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WAVE PROPAGATION IN A PRESTRESSED COMPRESSIBLE ELASTIC LAYER WITH CONSTRAINED BOUNDARIES

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The dynamic motion of a prestressed *compressible* elastic layer having constrained boundaries is considered. The dispersion relations which relate wave speed and wave number are obtained for both symmetric and antisymmetric motions. Both motions can be considered by formulating the incremental boundary-value problem based on the theory of incremental elastic deformations, and using the propagator matrix technique. The limiting phase speed at the low wave number limit of symmetric and antisymmetric waves is obtained. At the low wave number limit, depending on the prestress, for symmetric motion with slipping boundaries and for antisymmetric motion with vertically unconstrained boundaries, a finite phase speed may exist for the fundamental mode. Numerical results are presented for a Blatz–Ko material. The effects of the constrained boundaries are clearly seen in the dispersion curves.

1. Introduction

In general, wave propagation in prestressed *incompressible* elastic media has been studied before the analysis of waves in prestressed *compressible* elastic media. Surface wave propagation in an incompressible elastic half-space was studied in [Dowaikh and Ogden 1990] and the corresponding problem for a compressible elastic half-space was considered in [Dowaikh and Ogden 1991b]. Interfacial wave propagation in two joined incompressible elastic half-spaces was analyzed in [Dowaikh and Ogden 1991a] and the problem of compressible elastic half-spaces was considered in [Sotiropoulos 1998]. Vibration and stability analysis of a prestressed elastic layer was reported in [Ogden and Roxburgh 1993] for incompressible elastic materials and in [Roxburgh and Ogden 1994] for compressible elastic materials. Guided waves in a layered half-space were studied in [Ogden and Sotiropoulos 1995; 1996] for incompressible and compressible materials, respectively. Analyses of waves in an elastic layer bonded to two half-spaces were conducted in [Sotiropoulos and Sifniotopoulos 1995] for incompressible elastic materials and in [Sotiropoulos 2000] for compressible elastic materials. Propagation and reflection of plane waves in an incompressible elastic half-space has been considered in [Ogden and Sotiropoulos 1997], while the corresponding compressible elastic half-space problem has been reported in [Ogden and Sotiropoulos 1998].

The dispersive behavior of time harmonic in-plane waves in a prestressed *incompressible* symmetric layered composite with imperfect interface conditions has been analyzed for symmetric waves in [Leungvichcharoen and Wijeyewickrema 2003] and for antisymmetric waves in [Leungvichcharoen et al. 2004]. The corresponding problems for the perfectly bonded interface case were analyzed by [Rogerson and Sandiford 1997; 2000]. In the present paper, the effect of constrained boundaries on both symmetric

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and antisymmetric time harmonic plane waves in a prestressed *compressible* symmetric elastic layer are considered.

The basic equations of infinitesimal time harmonic wave propagation in prestressed, compressible, elastic media are given in Section 2. Using the propagator matrix, the dispersion relations for symmetric and antisymmetric motions are obtained in Section 3. The dispersion relation is analyzed in Section 4 and numerical results using Blatz–Ko material are presented in Section 5.

2. Basic equations

The equations for infinitesimal time-harmonic wave propagation in prestressed compressible elastic media are given in this section [Roxburgh and Ogden 1994; Ogden and Sotiropoulos 1998]. Consider a homogeneous, compressible, isotropic elastic body with an initial unstressed state $\mathfrak{B}_u$, which after being subjected to pure homogeneous strains has the new configuration $\mathfrak{B}_e$, the prestressed equilibrium state. A Cartesian coordinate system $Ox_1x_2x_3$, with axes coincident with the principal axes of strain is chosen for configuration $\mathfrak{B}_e$. Let $\boldsymbol{u}$ be a small, time dependent displacement superimposed on $\mathfrak{B}_e$. For the plane strain incremental problem considered here, the nonzero displacement components u_1 and u_2 are independent of x_3. The incremental equations of motion can be expressed as

$$\alpha_{11}u_{1,11}+\gamma_2 u_{1,22}+\delta u_{2,12}=\rho\ddot{u}_1, \qquad \gamma_1 u_{2,11}+\alpha_{22}u_{2,22}+\delta u_{1,12}=\rho\ddot{u}_2, \tag{1}$$

