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A GLOBAL COLLOCATION METHOD FOR TWO-DIMENSIONAL RECTANGULAR DOMAINS

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This paper proposes the use of a global collocation procedure in conjunction with a previously developed functional set suitable for the numerical solution of Poisson's equation in rectangular domains. We propose to expand the unknown variable in a bivariate series of monomials $x^i y^j$ that exist in Pascal's triangle. We also propose the use of the bivariate Gordon–Coons interpolation, apart from previous intuitive choices of the aforementioned monomials. The theory is sustained by two numerical examples of Dirichlet boundary conditions, in which we find that the approximate solution monotonically converges towards the exact solution.

1. Introduction

The use of collocation methods for the solution of partial differential equations (PDE) has become a subject of intensive interest in the past [Lanczos 1938; Ronto 1971; Russell and Shampine 1972; Finlayson 1972; De Boor and Swartz 1973; Diaz 1977; Houstis 1978; Botha and Pinder 1983]. While most research has focused on the local interpolation of a variable, not many works have appeared concerning the global collocation in two-dimensional problems [Frind and Pinder 1979; Hayes 1980; Van Blerk and Botha 1993]. In contrast, global approximations have been successfully applied in conjunction with the standard Galerkin–Ritz procedure either using the Gordon–Coons bivariate interpolation [Cavendish et al. 1976] or higher order p-methods [Szabó and Babuška 1991].

Moreover, closely related to the abovementioned p-methods [Szabó and Babuška 1991], an alternative Gordon–Coons method has been proposed for the static and dynamic analysis of structures [Provatidis 2006a; 2006b; 2006c; 2006d]. In these works, the standard Galerkin–Ritz procedure was applied, thus leading to stiffness and mass matrices in the form of integrals over the entire domain. This method has been called the *Coons patch macroelement* (CPM) approach [Provatidis 2006a; 2006b; 2006c; 2006d]. Although CPM works perfectly well, and can deal with even Π-shaped domains [Provatidis 2006a; 2006d], one could say that it has the disadvantage of fully populated matrices particularly when Lagrange polynomials (with no compact support) are used.

This paper contributes to overcoming the abovementioned shortcoming and modifies the previous CPM methodology first by preserving the global functional space, and second by moving from the Galerkin–Ritz to a *collocation* procedure in which no integral is needed. Quite recently, as a pilot study, this idea has been successfully applied to one-dimensional eigenvalue problems [Provatidis 2007].

It should be noted that the proposed collocation method could be formulated in terms of the global cardinal shape functions of the CPM approach, which are applicable even to curvilinear domains [Provatidis 2006a; 2006b; 2006d]. However, the aforementioned shape functions are derived through a numerical

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procedure in which the original information is lost. It is therefore preferable to work instead in conjunction with the basis functions. It has recently been shown that when the Gordon–Coons interpolation is applied to the reference $\xi\eta$-square ($0 \le \xi, \eta \le 1$), it leads to the functional space, $\xi^i \eta^j$, which consists of the well known monomials of Pascal's triangle [Provatidis 2006a; 2006b; 2006d]. Obviously, in the case of a rectangular domain, these monomials correspond to the usual $x^i y^j$ monomials that appear in the classical Taylor's series expansion. It is important to mention here that while an intuitively selected part (that is, relevant indices i, j) of a Taylor's series expansion may lead to a noninvertible equation matrix [Zienkiewicz 1977, p.151], the Gordon–Coons interpolation [Gordon 1971] always ensures a smooth approximation because it is based on the solid foundations of Computational Geometry [Farin 1990]. It is remarkable that the ideas just described can be extended to three-dimensional domains, that is, to the reference $\xi\eta\zeta$-cube [Provatidis 2006c].

In brief, this paper utilizes the advantage of the abovementioned complete functional set, and applies a global collocation that achieves solutions to boundary value problems in rectangular domains. In this work, the performance of the proposed method is shown in two examples under the Dirichlet boundary conditions.

2. The proposed global functional set

Let us consider a rectangular domain $\Omega = (ABCD) = [0, a] \times [0, b]$ in $\mathbb{R}^2$. The axis origin is chosen to be the corner A, and the Cartesian axes x and y lie on the sides AB and AD, respectively. The sides (AB, CD) and (BC, DA) are divided into p and q segments, respectively, thus leading to $(p+1)$ nodes along AB or CD, as well as $(q+1)$ nodes along BC or DA. As a result, the univariate function $u(x, 0)$ along the side AB can be interpolated through a polynomial of p-th degree in x (thus including the set $\{1, x, x^2, \ldots, x^p\}$), while the function $u(0, y)$ along the side DA can be interpolated through a polynomial of q-th degree in y (thus including the set $\{1, y, y^2, \ldots, y^q\}$).

