor Equation (48) is symbolically written, with respect to the basis of $\mathbf{E}_{rs}$ tensors, as

$$\mathbf{\Lambda} = \left\{ \begin{bmatrix} \bar{c}_{11} & \bar{c}_{12} & \bar{c}_{13} \\ \bar{c}_{21} & \bar{c}_{22} & \bar{c}_{23} \\ \bar{c}_{31} & \bar{c}_{32} & \bar{c}_{33} \end{bmatrix}, \bar{c}_{44}, \bar{c}_{55}, \bar{c}_{66} \right\}, \tag{56}$$

then its inverse, the elastic compliances tensor, can be symbolically written as [Walpole 1984]

$$\mathbf{\Lambda}^{-1} = \left\{ \begin{bmatrix} \bar{c}_{11} & \bar{c}_{12} & \bar{c}_{13} \\ \bar{c}_{21} & \bar{c}_{22} & \bar{c}_{23} \\ \bar{c}_{31} & \bar{c}_{32} & \bar{c}_{33} \end{bmatrix}^{-1}, \bar{c}_{44}^{-1}, \bar{c}_{55}^{-1}, \bar{c}_{66}^{-1} \right\}. \tag{57}$$

Thus,

$$\mathbf{\Lambda}^{-1} = \sum_{r,s=1}^{3} \bar{s}_{rs}\mathbf{E}_{rs} + \bar{s}_{44}\mathbf{E}_{44} + \bar{s}_{55}\mathbf{E}_{55} + \bar{s}_{66}\mathbf{E}_{66}, \tag{58}$$

where

$$\begin{aligned}
&\bar{s}_{11} = (\bar{c}_{22}\bar{c}_{33} - \bar{c}_{23}^2)/\bar{c}, && \bar{s}_{12} = \bar{s}_{21} = (\bar{c}_{31}\bar{c}_{23} - \bar{c}_{12}\bar{c}_{33})/\bar{c}, && \bar{s}_{22} = (\bar{c}_{11}\bar{c}_{33} - \bar{c}_{31}^2)/\bar{c}, \\
&\bar{s}_{23} = \bar{s}_{32} = (\bar{c}_{12}\bar{c}_{31} - \bar{c}_{11}\bar{c}_{23})/\bar{c}, && \bar{s}_{31} = \bar{s}_{13} = (\bar{c}_{12}\bar{c}_{23} - \bar{c}_{31}\bar{c}_{22})/\bar{c}, && \bar{s}_{33} = (\bar{c}_{11}\bar{c}_{22} - \bar{c}_{12}^2)/\bar{c}, \\
&\bar{s}_{44} = 1/\bar{c}_{44}, && \bar{s}_{55} = 1/\bar{c}_{55}, && \bar{s}_{66} = 1/\bar{c}_{66},
\end{aligned} \tag{59}$$

and

$$\bar{c} = \bar{c}_{11}\bar{c}_{22}\bar{c}_{33} + 2\bar{c}_{12}\bar{c}_{23}\bar{c}_{31} - \bar{c}_{11}\bar{c}_{23}^2 - \bar{c}_{22}\bar{c}_{31}^2 - \bar{c}_{33}\bar{c}_{12}^2. \tag{60}$$

The results for transversely isotropic materials can be recovered from the above results by taking $\bar{c}_{22} = \bar{c}_{11}$, $\bar{c}_{23} = \bar{c}_{31}$, and $\bar{c}_{55} = \bar{c}_{44} = \bar{c}_{11} - \bar{c}_{12}$.

The compliances tensor can also be derived by the inversion of the elastic moduli tensor expressed with respect to the set of base tensors used in Equation (55). This gives

$$\mathbf{\Lambda}^{-1} = \sum_{r,s=1}^{3} \hat{s}_{rs}\mathbf{E}_{rs} + \hat{s}_{44}\mathbf{A}\circ\boldsymbol{\delta} + \hat{s}_{55}\mathbf{B}\circ\boldsymbol{\delta} + \hat{s}_{66}\mathbf{C}\circ\boldsymbol{\delta}, \tag{61}$$

with the corresponding compliances

$$\begin{aligned}
&\hat{s}_{11} = \bar{s}_{11} - \bar{s}_{66} + \bar{s}_{44} - \bar{s}_{55}, && \hat{s}_{22} = \bar{s}_{22} - \bar{s}_{44} + \bar{s}_{55} - \bar{s}_{66}, && \hat{s}_{33} = \bar{s}_{33} - \bar{s}_{44} + \bar{s}_{66} - \bar{s}_{55}, \\
&\hat{s}_{12} = \bar{s}_{12}, && \hat{s}_{23} = \bar{s}_{23}, && \hat{s}_{31} = \bar{s}_{31}, \\
&\hat{s}_{44} = \tfrac{1}{2}\,(\bar{s}_{55} + \bar{s}_{66} - \bar{s}_{44}), && \hat{s}_{55} = \tfrac{1}{2}\,(\bar{s}_{44} + \bar{s}_{66} - \bar{s}_{55}), && \hat{s}_{66} = \tfrac{1}{2}\,(\bar{s}_{44} + \bar{s}_{55} - \bar{s}_{66}).
\end{aligned} \tag{62}$$