where ρ is the current material density, the superimposed dot and comma indicate differentiation with respect to time t and the spatial coordinate component in $\mathfrak{B}_e$, respectively, and $\alpha_{ij}=\mathscr{A}_{0iijj}$ $(i=1,2)$, $\gamma_1=\mathscr{A}_{01212}$, $\gamma_2=\mathscr{A}_{02121}$, $\delta=\alpha_{12}+\gamma_2-\sigma_2=\alpha_{12}+\gamma_1-\sigma_1$, in which $\mathscr{A}_{0ijkl}$ are the components of the fourth-order tensor of first-order instantaneous moduli of isotropic elastic material which relates the nominal stress increment tensor and the deformation gradient increment tensor and σ_i are the principal Cauchy stress in x_i-direction. The instantaneous elastic moduli α_{ij}, γ_i and the principal Cauchy stress σ_i can be expressed in terms of the strain energy function W per unit reference volume and principal stretches λ_i as in [Ogden and Sotiropoulos 1998]:

$$J\alpha_{ij}=\lambda_i\lambda_j W_{ij}, \qquad J\gamma_i=\frac{\lambda_1 W_1-\lambda_2 W_2}{\lambda_1^2-\lambda_2^2}\lambda_i^2, \qquad J\sigma_1=\lambda_1 W_1, \quad J\sigma_2=\lambda_2 W_2, \tag{2}$$

where $W_i=\partial W/\partial\lambda_i$, $W_{ij}=\partial^2 W/\partial\lambda_i\partial\lambda_j$ $(i,j=1,2)$ and $J=\lambda_1\lambda_2\lambda_3$. Note that the properties $\alpha_{12}=\alpha_{21}$ and $\gamma_1-\sigma_1=\gamma_2-\sigma_2$ have been used in obtaining (1). In the case of equibiaxial deformation when $\lambda_1=\lambda_2$, (2) reduces to

$$\begin{aligned} J\alpha_{11}&=J\alpha_{22}=\lambda_1^2 W_{11}, & J\alpha_{12}&=\lambda_1^2 W_{12},\\ J\gamma_1&=J\gamma_2=\tfrac{1}{2}\lambda_1(\lambda_1 W_{11}-\lambda_1 W_{12}+W_1), & J\sigma_1&=J\sigma_2=\lambda_1 W_1,\end{aligned}$$

that is $\alpha_{11}=\alpha_{22}=\alpha_{12}+2\gamma_2-\sigma_2$. In addition, in the configuration $\mathfrak{B}_u$, $\alpha_{11}=\lambda+2\mu$, $\alpha_{12}=\lambda$, $\gamma_1=\gamma_2=\mu$, where λ and μ are the classical Lamé moduli of the material. In the analysis of wave propagation, the strong ellipticity conditions given by [Ogden and Sotiropoulos 1998] are assumed, that is, $\alpha_{ii}>0$, $\gamma_i>0$ $(i=1,2)$, $\beta_c+\sqrt{\alpha_{11}\alpha_{22}\gamma_1\gamma_2}>0$, where $2\beta_c=\alpha_{11}\alpha_{22}+\gamma_1\gamma_2-\delta^2$.

The relevant components of the nominal stress increment tensor in the configuration $\mathfrak{B}_e$ can be expressed as

$$\begin{aligned} s_{021}(x_1, x_2, t) &= \gamma_2 u_{1,2} + (\gamma_2 - \sigma_2) u_{2,1}, \\ s_{022}(x_1, x_2, t) &= \alpha_{12} u_{1,1} + \alpha_{22} u_{2,2}. \end{aligned} \tag{3}$$

3. Formulation of problem and dispersion relations

The prestressed isotropic compressible elastic layer with constrained boundaries is shown in Figure 1. The Cartesian coordinate system is chosen such that x_1 and x_2-axes are coincident with the principal axes, the x_2-direction is normal to the midplane of the layer, the time harmonic wave propagation is in x_1-direction and the origin O lies at the midplane of the layer. The thickness of the layer is $2h$. The homogeneous layer has material parameters $\alpha_{11}, \alpha_{12}, \alpha_{22}, \gamma_1, \gamma_2$ and mass density ρ.

The superimposed infinitesimal displacements may be expressed as

$$(u_1, u_2) = (A_1, A_2) e^{\tilde{q} k x_2} e^{ik(x_1 - vt)}, \tag{4}$$

where k is the wave number, v is the phase speed, A_1 and A_2 are arbitrary constants, and the parameter $\tilde{q}$ is to be determined. (This is the notation we use for the compressible case; q and q^* were used in [Leungvichcharoen and Wijeyewickrema 2003; Leungvichcharoen et al. 2004] for related problems in the incompressible case.)