2.1. ***Boundary functions.*** Given the univariate polynomial degrees p and q, one of the most apparent extensions in the two-dimensional approximation is perhaps the use of Coons' interpolation [Farin 1990], which assumes a linear blending of the aforementioned univariate polynomials. If $u \in C^{2,2}(\Omega_{st})$ where $\Omega_{st} = [0, 1] \times [0, 1]$ is the *standard* reference square, then in terms of normalized coordinates, ξ and η ($0 \le \xi, \eta \le 1$), Coons interpolation can be written as:

$$\tilde{u}(\xi, \eta) = (1-\xi)u(0, \eta) + \xi u(1, \eta) + (1-\eta)u(\xi, 0) + \eta u(\xi, 1) - \tilde{\tilde{u}}(\xi, \eta), \tag{1}$$

where

$$\tilde{\tilde{u}}(\xi, \eta) = (1-\xi)(1-\eta)u(0, 0) + \xi(1-\eta)u(1, 0) + (1-\xi)\eta u(0, 1) + \xi\eta u(1, 1).$$

For the rectangular domain, it holds that $x = a\xi$ and $y = b\eta$, and hereafter we replace the normalized coordinates by the Cartesian ones. Therefore, the polynomial set $\{1, x, x^2, \ldots, x^p\}$ along $AB(\eta = 0, y = 0)$ is projected by the blending function $(1 - y/b)$ towards the y-axis, thus including the monomials involved in the product $\{1, x, x^2, \ldots, x^p\}(1 - y/b)$. In an analogous way, the interpolation along DA ($\xi = 0, x = 0$) leads to the terms involved in the product $\{1, y, y^2, \ldots, y^q\}(1 - x/a)$. In summary, for the case of nodal points arranged along the boundary of the rectangular domain $ABCD$, the variable $u(x, y)$ is expanded in a series (Equation (3)) comprising the aforementioned $2(p+q)$ monomials that compose a set, here

denoted as S_0, which constitutes the 'legs' of Pascal's triangle with a surplus ('thickness') of two layers of terms. The particular case of this scheme for $p = q$ can be found in standard textbooks such as [Zienkiewicz 1977, p.160].

The above justified choice constitutes a subset of the space $S^p(\Omega_{st}^{(q)})$ of polynomials on the unit reference square, $\Omega_{st}^{(q)}$, which was introduced by Szabó and Babuška in their pioneering works concerning higher-order p-methods [Szabó and Babuška 1991]. The space $S^p(\Omega_{st}^{(q)})$ is spanned by the set of monomials

$$x^i y^j, \quad i, j = 0, 1, \ldots, p, \quad i + j = 0, 1, \ldots, p,$$

supplemented by the following monomials:

(a) in the case $p = 1$, the monomial xy;

(b) in the case $p \geq 2$, the monomials $x^p y$ and xy^p [Szabó and Babuška 1991, p.97].

In the particular case that $p = q$, it is trivial to verify that the proposed set of equations (Equation (1)) does *not* coincide with that proposed by Szabó and Babuška [1991]. For example, in the case of $p = 4$, the proposed Coons-based functional set misses the term x^2y^2 (it belongs to the third layer parallel to the 'legs' of Pascal's triangle), which can, however, easily be obtained in the sequence by considering one internal node (see Section 2.2).

2.2. ***Internal functions.*** In order to get full polynomials, it is necessary to use internal nodes where the PDE will be collocated. One extreme case is to use $(p-1) \times (q-1)$ internal nodes, a choice that corresponds to the well known Lagrange type finite element. The corresponding space is denoted by $S^{p,q}(\Omega_{st}^{(q)})$ [Szabó and Babuška 1991, p.97] and it is spanned by the set of all monomials

$$x^i y^j, \quad i = 0, 1, 2, \ldots, p, \quad j = 0, 1, 2, \ldots, q.$$

Alternatively, the author has recently presented a general systematic and automatic procedure to deal with any number of internal nodes, preferably distributed at equidistant locations, and the number of these nodes may be smaller or even larger than those in the Lagrange type element that occupies the entire domain Ω [Provatidis 2006a]. In more detail, the Gordon–Coons transfinite interpolation is applied so that the boundary (and interboundary) data are blended through higher order functions, thus leading to monomials outside the 'legs' of Pascal's triangle with a surplus of two. For reasons of simplicity, in this paper we proceed in a slightly different but equivalent way. Apart from the monomials mentioned in Section 2.1, we consider up to nine additional monomials as follows. The first subset,