3.2. ***Relationships to engineering constants in Voigt notation.*** The elastic moduli c_r are related to commonly used engineering moduli C_{ij}, appearing in Equation (25), by

$$\begin{aligned} &c_1 = 2(C_{44}+C_{55}-C_{66}), && c_2 = C_{33}-c_1, && c_3 = c_4 = C_{31}-c_2, \\ &c_5 = c_6 = C_{23}-c_2, && c_7 = 2(C_{66}-C_{44}), && c_8 = 2(C_{66}-C_{55}), \\ &c_9 = C_{11}+C_{33}-2C_{31}-4C_{55}, && c_{10} = C_{22}+C_{33}-2C_{23}-4C_{44}, && c_{11} = c_{12} = c_2+C_{12}-C_{23}-C_{31}. \end{aligned} \tag{63}$$

The relationships between $\bar{c}_{ij}$ and C_{ij} are

$$\begin{aligned} &\bar{c}_{11} = C_{11}, && \bar{c}_{22} = C_{22}, && \bar{c}_{33} = C_{33}, \\ &\bar{c}_{12} = C_{12}, && \bar{c}_{23} = C_{23}, && \bar{c}_{31} = C_{31}, \\ &\bar{c}_{44} = 2C_{44}, && \bar{c}_{55} = 2C_{55}, && \bar{c}_{66} = 2C_{66}. \end{aligned} \tag{64}$$

To derive the relationship between the moduli $\hat{c}_{ij}$ appearing the stiffness representation Equation (55) and the moduli $\bar{c}_{ij}$ or C_{ij}, we first note that

$$\mathbf{C} \circ \boldsymbol{\delta} = (\boldsymbol{\delta} - \mathbf{A} - \mathbf{B}) \circ \boldsymbol{\delta} = 2\mathbf{I}_1 - \mathbf{I}_7 - \mathbf{I}_8. \tag{65}$$

It then readily follows that

$$\begin{aligned} &\hat{c}_{11} = \bar{c}_{11} - \bar{c}_{66} + \bar{c}_{44} - \bar{c}_{55}, && \hat{c}_{12} = \bar{c}_{12}, && \hat{c}_{23} = \bar{c}_{23}, \\ &\hat{c}_{22} = \bar{c}_{22} - \bar{c}_{44} + \bar{c}_{55} - \bar{c}_{66}, && \hat{c}_{44} = \tfrac{1}{2}(\bar{c}_{55} + \bar{c}_{66} - \bar{c}_{44}), && \hat{c}_{55} = \tfrac{1}{2}(\bar{c}_{44} + \bar{c}_{66} - \bar{c}_{55}), \\ &\hat{c}_{33} = \bar{c}_{33} - \bar{c}_{44} + \bar{c}_{66} - \bar{c}_{55}, && \hat{c}_{31} = \bar{c}_{31}, && \hat{c}_{66} = \tfrac{1}{2}(\bar{c}_{44} + \bar{c}_{55} - \bar{c}_{66}), \end{aligned} \tag{66}$$

in agreement with the corresponding expressions derived by other means in [Boehler 1987].

The compliance coefficients $\bar{s}_{ij}$ are related to the usual engineering constants of orthotropic materials (defined with respect to its principal axes of orthotropy **a**, **b**, and **c**) by

$$\begin{aligned} &\bar{s}_{11} = \frac{1}{E_a}, && \bar{s}_{12} = -\frac{\nu_{ab}}{E_a}, && \bar{s}_{13} = -\frac{\nu_{ac}}{E_a}, \\ &\bar{s}_{22} = \frac{1}{E_b}, && \bar{s}_{23} = -\frac{\nu_{bc}}{E_b}, && \bar{s}_{33} = \frac{1}{E_c}, \\ &\bar{s}_{44} = \frac{1}{2G_{bc}}, && \bar{s}_{55} = \frac{1}{2G_{ac}}, && \bar{s}_{66} = \frac{1}{2G_{ab}}. \end{aligned} \tag{67}$$

The compliances $\bar{s}_{ij}$ are actually equal to S_{ij}, except for the shear compliances which are related by the coefficient of 2 (that is, $S_{44} = 2\bar{s}_{44}$, $S_{55} = 2\bar{s}_{55}$, and $S_{66} = 2\bar{s}_{66}$).