Substituting (4) into (1) yields a system of homogeneous equations for which a nontrivial solution exists provided that

$$\alpha_{22}\gamma_2 \tilde{q}^4 - (\alpha'_{11}\alpha_{22} + \gamma'_1\gamma_2 - \delta^2)\tilde{q}^2 + \alpha'_{11}\gamma'_1 = 0, \tag{5}$$

where $\alpha'_{11} = \alpha_{11} - \rho v^2$ and $\gamma'_1 = \gamma_1 - \rho v^2$. Let $\tilde{q}_1^2$ and $\tilde{q}_2^2$ be the roots of the quadratic equation (5); then

$$\tilde{q}_1^2 + \tilde{q}_2^2 = \frac{\alpha'_{11}\alpha_{22} + \gamma'_1\gamma_2 - \delta^2}{\alpha_{22}\gamma_2}, \qquad \tilde{q}_1^2 \tilde{q}_2^2 = \frac{\alpha'_{11}\gamma'_1}{\alpha_{22}\gamma_2}.$$

Define the *squared phase speed* as the nondimensional quantity $\xi = \rho v^2/\gamma_2$, and set

$$\bar{\alpha}_{ij} = \frac{\alpha_{ij}}{\gamma_2} \quad (i, j = 1, 2), \qquad \bar{\alpha}'_{11} = \frac{\alpha'_{11}}{\gamma_2}, \qquad \bar{\delta} = \frac{\delta}{\gamma_2}, \qquad \bar{\gamma}_1 = \frac{\gamma_1}{\gamma_2}, \qquad \bar{\sigma}_2 = \frac{\sigma_2}{\gamma_2}. \tag{6}$$

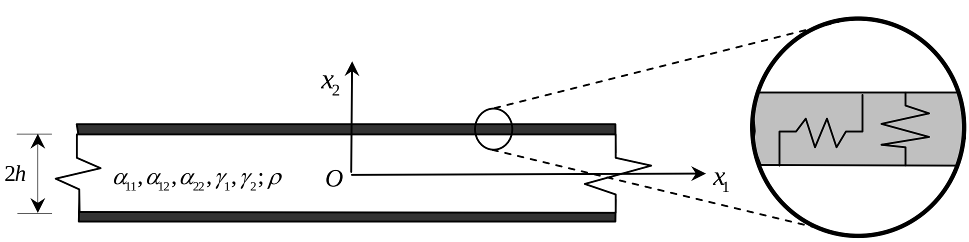

Figure 1. Prestressed compressible elastic layer with constrained boundaries.

To obtain the propagator matrix, the incremental displacements and stresses in (3) and (4) are written in the form of a 4×1 vector as

$$(u_1, u_2, s_{021}, s_{022})^T = \big[U_1(x_2), U_2(x_2), S_{021}(x_2), S_{022}(x_2)\big]^T e^{ik(x_1 - vt)}. \tag{7}$$

From (7) it can be shown after some manipulation that

$$\boldsymbol{y}(x_2) = \boldsymbol{H}\tilde{\boldsymbol{E}}(x_2)\boldsymbol{a}, \tag{8}$$

where

$$\boldsymbol{y}(x_2) = \left[-iU_1(x_2), U_2(x_2), \frac{S_{021}(x_2)}{ik}, \frac{S_{022}(x_2)}{k}\right]^T$$

is the displacement-stress increment vector,

$$\tilde{\boldsymbol{E}}(x_2) = \mathrm{diag}\big(e^{\tilde{q}_1 k x_2}, e^{-\tilde{q}_1 k x_2}, e^{\tilde{q}_2 k x_2}, e^{-\tilde{q}_2 k x_2}\big),$$

is a diagonal matrix,

$$\boldsymbol{a} = \left[\frac{iA_1^{(1)}}{\tilde{q}_1\bar{\delta}}, \frac{-iA_1^{(2)}}{\tilde{q}_1\bar{\delta}}, \frac{iA_1^{(3)}}{\tilde{q}_2\bar{\delta}}, \frac{-iA_1^{(4)}}{\tilde{q}_2\bar{\delta}}\right]^T$$