$$S_1 = \{x^2y^2\}$$

is related to a central node ($x = a/2$, $y = b/2$); the second set,

$$S_2 = S_1 \cup \{x^3y^2, x^2y^3, x^3y^3\}$$

to a tensor product of 2×2 (four) points, while the third set,

$$S_3 = S_1 \cup S_2 \cup \{x^4y^2, x^2y^4, x^4y^3, x^3y^4, x^4y^4\},$$

is related to a tensor product of 3×3 (nine) points in the interior. Finally, as an extreme case, for a certain boundary discretization through p segments per side, the abovementioned space of Lagrange-type

finite elements, $S^{p,q}(\Omega_{st}^{(q)})$, is also considered as a candidate approximation (in the proposed collocation method) of which the performance will be validated.

3. The proposed global collocation procedure

3.1. ***General.*** In any collocation method, for a given PDE:

$$D(u) - f(x, y) = 0, \qquad \text{in} \quad \Omega = [0, a] \times [0, b], \tag{2}$$

we generally seek an approximate solution to Equation (2) which is a linear combination of the global *basis* functions $\{\phi_i(x, y)\},\ i = 1, 2, \ldots, n$:

$$\tilde{u}(x, y) = \sum_{j=1}^{n} \alpha_j \phi_j(x, y). \tag{3}$$

For the sake of simplicity, in this work we deal with the Dirichlet conditions that hold along the entire boundary $\partial\Omega$.

For a given number of uniform boundary subdivisions, that is, p and q towards the x and y axes respectively, the boundary includes $n_b = 2(p+q)$ nodes. In order to achieve a complete numerical solution we also use n_I internal nodes [Provatidis 2006a; 2006b]. Then, we solve for both the n_b unknown fluxes $(\partial u/\partial n)$ along $\partial\Omega$ and for the unknown variable u at the n_I internal nodes, a total of $n = n_b + n_I$ unknowns. Therefore, we need to form n linearly independent equations, one for each node of the boundary and the interior. Afterwards, the n coefficients $\alpha_j,\ j = 1, 2, \ldots, n$, which appear in Equation (3), are determined.

Without loss of generality, the proposed methodology below refers to Poisson's equation,

$$(D(u) - f = \nabla^2 u - f = 0).$$

3.2. ***Matrix formulation.*** Because the basis functions do not fulfill any special condition, regardless of whether the Dirichlet boundary conditions are or are not homogeneous, Equation (3) is written for all boundary nodes:

$$\bar{\boldsymbol{u}} = \boldsymbol{C}_0 \boldsymbol{\alpha}, \tag{4}$$

where $\bar{\boldsymbol{u}}$ is a vector of n_b elements, including the nodal values of the prescribed variable u along $\partial\Omega$, $\boldsymbol{C}_0$ is a $n_b \times n$ matrix with elements $[C_0]_{ij} = \phi_j(x_i, y_i),\ i = 1, \ldots, n_b,\ j = 1, \ldots, n$, and $\boldsymbol{\alpha}$ is the vector of the n unknown coefficients appearing in Equation (3).

By taking the Laplacian of both parts in Equation (3), one arrives at the following approximation:

$$\nabla^2 \tilde{u}(x, y) = \sum_{j=1}^{n} \alpha_j \nabla^2 \phi_j(x, y).$$

The collocation method enforces the residual

$$r(x, y) = \nabla^2 \tilde{u}(x, y) - f(x, y), \tag{5}$$

to become zero at specific points (x_i, y_i). In the proposed method, it is suggested that these points be chosen so as to coincide with the abovementioned n_I internal nodes. Therefore, by applying Equations

(5) and (6) at the internal nodes, one obtains:

$$\boldsymbol{r} = \boldsymbol{C}_2\boldsymbol{\alpha} - \bar{\boldsymbol{f}} = \boldsymbol{0}, \tag{6}$$

where $\boldsymbol{C}_2$ is an $n_I \times n$ matrix with elements $[C_2]_{ij} = \nabla^2\phi_j(x_i, y_i)$, $i = 1, \ldots, n_I$, $j = 1, \ldots, n$, $\boldsymbol{r}$ is the vanishing vector of residuals, and $\bar{\boldsymbol{f}}$ is a vector of n_I elements that consists of the values of the functions $f(x, y)$ at the collocation points (x_i, y_i) in the interior.