To express the elastic compliances tensor in terms of the original base tensors, we first express the tensors $\mathbf{I}_r$ in terms of the tensors $\mathbf{E}_{rs}$. The connections are

$$\begin{aligned}
&\mathbf{I}_1 = \mathbf{E}_{11} + \mathbf{E}_{22} + \mathbf{E}_{33} + \mathbf{E}_{44} + \mathbf{E}_{55} + \mathbf{E}_{66},\\
&\mathbf{I}_2 = \mathbf{E}_{11} + \mathbf{E}_{12} + \mathbf{E}_{13} + \mathbf{E}_{21} + \mathbf{E}_{22} + \mathbf{E}_{23} + \mathbf{E}_{31} + \mathbf{E}_{32} + \mathbf{E}_{33},\\
&\mathbf{I}_3 = \mathbf{E}_{11} + \mathbf{E}_{12} + \mathbf{E}_{13}, \quad \mathbf{I}_4 = \mathbf{E}_{11} + \mathbf{E}_{21} + \mathbf{E}_{31},\\
&\mathbf{I}_5 = \mathbf{E}_{21} + \mathbf{E}_{22} + \mathbf{E}_{23}, \quad \mathbf{I}_6 = \mathbf{E}_{12} + \mathbf{E}_{22} + \mathbf{E}_{32},\\
&\mathbf{I}_7 = 2\mathbf{E}_{11} + \mathbf{E}_{55} + \mathbf{E}_{66}, \quad \mathbf{I}_8 = 2\mathbf{E}_{22} + \mathbf{E}_{44} + \mathbf{E}_{66},\\
&\mathbf{I}_9 = \mathbf{E}_{11}, \quad \mathbf{I}_{10} = \mathbf{E}_{22}, \quad \mathbf{I}_{11} = \mathbf{E}_{12}, \quad \mathbf{I}_{12} = \mathbf{E}_{21},
\end{aligned} \tag{68}$$

with the inverse expressions

$$\begin{aligned}
&\mathbf{E}_{11} = \mathbf{I}_9, \quad \mathbf{E}_{12} = \mathbf{I}_{11}, \quad \mathbf{E}_{13} = \mathbf{I}_3 - \mathbf{I}_9 - \mathbf{I}_{11},\\
&\mathbf{E}_{21} = \mathbf{I}_{12}, \quad \mathbf{E}_{22} = \mathbf{I}_{10}, \quad \mathbf{E}_{23} = \mathbf{I}_5 - \mathbf{I}_{10} - \mathbf{I}_{12},\\
&\mathbf{E}_{31} = \mathbf{I}_4 - \mathbf{I}_9 - \mathbf{I}_{12}, \quad \mathbf{E}_{32} = \mathbf{I}_6 - \mathbf{I}_{10} - \mathbf{I}_{11},\\
&\mathbf{E}_{33} = \mathbf{I}_2 - \mathbf{I}_3 - \mathbf{I}_4 - \mathbf{I}_5 - \mathbf{I}_6 + \mathbf{I}_9 + \mathbf{I}_{10} + \mathbf{I}_{11} + \mathbf{I}_{12},\\
&\mathbf{E}_{44} = \mathbf{I}_1 - \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4 + \mathbf{I}_5 + \mathbf{I}_6 - \mathbf{I}_7 - 2\mathbf{I}_{10} - \mathbf{I}_{11} - \mathbf{I}_{12},\\
&\mathbf{E}_{55} = \mathbf{I}_1 - \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4 + \mathbf{I}_5 + \mathbf{I}_6 - \mathbf{I}_8 - 2\mathbf{I}_9 - \mathbf{I}_{11} - \mathbf{I}_{12}\\
&\mathbf{E}_{66} = -\mathbf{I}_1 + \mathbf{I}_2 - \mathbf{I}_3 - \mathbf{I}_4 - \mathbf{I}_5 - \mathbf{I}_6 + \mathbf{I}_7 + \mathbf{I}_8 + \mathbf{I}_{11} + \mathbf{I}_{12}.
\end{aligned} \tag{69}$$

The substitution of Equation (69) into (58) then gives

$$\boldsymbol{\Lambda}^{-1} = \sum_{r=1}^{12} s_r \mathbf{I}_r. \tag{70}$$

The corresponding elastic compliances are

$$\begin{aligned}
&s_1 = \bar{s}_{55} + \bar{s}_{66} - \bar{s}_{44}, && s_2 = \bar{s}_{33} + \bar{s}_{44} - \bar{s}_{55} - \bar{s}_{66},\\
&s_3 = s_4 = \bar{s}_{13} - \bar{s}_{33} - \bar{s}_{44} + \bar{s}_{55} + \bar{s}_{66}, && s_5 = s_6 = \bar{s}_{23} - \bar{s}_{33} - \bar{s}_{44} + \bar{s}_{55} + \bar{s}_{66},\\
&s_7 = \bar{s}_{44} - \bar{s}_{55}, && s_8 = \bar{s}_{44} - \bar{s}_{66},\\
&s_9 = \bar{s}_{11} - 2\bar{s}_{31} + \bar{s}_{33} - 2\bar{s}_{66}, && s_{10} = \bar{s}_{22} - 2\bar{s}_{23} + \bar{s}_{33} - 2\bar{s}_{55},\\
&s_{11} = s_{12} = \bar{s}_{21} - \bar{s}_{23} - \bar{s}_{31} + \bar{s}_{33} + \bar{s}_{44} - \bar{s}_{55} - \bar{s}_{66}.
\end{aligned} \tag{71}$$

Since $\bar{s}_{rs}$ are specified in terms of $\bar{c}_{rs}$ by Equation (59), and $\bar{c}_{rs}$ in terms of c_r by Equation (49), the compliances s_r in Equation (71) are all expressed in terms of the elastic moduli c_r.