is a vector of constants, and $\boldsymbol{H}$ is a 4×4 matrix independent of x_2 and defined by

$$\boldsymbol{H} = \begin{bmatrix} -\tilde{q}_1\bar{\delta} & \tilde{q}_1\bar{\delta} & -\tilde{q}_2\bar{\delta} & \tilde{q}_2\bar{\delta} \\ \tilde{f}(\tilde{q}_1) & \tilde{f}(\tilde{q}_1) & \tilde{f}(\tilde{q}_2) & \tilde{f}(\tilde{q}_2) \\ -\gamma_2\tilde{f}(\tilde{q}_1)\tilde{g}(\tilde{q}_2)/\bar{\delta} & -\gamma_2\tilde{f}(\tilde{q}_1)\tilde{g}(\tilde{q}_2)/\bar{\delta} & -\gamma_2\tilde{f}(\tilde{q}_2)\tilde{g}(\tilde{q}_1)/\bar{\delta} & -\gamma_2\tilde{f}(\tilde{q}_2)\tilde{g}(\tilde{q}_1)/\bar{\delta} \\ \gamma_2\tilde{q}_1\tilde{g}(\tilde{q}_1) & -\gamma_2\tilde{q}_1\tilde{g}(\tilde{q}_1) & \gamma_2\tilde{q}_2\tilde{g}(\tilde{q}_2) & -\gamma_2\tilde{q}_2\tilde{g}(\tilde{q}_2); \end{bmatrix}$$

with

$$\tilde{f}(\tilde{q}_m) = \tilde{q}_m^2 - \bar{\alpha}'_{11}, \quad \tilde{g}(\tilde{q}_m) = \bar{\alpha}_{22}\tilde{f}(\tilde{q}_m) + \bar{\delta}\bar{\alpha}_{12} \qquad (m = 1, 2). \tag{9}$$

The vector $\boldsymbol{a}$ is eliminated from (8) by introducing the vector $\boldsymbol{y}(\bar{x}_2)$ at an arbitrary location $x_2 = \bar{x}_2$ and finally obtain the expression

$$\boldsymbol{y}(x_2) = \boldsymbol{H}\tilde{\boldsymbol{E}}(x_2 - \bar{x}_2)\boldsymbol{H}^{-1}\boldsymbol{y}(\bar{x}_2) = \boldsymbol{P}(x_2 - \bar{x}_2)\boldsymbol{y}(\bar{x}_2). \tag{10}$$

involving the propagator matrix $\boldsymbol{P}$, whose entries are given by

$$\begin{aligned}
P_{11} &= P_{33} = \tilde{q}_1\tilde{q}_2\bar{\delta}^2\tilde{f}(\tilde{q}_1)\tilde{f}(\tilde{q}_2)\big[\tilde{g}(\tilde{q}_1)\tilde{C}_2 - \tilde{g}(\tilde{q}_2)\tilde{C}_1\big]\tilde{\kappa}^{-1}, \\
P_{12} &= \tilde{q}_1\tilde{q}_2\bar{\delta}^3\big[\tilde{q}_2\tilde{f}(\tilde{q}_1)\tilde{g}(\tilde{q}_2)\tilde{S}_2 - \tilde{q}_1\tilde{f}(\tilde{q}_2)\tilde{g}(\tilde{q}_1)\tilde{S}_1\big]\tilde{\kappa}^{-1}, \\
P_{13} &= \tilde{q}_1\tilde{q}_2\bar{\delta}^4\big[\tilde{q}_2\tilde{f}(\tilde{q}_1)\tilde{S}_2 - \tilde{q}_1\tilde{f}(\tilde{q}_2)\tilde{S}_1\big](\gamma_2\tilde{\kappa})^{-1}, \\
P_{14} &= -P_{23} = \tilde{q}_1\tilde{q}_2\bar{\delta}^3\tilde{f}(\tilde{q}_1)\tilde{f}(\tilde{q}_2)\big[\tilde{C}_2 - \tilde{C}_1\big](\gamma_2\tilde{\kappa})^{-1}, \\
P_{21} &= -P_{34} = \bar{\delta}\tilde{f}(\tilde{q}_1)\tilde{f}(\tilde{q}_2)\big[\tilde{q}_2\tilde{f}(\tilde{q}_1)\tilde{g}(\tilde{q}_2)\tilde{S}_1 - \tilde{q}_1\tilde{f}(\tilde{q}_2)\tilde{g}(\tilde{q}_1)\tilde{S}_2\big]\tilde{\kappa}^{-1}, \\
P_{22} &= P_{44} = \tilde{q}_1\tilde{q}_2\bar{\delta}^2\tilde{f}(\tilde{q}_1)\tilde{f}(\tilde{q}_2)\big[\tilde{g}(\tilde{q}_1)\tilde{C}_1 - \tilde{g}(\tilde{q}_2)\tilde{C}_2\big]\tilde{\kappa}^{-1}, \\
P_{24} &= \bar{\delta}^2\tilde{f}(\tilde{q}_1)\tilde{f}(\tilde{q}_2)\big[\tilde{q}_2\tilde{f}(\tilde{q}_1)\tilde{S}_1 - \tilde{q}_1\tilde{f}(\tilde{q}_2)\tilde{S}_2\big](\gamma_2\tilde{\kappa})^{-1},
\end{aligned}$$