Obviously, Equations (4) and (6) constitute a linear system of n equations with n unknowns,

$$\begin{bmatrix} \boldsymbol{C}_0 \\ \boldsymbol{C}_2 \end{bmatrix} \boldsymbol{\alpha} = \begin{Bmatrix} \bar{\boldsymbol{u}} \\ \bar{\boldsymbol{f}} \end{Bmatrix},$$

which can be easily solved in $\boldsymbol{\alpha}$.

3.3. ***Numerical implementation.*** A FORTRAN-77 code was developed on a usual PC Pentium IV. A subroutine was written for the automatic determination of the abovementioned $2(p + q)$ monomials involved in the boundary discretization, Equation (1). Then, two choices are available as follows. The first concerns the use of Lagrange-type interpolation ($(p - 1) \times (q - 1)$ terms: $x^i y^j$; $i = 0, 1, \ldots, p$, $j = 0, 1, \ldots, q$), and the second, the sequence of previously defined S_1, S_2 and S_3 sets (see Section 2.2). As we are concerned with second order problems, (Poisson's equation) we have limited this subroutine only to the second derivatives in x and y.

4. Numerical results

The theory is elucidated by two examples taken from literature [Elansari et al. 2001], in which the exact solution is not of polynomial form. The quality of the numerical solution is evaluated in terms of an L_2-norm, which is defined as follows:

$$L_2 = \frac{\sqrt{\int_\Omega (\tilde{u} - u_{\text{exact}})^2 d\Omega}}{\sqrt{\int_\Omega u_{\text{exact}}^2 d\Omega}} \times 100\%.$$

Both examples are defined on the square domain $\Omega = [0, 2]^2$ in $\mathbb{R}^2$, and refer to the Poisson's equation:

$$\nabla^2 u = f(x, y).$$

Example 1.

$$f(x, y) = -2\sin(x)\sin(y).$$

In this case, the exact solution is

$$u(x, y) = \sin(x)\sin(y). \tag{7}$$

The Dirichlet boundary condition is calculated using Equation (7) and prescribed on the entire boundary. In the beginning, the solution is approximated by applying Coons interpolation in conjunction with only boundary data (Equation (1)). Then, one (S_1-space), four (S_2-space) and nine (S_3-space) internal nodes were introduced. For the case of four internal nodes, two options were tested: the first using equidistant positions and the second using the locations occupied in the usual Gaussian quadrature (the

latter was initially suggested in one-dimensional problems [De Boor and Swartz 1973]). For the case of nine internal nodes, only equidistant locations were tested. In addition, for each boundary discretization (p), the corresponding $(p-1)^2$ internal nodes, which also exist in a Lagrange-type finite element (that occupies the entire domain), are tested. The results of this example are shown in Table 1 and suggest the following:

- The bivariate Coons interpolation (Equation (1)) is incapable of approximating the solution and converges to an error norm L_2 about 13.2%.
- The introduction of only one internal node at the center of the domain induces severe improvement in the solution (L_2 reduces to about 2.2%).
- The introduction of four internal nodes causes a further reduction of L_2 to about 0.6%. In contrast to prior observations [Carey and Finlayson 1975; Carey and Oden 1983; Elansari et al. 2001], when the internal points are distributed in equal distances, the solution becomes slightly better than that using locations of Gaussian quadrature.
- The introduction of nine equidistantly distributed internal points causes a further improvement (0.04%).
- The locations of uniformly distributed points according to the Lagrange-type finite element lead to the best approximate solution tested.
- In the abovementioned case of using the set of monomials found in the Lagrange-type finite element, it was not possible to use more than nine segments per side ($p = q = 9$), due to truncation errors involved.
- When $p > 7$, the L_2-norm does not further decrease. This finding is consistent with the practice of Szabó and Babuška [1991, p.355], who reduce the use of Legendre polynomials up to the seventh degree.

Example 2.

$$f(x, y) = 2e^{x+y}.$$

Now, the exact solution is

$$u(x, y) = e^{x+y} + e^{y} \sin x. \tag{8}$$

The Dirichlet boundary condition is calculated using Equation (8) and prescribed on the entire boundary. In an analogous way with Example 1, the results are shown in Table 2 and indicate a very similar quality. All remarks in the bullets of Example 1 are also valid here.