The relationships Equation (71) reveal the connections between the original moduli appearing in Equation (41) and the engineering moduli. The resulting expressions in an explicit form are lengthy, but their numerical evaluations are simple. For example, the Young's modulus in the direction of principal orthotropy **a** is

$$E_a = \frac{1}{\bar{s}_{11}} = \frac{\bar{c}}{\bar{c}_{22}\bar{c}_{33} - \bar{c}_{23}^2}, \tag{72}$$

c_1	c_2	$c_3=c_4$	$c_5=c_6$	c_7	c_8	c_9	c_{10}	$c_{11}=c_{12}$
14.64	12.96	−2.86	−2.26	−3.42	−2.18	2.96	1.48	2.14

Table 6. Elastic moduli c_i (in units of GPa).

where $\bar{c}_{ij}$ are given by Equation (49) and $\bar{c}$ by (60). In the case of transverse isotropy, with the axis of isotropy parallel to **c** ($\bar{c}_{22}=\bar{c}_{11}$, $\bar{c}_{23}=\bar{c}_{31}$, and $\bar{c}_{55}=\bar{c}_{44}=\bar{c}_{11}-\bar{c}_{12}$), this reduces to Equation (72), while in the isotropic case ($\alpha_1=\alpha_2=\beta_1=\beta_2=\beta_3=0$ and $\mu_1=\mu_2=\mu$), we recover the result $E=\mu(3\lambda+2\mu)/(\lambda+\mu)$.

3.3. ***Elastic constants of a human femur.*** Elastic moduli C_{ij} of a human femural bone, determined by ultrasound measurement technique, were reported in Ashman and Buskirk (1987) as

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 18 & 9.98 & 10.1 \\ 9.98 & 20.2 & 10.7 \\ 10.1 & 10.7 & 27.6 \end{bmatrix}, \qquad \begin{bmatrix} C_{44} \\ C_{55} \\ C_{66} \end{bmatrix} = \begin{bmatrix} 6.23 \\ 5.61 \\ 4.52 \end{bmatrix} \qquad \text{(GPa)}.$$

The corresponding moduli $\bar{c}_{ij}$ and $\hat{c}_{ij}$ follow from (64) and (66). In units of GPa, they are

$$\begin{bmatrix} \bar{c}_{11} & \bar{c}_{12} & \bar{c}_{13} \\ \bar{c}_{21} & \bar{c}_{22} & \bar{c}_{23} \\ \bar{c}_{31} & \bar{c}_{32} & \bar{c}_{33} \end{bmatrix} = \begin{bmatrix} 18 & 9.98 & 10.1 \\ 9.98 & 20.2 & 10.7 \\ 10.1 & 10.7 & 27.6 \end{bmatrix}, \qquad \begin{bmatrix} \bar{c}_{44} \\ \bar{c}_{55} \\ \bar{c}_{66} \end{bmatrix} = \begin{bmatrix} 12.46 \\ 11.22 \\ 9.04 \end{bmatrix},$$

$$\begin{bmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} \\ \hat{c}_{21} & \hat{c}_{22} & \hat{c}_{23} \\ \hat{c}_{31} & \hat{c}_{32} & \hat{c}_{33} \end{bmatrix} = \begin{bmatrix} 10.2 & 9.98 & 10.1 \\ 9.98 & 9.92 & 10.7 \\ 10.1 & 10.7 & 12.96 \end{bmatrix}, \qquad \begin{bmatrix} \hat{c}_{44} \\ \hat{c}_{55} \\ \hat{c}_{66} \end{bmatrix} = \begin{bmatrix} 3.9 \\ 5.14 \\ 7.32 \end{bmatrix}.$$

The moduli c_i, calculated from Equation (63), are listed in Table 6.

The elastic compliances S_{ij} (in units of TPa^{-1}), corresponding to elastic moduli C_{ij} given above, are

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \begin{bmatrix} 83.24 & -31.45 & -18.27 \\ -31.45 & 74.18 & -17.25 \\ -18.27 & -17.25 & 49.60 \end{bmatrix}, \qquad \begin{bmatrix} S_{44} \\ S_{55} \\ S_{66} \end{bmatrix} = \begin{bmatrix} 160.52 \\ 178.26 \\ 221.24 \end{bmatrix}.$$