$$
\begin{aligned}
P_{31} &= \gamma_2 \tilde{f}(\tilde{q}_1)\tilde{f}(\tilde{q}_2)\big[\tilde{q}_1 \tilde{f}(\tilde{q}_2)\tilde{g}(\tilde{q}_1)^2 \tilde{S}_2 - \tilde{q}_2 \tilde{f}(\tilde{q}_1)\tilde{g}(\tilde{q}_2)^2 \tilde{S}_1\big]\tilde{\kappa}^{-1},\\
P_{32} &= -P_{41} = \gamma_2 \tilde{q}_1 \tilde{q}_2 \bar{\delta} \tilde{f}(\tilde{q}_1)\tilde{f}(\tilde{q}_2)\tilde{g}(\tilde{q}_1)\tilde{g}(\tilde{q}_2)\big[\tilde{C}_2 - \tilde{C}_1\big]\tilde{\kappa}^{-1},\\
P_{42} &= \gamma_2 \tilde{q}_1 \tilde{q}_2 \bar{\delta}^2\big[\tilde{q}_1 \tilde{f}(\tilde{q}_2)\tilde{g}(\tilde{q}_1)^2 \tilde{S}_1 - \tilde{q}_2 \tilde{f}(\tilde{q}_1)\tilde{g}(\tilde{q}_2)^2 \tilde{S}_2\big]\tilde{\kappa}^{-1},\\
P_{43} &= \tilde{q}_1 \tilde{q}_2 \bar{\delta}^3\big[\tilde{q}_1 \tilde{f}(\tilde{q}_2)\tilde{g}(\tilde{q}_1) \tilde{S}_1 - \tilde{q}_2 \tilde{f}(\tilde{q}_1)\tilde{g}(\tilde{q}_2) \tilde{S}_2\big]\tilde{\kappa}^{-1},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{S}_m &= \sinh[\tilde{q}_m kh], \quad \tilde{C}_m = \cosh[\tilde{q}_m kh] \qquad (m = 1, 2),\\
\tilde{\kappa} &= \tilde{q}_1 \tilde{q}_2 \bar{\delta}^2 \tilde{f}(\tilde{q}_1)\tilde{f}(\tilde{q}_2)\big[\tilde{g}(\tilde{q}_1) - \tilde{g}(\tilde{q}_2)\big].
\end{aligned} \tag{11}
$$

For in-plane wave propagation in the x_1-direction, since the wave motion can be decoupled into symmetric and antisymmetric modes, it is convenient to analyze the *symmetric* and *antisymmetric* waves separately, and it is sufficient to consider only the upper half of the layer ($0 \le x_2 \le h$).

The midplane conditions for symmetric and antisymmetric waves can be written as

$$
\text{symmetric waves:} \quad U_2(0) = S_{021}(0) = 0, \tag{12}
$$

$$
\text{antisymmetric waves:} \quad U_1(0) = S_{022}(0) = 0. \tag{13}
$$

At the boundary ($x_2 = h$), the shear stress increment is assumed to be proportional to the displacement increment in the x_1-direction and the normal stress increment is assumed to be proportional to the displacement increment in the x_2-direction, that is,

$$
S_{021}(h) = \frac{k_1\gamma_2}{h} U_1(h), \qquad S_{022}(h) = \frac{k_2\gamma_2}{h} U_2(h), \tag{14}
$$

where k_1, k_2 are the nondimensional spring parameters.