5. Discussion and conclusions

The global collocation method has been previously applied mainly to one-dimensional problems in conjunction with trigonometrical, Chebyshev, Legendre, and similar polynomials used in spectral methods. In the last cases, a complete space of monomials is hidden. For example, it was recently shown that either of Lagrange, Chebyshev, or Berstein polynomials led to the same results with those obtained through the Taylor's series expansion [Provatidis 2007]. Moreover, the finite element method has intuitively used monomials related to truncated Taylor's series expansion and Pascal's triangle. Therefore, it is an

p	L_2-norm (in %)					
	Number of monomials related to internal nodes (n_I)					
	0 Equation (1)	1 (S_1)	4 (S_2)		9 (S_3)	Lagrange-type $S^{p,q}(\Omega_{st}^{(q)})$
			Equidistantly	At Gauss points		
1	57.29	98.22	683.29	123.29	61.77	—
2	13.32	5.49	3.21	3.42	23.36	5.49
3	13.09	3.56	1.58	3.08	0.54	1.58
4	13.18	2.18	0.63	1.28	0.10	0.10
5	13.19	2.19	0.62	1.29	0.06	2.86E-02
6	13.18	2.17	0.63	1.24	0.04	1.31E-03
7	13.18	2.17	0.63	1.24	0.04	2.94E-04
8	13.18	2.17	0.63	1.24	0.04	5.06E-04
9	13.18	2.17	0.63	1.24	0.04	1.44E-02

Table 1. Example 1: accuracy of the proposed method for several boundary dicretizations ($p = q$) and several monomials.

p	L_2-norm (in %)					
	Number of monomials related to internal nodes (n_I)					
	0 Equation (1)	1 (S_1)	4 (S_2)		9 (S_3)	Lagrange-type $S^{p,q}(\Omega_{st}^{(q)})$
			Equidistantly	At Gauss points		
1	49.58	12.27	208.18	63.08	67.70	—
2	6.95	5.39	2.91	2.93	9.41	5.39
3	5.45	1.39	1.11	1.91	3.19E-01	1.11
4	5.58	1.25	0.34	0.69	1.05E-01	1.05E-01
5	5.58	1.27	0.33	0.69	3.43E-02	1.51E-02
6	5.58	1.26	0.34	0.64	2.83E-02	1.17E-03
7	5.58	1.26	0.34	0.64	2.92E-02	1.92E-04
8	5.58	1.26	0.34	0.64	2.93E-02	2.60E-04
9	5.58	1.26	0.34	0.64	2.93E-02	9.11E-03

Table 2. Example 2: accuracy of the proposed method for several boundary dicretizations ($p = q$) and monomials.

old idea used to approximate the solution within a domain through monomials. However, difficulties in inverting the matrix of coefficients $\boldsymbol{C}_0$ (see Equation (4)) have been extensively discussed in the 1960s. Zienkiewicz [1977, p.151] writes that "occasionally an inverse of $\boldsymbol{C}_0$ may not exist and *always* considerable algebraic difficulty is experienced in obtaining an inverse in general terms suitable for all element geometries".

Unlike the previous work mentioned above, we show here that instead of an intuitive choice of monomials, the well known bivariate Gordon–Coons transfinite interpolation can be used. Since this method reflects the extension of linear interpolation from a one- to a two-dimensional field, and it is based on the solid background of Computational Geometry, in the case of a smooth patch such as a rectangle, the invertability of $\boldsymbol{C}_0$ is always anticipated.

We have shown here that a complete solution requires the use of internal nodes that should be generally located at equidistant locations like an imaginary uniform mesh. For a certain number of boundary discretizations, when progressively increasing the number of internal nodes up to $p = 7$ subdivisions, the solution monotonically improves. For $p > 7$, probably due to truncation errors, the solution becomes ill-conditioned and gives rise to a slightly increasing error norm (see, for example, the bottom of Table 1 and Table 2).

As mentioned in Section 1, the proposed collocation method can be applied also to nonrectangular domains. However, ongoing research shows that the method should be reformulated by replacing monomials with the global cardinal shape functions of which closed form expressions have been previously reported [Provatidis 2006a; 2006b; 2006c; 2006d]. In case of a boundary under partially Dirichlet and partially Neumann conditions, the procedure is straightforward, and analogous one-dimensional applications can be found in [Provatidis 2007] (both ends fixed, one fixed the other free). Finally, it is well known that one weakness of a global interpolation method is its inability to handle problems having discontinuous boundary conditions. Indeed, the proposed method is no exception.

In conclusion, the proposed method is a global collocation technique in which no domain or boundary integrals need to be computed and no least-squares procedure is required (at least for the Dirichlet conditions tested). Obviously, the extension of the proposed approach to other types of partial differential operators and/or three-dimensional problems is straightforward.

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