The compliances $\bar{s}_{ij}$ and $\hat{s}_{ij}$ follow from (59) and (62). In units of TPa^{-1}, they are

$$\begin{bmatrix} \bar{s}_{11} & \bar{s}_{12} & \bar{s}_{13} \\ \bar{s}_{21} & \bar{s}_{22} & \bar{s}_{23} \\ \bar{s}_{31} & \bar{s}_{32} & \bar{s}_{33} \end{bmatrix} = \begin{bmatrix} 83.24 & -31.45 & -18.27 \\ -31.45 & 74.18 & -17.25 \\ -18.27 & -17.25 & 49.60 \end{bmatrix}, \qquad \begin{bmatrix} \bar{s}_{44} \\ \bar{s}_{55} \\ \bar{s}_{66} \end{bmatrix} = \begin{bmatrix} 80.26 \\ 89.13 \\ 110.62 \end{bmatrix},$$

$$\begin{bmatrix} \hat{s}_{11} & \hat{s}_{12} & \hat{s}_{13} \\ \hat{s}_{21} & \hat{s}_{22} & \hat{s}_{23} \\ \hat{s}_{31} & \hat{s}_{32} & \hat{s}_{33} \end{bmatrix} = \begin{bmatrix} -36.24 & -31.45 & -18.27 \\ -31.45 & -27.57 & -17.25 \\ -18.27 & -17.25 & -9.16 \end{bmatrix}, \qquad \begin{bmatrix} \hat{s}_{44} \\ \hat{s}_{55} \\ \hat{s}_{66} \end{bmatrix} = \begin{bmatrix} 59.74 \\ 50.87 \\ 29.38 \end{bmatrix}.$$

The above results show that for human femur S_{ij} and $\bar{s}_{ij}$ are negative for $i \neq j$, while $\hat{s}_{ij}$ are negative for all $i,\ j=1,2,3$. The compliance coefficients s_i, calculated from Equation (71), are listed in Table 7. The longitudinal and shear moduli and the Poisson ratios, with respect to the principal axes of

s_1	s_2	$s_3{=}s_4$	$s_5{=}s_6$	s_7	s_8	s_9	s_{10}	$s_{11}{=}s_{12}$
119.49	−69.88	51.61	52.63	−8.87	−30.36	−51.85	−19.97	−65.81

Table 7. Elastic compliances s_i (in units of TPa^{-1}).

E_a	E_b	E_c	G_{ab}	G_{bc}	G_{ac}	ν_{ab}	ν_{bc}	ν_{ac}
12.01	13.48	20.16	4.52	6.23	5.61	0.378	0.233	0.219

Table 8. Longitudinal and shear moduli (in units of GPa) and the Poisson ratios.

orthotropy, follow from Equation (67) and are given in Table 8. The remaining three Poisson's ratios are determined from the well known connections $\nu_{ba} = \nu_{ab}E_b/E_a = 0.424$, $\nu_{cb} = \nu_{bc}E_c/E_b = 0.348$, and $\nu_{ca} = \nu_{ac}E_c/E_a = 0.368$.

4. Conclusion

In this paper we have derived the relationships between the elastic moduli and compliances of transversely isotropic materials, which correspond to two different sets of linearly independent fourth-order base tensors used to cast the tensorial representation of the elastic moduli and compliances tensors. The two sets of elastic constants are related to engineering constants defined with respect to the coordinate system in which one of the axes is parallel to the axis of material symmetry. Extending the analysis to orthotropic materials, three different representations of the elastic moduli tensor are constructed by choosing three appealing sets of twelve linearly independent fourth-order base tensors. This was accomplished on the basis of the representation theorems for orthotropic tensor functions of a symmetric second-order tensor and the structural tensors associated with the principal axes of orthotropy. The three sets of the corresponding elastic moduli are related to each other. The compliances tensor is deduced by an explicit inversion of the stiffness tensor for all considered sets of the base tensors. The different compliances are related to each other, and to classical engineering constants expressed in the Voigt notation. The formulas are applied to calculate the elastic constants for a transversely isotropic monocrystalline zinc and an orthotropic human femural bone. Apart from the analytical point of view, the derived results may be of interest for the analysis of anisotropic elastic and inelastic material response (with an elastic component of strain or stress) in the mechanics of fiber reinforced composite materials [Hyer 1998; Vasiliev and Morozov 2001], creep mechanics [Drozdov 1998; Betten 2002], damage-elastoplasticity [Lubarda 1994], mechanics of brittle materials weakened by anisotropic crack distributions [Krajcinovic 1996; Voyiadjis and Kattan 1999], and biological materials (membranes or tissues) with embedded filament networks [Evans and Skalak 1980; Fung 1990; Humphrey 2002; Lubarda and Hoger 2002; Asaro and Lubarda 2006].