Substituting the boundary conditions (14) and the midplane conditions, (12), into (10) with $x_2 = h$ and $\bar{x}_2 = 0$, the dispersion relation for symmetric waves in a layer with constrained boundaries is obtained as

$$
\begin{aligned}
&\{P_{31}P_{44} - P_{34}P_{41}\} + \frac{\gamma_2 k_1}{kh}\{P_{11}P_{44} - P_{14}P_{41}\}\\
&\qquad + \frac{\gamma_2 k_2}{kh}\{P_{31}P_{24} - P_{34}P_{21}\} + \frac{\gamma_2 k_1}{kh}\frac{\gamma_2 k_2}{kh}\{P_{11}P_{24} - P_{14}P_{21}\} = 0.
\end{aligned} \tag{15}
$$

Similarly the dispersion relation for antisymmetric waves can be obtained as

$$
\begin{aligned}
&\{P_{32}P_{43} - P_{33}P_{42}\} + \frac{\gamma_2 k_1}{kh}\{P_{12}P_{43} - P_{13}P_{42}\}\\
&\qquad + \frac{\gamma_2 k_2}{kh}\{P_{23}P_{32} - P_{22}P_{33}\} + \frac{\gamma_2 k_1}{kh}\frac{\gamma_2 k_2}{kh}\{P_{12}P_{23} - P_{13}P_{22}\} = 0.
\end{aligned} \tag{16}
$$

For symmetric waves we substitute the elements of $\boldsymbol{P}(h)$ into (15) and remove the common factor $\gamma_2 \tilde{q}_1 \tilde{q}_2 \bar{\delta}^2[\tilde{g}(\tilde{q}_1) - \tilde{g}(\tilde{q}_2)]$ from the denominator. The removal of this common factor leads to spurious roots in the resulting relation

$$
\begin{aligned}
&\bar{\delta}^2\big(\tilde{q}_1 \tilde{f}(\tilde{q}_2)\tilde{S}_2\tilde{C}_1 - \tilde{q}_2 \tilde{f}(\tilde{q}_1)\tilde{S}_1\tilde{C}_2\big) + \frac{kh}{k_1}\tilde{f}(\tilde{q}_1)\tilde{f}(\tilde{q}_2)\tilde{S}_1\tilde{S}_2\big(\tilde{g}(\tilde{q}_2) - \tilde{g}(\tilde{q}_1)\big)\\
&\quad + \frac{kh}{k_2}\bar{\delta}^2\tilde{q}_1\tilde{q}_2\tilde{C}_1\tilde{C}_2\big(\tilde{g}(\tilde{q}_2) - \tilde{g}(\tilde{q}_1)\big) + \frac{(kh)^2}{k_1k_2}\big(\tilde{q}_2 \tilde{f}(\tilde{q}_1)\tilde{g}(\tilde{q}_2)^2\tilde{S}_1\tilde{C}_2 - \tilde{q}_1 \tilde{f}(\tilde{q}_2)\tilde{g}(\tilde{q}_1)^2\tilde{S}_2\tilde{C}_1\big) = 0,
\end{aligned} \tag{17}
$$

where $\tilde{f}(\tilde{q}_m)$ and $\tilde{g}(\tilde{q}_m)$ are defined in (9) and $\tilde{C}_m$ and $\tilde{S}_m$ are defined in (11).

The dispersion relation for antisymmetric waves can be similarly obtained as

$$\bar{\delta}^2\big(\tilde{q}_2\tilde{f}(\tilde{q}_1)\tilde{S}_2\tilde{C}_1-\tilde{q}_1\tilde{f}(\tilde{q}_2)\tilde{S}_1\tilde{C}_2\big)+\frac{kh}{k_1}\tilde{f}(\tilde{q}_1)\tilde{f}(\tilde{q}_2)\tilde{C}_1\tilde{C}_2\big(\tilde{g}(\tilde{q}_1)-\tilde{g}(\tilde{q}_2)\big) + \frac{kh}{k_2}\bar{\delta}^2\tilde{q}_1\tilde{q}_2\tilde{S}_1\tilde{S}_2\big(\tilde{g}(\tilde{q}_1)-\tilde{g}(\tilde{q}_2)\big)+\frac{(kh)^2}{k_1k_2}\big(\tilde{q}_1\tilde{f}(\tilde{q}_2)\tilde{g}(\tilde{q}_1)^2\tilde{S}_1\tilde{C}_2-\tilde{q}_2\tilde{f}(\tilde{q}_1)\tilde{g}(\tilde{q}_2)^2\tilde{S}_2\tilde{C}_1\big)=0, \quad (18)$$

where the common factor removed from the denominator of (16) after substitution of the components of the propagator matrix is $\gamma_2\tilde{f}(\tilde{q}_1)\tilde{f}(\tilde{q}_2)[\tilde{g}(\tilde{q}_1)-\tilde{g}(\tilde{q}_2)]$.