↱	$\mathbf{I}_1$	$\mathbf{I}_2$	$\mathbf{I}_3$	$\mathbf{I}_4$	$\mathbf{I}_5$	$\mathbf{I}_6$	$\mathbf{I}_7$	$\mathbf{I}_8$	$\mathbf{I}_9$	$\mathbf{I}_{10}$	$\mathbf{I}_{11}$	$\mathbf{I}_{12}$
$\mathbf{I}_1$	$\mathbf{I}_1$	$\mathbf{I}_2$	$\mathbf{I}_3$	$\mathbf{I}_4$	$\mathbf{I}_5$	$\mathbf{I}_6$	$\mathbf{I}_7$	$\mathbf{I}_8$	$\mathbf{I}_9$	$\mathbf{I}_{10}$	$\mathbf{I}_{11}$	$\mathbf{I}_{12}$
$\mathbf{I}_2$	$\mathbf{I}_2$	$3\mathbf{I}_2$	$\mathbf{I}_2$	$3\mathbf{I}_4$	$\mathbf{I}_2$	$3\mathbf{I}_6$	$2\mathbf{I}_4$	$2\mathbf{I}_6$	$\mathbf{I}_4$	$\mathbf{I}_6$	$\mathbf{I}_6$	$\mathbf{I}_4$
$\mathbf{I}_3$	$\mathbf{I}_3$	$3\mathbf{I}_3$	$\mathbf{I}_3$	$3\mathbf{I}_9$	$\mathbf{I}_3$	$3\mathbf{I}_{11}$	$2\mathbf{I}_9$	$2\mathbf{I}_{11}$	$\mathbf{I}_9$	$\mathbf{I}_{11}$	$\mathbf{I}_{11}$	$\mathbf{I}_9$
$\mathbf{I}_4$	$\mathbf{I}_4$	$\mathbf{I}_2$	$\mathbf{I}_2$	$\mathbf{I}_4$	0	$\mathbf{I}_6$	$2\mathbf{I}_4$	0	$\mathbf{I}_4$	0	$\mathbf{I}_6$	0
$\mathbf{I}_5$	$\mathbf{I}_5$	$3\mathbf{I}_5$	$\mathbf{I}_5$	$3\mathbf{I}_{12}$	$\mathbf{I}_5$	$3\mathbf{I}_{10}$	$2\mathbf{I}_{12}$	$2\mathbf{I}_{10}$	$\mathbf{I}_{12}$	$\mathbf{I}_{10}$	$\mathbf{I}_{10}$	$\mathbf{I}_{12}$
$\mathbf{I}_6$	$\mathbf{I}_6$	$\mathbf{I}_2$	0	$\mathbf{I}_4$	$\mathbf{I}_2$	$\mathbf{I}_6$	0	$2\mathbf{I}_6$	0	$\mathbf{I}_6$	0	$\mathbf{I}_4$
$\mathbf{I}_7$	$\mathbf{I}_7$	$2\mathbf{I}_3$	$2\mathbf{I}_3$	$2\mathbf{I}_9$	0	$2\mathbf{I}_{11}$	$\mathbf{I}_7 + 2\mathbf{I}_9$	$\mathbf{A} \circ \mathbf{B}$	$2\mathbf{I}_9$	0	$2\mathbf{I}_{11}$	0
$\mathbf{I}_8$	$\mathbf{I}_8$	$2\mathbf{I}_5$	0	$2\mathbf{I}_{12}$	$2\mathbf{I}_5$	$2\mathbf{I}_{10}$	$\mathbf{A} \circ \mathbf{B}$	$\mathbf{I}_8 + 2\mathbf{I}_{10}$	0	$2\mathbf{I}_{10}$	0	$2\mathbf{I}_{12}$
$\mathbf{I}_9$	$\mathbf{I}_9$	$\mathbf{I}_3$	$\mathbf{I}_3$	$\mathbf{I}_9$	0	$\mathbf{I}_{11}$	$2\mathbf{I}_9$	0	$\mathbf{I}_9$	0	$\mathbf{I}_{11}$	0
$\mathbf{I}_{10}$	$\mathbf{I}_{10}$	$\mathbf{I}_5$	0	$\mathbf{I}_{12}$	$\mathbf{I}_5$	$\mathbf{I}_{10}$	0	$2\mathbf{I}_{10}$	0	$\mathbf{I}_{10}$	0	$\mathbf{I}_{12}$
$\mathbf{I}_{11}$	$\mathbf{I}_{11}$	$\mathbf{I}_3$	0	$\mathbf{I}_9$	$\mathbf{I}_3$	$\mathbf{I}_{11}$	0	$2\mathbf{I}_{11}$	0	$\mathbf{I}_{11}$	0	$\mathbf{I}_9$
$\mathbf{I}_{12}$	$\mathbf{I}_{12}$	$\mathbf{I}_5$	$\mathbf{I}_5$	$\mathbf{I}_{12}$	0	$\mathbf{I}_{10}$	$2\mathbf{I}_{12}$	0	$\mathbf{I}_{12}$	0	$\mathbf{I}_{10}$	0

Table 9. Table of products $\mathbf{I}_r : \mathbf{I}_s$.

↱	$\mathbf{E}_{11}$	$\mathbf{E}_{12}$	$\mathbf{E}_{13}$	$\mathbf{E}_{21}$	$\mathbf{E}_{22}$	$\mathbf{E}_{23}$	$\mathbf{E}_{31}$	$\mathbf{E}_{32}$	$\mathbf{E}_{33}$	$\mathbf{E}_{44}$	$\mathbf{E}_{55}$	$\mathbf{E}_{66}$
$\mathbf{E}_{11}$	$\mathbf{E}_{11}$	$\mathbf{E}_{12}$	$\mathbf{E}_{13}$	0	0	0	0	0	0	0	0	0
$\mathbf{E}_{12}$	0	0	0	$\mathbf{E}_{11}$	$\mathbf{E}_{12}$	$\mathbf{E}_{13}$	0	0	0	0	0	0
$\mathbf{E}_{13}$	0	0	0	0	0	0	$\mathbf{E}_{11}$	$\mathbf{E}_{12}$	$\mathbf{E}_{13}$	0	0	0
$\mathbf{E}_{21}$	$\mathbf{E}_{21}$	$\mathbf{E}_{22}$	$\mathbf{E}_{23}$	0	0	0	0	0	0	0	0	0
$\mathbf{E}_{22}$	0	0	0	$\mathbf{E}_{21}$	$\mathbf{E}_{22}$	$\mathbf{E}_{23}$	0	0	0	0	0	0
$\mathbf{E}_{23}$	0	0	0	0	0	0	$\mathbf{E}_{21}$	$\mathbf{E}_{22}$	$\mathbf{E}_{23}$	0	0	0
$\mathbf{E}_{31}$	$\mathbf{E}_{31}$	$\mathbf{E}_{32}$	$\mathbf{E}_{33}$	0	0	0	0	0	0	0	0	0
$\mathbf{E}_{32}$	0	0	0	$\mathbf{E}_{31}$	$\mathbf{E}_{32}$	$\mathbf{E}_{33}$	0	0	0	0	0	0
$\mathbf{E}_{33}$	0	0	0	0	0	0	$\mathbf{E}_{31}$	$\mathbf{E}_{32}$	$\mathbf{E}_{33}$	0	0	0
$\mathbf{E}_{44}$	0	0	0	0	0	0	0	0	0	$\mathbf{E}_{44}$	0	0
$\mathbf{E}_{55}$	0	0	0	0	0	0	0	0	0	0	$\mathbf{E}_{55}$	0
$\mathbf{E}_{66}$	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{E}_{66}$

Table 10. Table of products $\mathbf{E}_{rs} : \mathbf{E}_{pq}$.

Appendix: Multiplication tables for tensors $\mathbf{I}_r$ and $\mathbf{E}_{rs}$

The trace products $\mathbf{I}_r : \mathbf{I}_s$ are listed in Table 9. The tensors $\mathbf{I}_r$ are defined in Equation (42). The symbol ↱ indicates that the products are in the order: *a tensor from the left column* traced with *a tensor from the top row*. The products $\mathbf{E}_{rs} : \mathbf{E}_{pq}$, where the tensors $\mathbf{E}_{rs}$ are defined in Equation (50) and (51), are listed in Table 10.

References

[Asaro and Lubarda 2006] R. J. Asaro and V. A. Lubarda, *Mechanics of solids and materials*, Cambridge University Press, Cambridge, 2006.

[Betten 1987] J. Betten, "Invariants of fourth-order tensors", pp. 203–226 in *Applications of tensor functions in solid mechanics*, edited by J. P. Boehler, CISM Courses and Lectures **292**, Springer-Verlag, Wien, 1987.

[Betten 2002] J. Betten, *Creep mechanics*, Springer-Verlag, Berlin, 2002.

[Boehler 1987] J. P. Boehler, "Representations for isotropic and anisotropic non-polynomial tensor functions", pp. 31–53 in *Applications of tensor functions in solid mechanics*, edited by J. P. Boehler, CISM Courses and Lectures **292**, Springer, Wien, 1987.

[Cowin and Mehrabadi 1995] S. C. Cowin and M. M. Mehrabadi, "Anisotropic symmetries in linear elasticity", *Appl. Mech. Rev.* **48** (1995), 247–285.

[Drozdov 1998] A. D. Drozdov, *Mechanics of Viscoelastic Solids*, John Wiley & Sons, New York, 1998.

[Evans and Skalak 1980] E. A. Evans and R. Skalak, *Mechanics and thermodynamics of biomembranes*, CRC Press, Boca Raton, Florida, 1980.

[Fung 1990] Y.-C. Fung, *Biomechanics: motion, flow, stress, and growth*, Springer-Verlag, New York, 1990.

[Gangi 2000] A. F. Gangi, "Fourth-order elastic-moduli tensors by inspection", pp. 1–10 in *Fractures, converted waves and case studies–proceedings of the ninth international workshop on seismic anisotropy*, Tulsa, OK, 2000.

[Hearmon 1961] R. F. E. Hearmon, *An introduction to applied anisotropic elasticity*, Clarendon Press, Oxford, 1961.

[Humphrey 2002] J. D. Humphrey, *Cardiovascular solid mechanics: cells, tissues, and organs*, Springer-Verlag, New York, 2002.

[Hyer 1998] M. W. Hyer, *Stress analysis of fiber-reinforced composite materials*, McGraw-Hill, Boston, 1998.

[Krajcinovic 1996] D. Krajcinovic, *Damage mechanics*, Elsevier, New York, 1996.

[Kunin 1981] I. A. Kunin, "An algebra of tensor operations and its applications to elasticity", *Int. J. Engng. Sci.* **19** (1981), 1551–1561.

[Kunin 1983] I. A. Kunin, *Elastic media with microstructure, part II*, Springer-Verlag, Berlin, 1983.

[Landolt-Börnstein 1979] Landolt-Börnstein, *Numerical data and functional relationships in science and technology*, vol. 11, edited by K.-H. Hellwege and A. M. Hellwege, Group III, Springer-Verlag, Berlin, 1979.

[Lekhnitskii 1981] S. G. Lekhnitskii, *Theory of elasticity of an anisotropic elastic body*, Mir Publishers, Moscow, 1981.

[Lubarda 1994] V. A. Lubarda, "An analysis of large-strain damage elastoplasticity", *Int. J. Solids Struct.* **31** (1994), 2951–2964.

[Lubarda and Hoger 2002] V. A. Lubarda and A. Hoger, "On the mechanics of solids with a growing mass", *Int. J. Solids Struct.* **39** (2002), 4627–4664.

[Lubarda and Krajcinovic 1994] V. A. Lubarda and D. Krajcinovic, "Tensorial representation of the effective elastic properties of the damaged material", *Int. J. Damage Mech.* **3** (1994), 38–56.

[Lubarda and Meyers 1999] V. A. Lubarda and M. A. Meyers, "On the negative Poisson ratio in monocrystalline zinc", *Scripta Mater.* **40** (1999), 975–977.

[Mehrabadi and Cowin 1990] M. M. Mehrabadi and S. C. Cowin, "Eigentensors of linear anisotropic elastic materials", *Q. Jl. Mech. Appl. Math.* **43** (1990), 15–41.

[Nadeau and Ferrari 1998] J. C. Nadeau and M. Ferrari, "Invariant tensor-to-matrix mappings for evaluation of tensorial expressions", *J. Elasticity* **52** (1998), 43–61.

[Nye 1964] J. F. Nye, *Physical properties of crystals: their representation by tensors and matrices*, Clarendon Press, Oxford, 1964.

[Rychlewski 1984] J. Rychlewski, "On Hooke's law", *J. Appl. Mech. Math.* **48** (1984), 303–314.

[Spencer 1982] A. J. M. Spencer, "The formulation of constitutive equation for anisotropic solids", pp. 2–26 in *Mechanical behavior of anisotropic solids*, edited by J. P. Boehler, Martinus Nijhoff Publishers, The Hague, 1982.

[Srinivasan and Nigam 1969] T. P. Srinivasan and S. D. Nigam, "Invariant elastic constants for crystals", *J. Math. Mech.* **19** (1969), 411–420.

[Sutcliffe 1992] S. Sutcliffe, "Spectral decomposition of the elasticity tensor", *J. Appl. Mech.* **59** (1992), 762–773.

[Theocaris and Philippidis 1989] P. S. Theocaris and T. P. Philippidis, "Elastic eigenstates of a medium with transverse isotropy", *Arch. Mech.* **41** (1989), 717–724.

[Theocaris and Sokolis 2000] P. S. Theocaris and D. P. Sokolis, "Spectral decomposition of the compliance fourth-rank tensor for orthotropic materials", *Arch. Appl. Mech.* **70** (2000), 289–306.

[Ting 1987] T. C. T. Ting, "Invariants of anisotropic elastic constants", *Q. Jl. Mech. Appl. Math.* **40** (1987), 431–448.

[Ting 2000] T. C. T. Ting, "Anisotropic elastic constants that are structurally invariant", *Q. Jl. Mech. Appl. Math.* **53** (2000), 511–523.

[Ting and He 2006] T. C. T. Ting and Q.-C. He, "Decomposition of elasticity tensors and tensors that are structurally invariant in three dimensions", *Q. Jl. Mech. Appl. Math.* **59** (2006), 323–341.

[Vasiliev and Morozov 2001] V. V. Vasiliev and E. V. Morozov, *Mechanics and analysis of composite materials*, Elsevier, Amsterdam, 2001.

[Voigt 1928] W. Voigt, *Lehrbuch der Kristallphysik*, Johnson Reprint Corporation, New York, 1928. (1966 reprint).

[Voyiadjis and Kattan 1999] G. Z. Voyiadjis and P. I. Kattan, *Advances in damage mechanics: metals and metal matrix composites*, Elsevier, New York, 1999.

[Walpole 1984] L. J. Walpole, "Fourth-rank tensors of the thirty-two crystal classes: multiplication tables", *Proc. Royal Soc. London A* **391** (1984), 149–179.

[Zheng 1994] Q.-S. Zheng, "Theory of representations for tensor functions—a unified invariant approach to constitutive equations", *Appl. Mech. Rev.* **47** (1994), 545–587.

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VLADO A. LUBARDA: vlubarda@ucsd.edu
Department of Mechanical and Aerospace Engineering, University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0411, United States

MICHELLE C. CHEN: mcchen@ucsd.edu
Department of Structural Engineering, University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0085, United States