The dispersion relations for the fixed boundary case and traction free boundary case can be deduced from the dispersion relations for the constrained boundary case.

When $k_1, k_2 \to \infty$, the dispersion relation for waves propagating in a layer with fixed boundaries is obtained for symmetric waves from (17) as

$$\bar{\delta}^2\big(\tilde{q}_1\tilde{f}(\tilde{q}_2)\tilde{S}_2\tilde{C}_1-\tilde{q}_2\tilde{f}(\tilde{q}_1)\tilde{S}_1\tilde{C}_2\big)=0, \quad (19)$$

and for antisymmetric waves propagating from (18) as

$$\bar{\delta}^2\big(\tilde{q}_2\tilde{f}(\tilde{q}_1)\tilde{S}_2\tilde{C}_1-\tilde{q}_1\tilde{f}(\tilde{q}_2)\tilde{S}_1\tilde{C}_2\big)=0.$$

When $k_1, k_2 \to 0$, the dispersion relation for waves propagating in a layer with traction free boundaries is obtained for symmetric waves from (17) as

$$\big(\tilde{q}_2\tilde{f}(\tilde{q}_1)\tilde{g}(\tilde{q}_2)^2\tilde{S}_1\tilde{C}_2-\tilde{q}_1\tilde{f}(\tilde{q}_2)\tilde{g}(\tilde{q}_1)^2\tilde{S}_2\tilde{C}_1\big)=0, \quad (20)$$

and for antisymmetric waves propagating from (18) as

$$\big(\tilde{q}_1\tilde{f}(\tilde{q}_2)\tilde{g}(\tilde{q}_1)^2\tilde{S}_1\tilde{C}_2-\tilde{q}_2\tilde{f}(\tilde{q}_1)\tilde{g}(\tilde{q}_2)^2\tilde{S}_2\tilde{C}_1\big)=0. \quad (21)$$

After some manipulation (20) and (21) can be rewritten respectively as

$$\frac{\tanh[kh\tilde{q}_1]}{\tanh[kh\tilde{q}_2]}=\frac{\tilde{q}_1\big[\bar{\alpha}'_{11}\big(\gamma'_1-(1-\bar{\sigma}_2)^2\big)-\tilde{q}_2^2(\bar{\alpha}_{12}^2-\bar{\alpha}'_{11}\bar{\alpha}_{22})\big]}{\tilde{q}_2\big[\bar{\alpha}'_{11}\big(\gamma'_1-(1-\bar{\sigma}_2)^2\big)-\tilde{q}_1^2(\bar{\alpha}_{12}^2-\bar{\alpha}'_{11}\bar{\alpha}_{22})\big]}, \quad (22)$$

and

$$\frac{\tanh[kh\tilde{q}_1]}{\tanh[kh\tilde{q}_2]}=\frac{\tilde{q}_2\big[\bar{\alpha}'_{11}\big(\gamma'_1-(1-\bar{\sigma}_2)^2\big)-q_1^2(\bar{\alpha}_{12}^2-\bar{\alpha}'_{11}\bar{\alpha}_{22})\big]}{\tilde{q}_1\big[\bar{\alpha}'_{11}\big(\gamma'_1-(1-\bar{\sigma}_2)^2\big)-q_2^2(\bar{\alpha}_{12}^2-\bar{\alpha}'_{11}\bar{\alpha}_{22})\big]}. \quad (23)$$

These agree with [Roxburgh and Ogden 1994, (4.24) and (4.22)].

4. Analysis of dispersion relations

The similarity of the dispersion relation for symmetric and antisymmetric waves shown in Section 3 results in similar behavior for these two kinds of waves, as will be discussed in this section. Equation (5) can be rewritten in nondimensional form as

$$\bar{\alpha}_{22}\tilde{q}^4-(\bar{\alpha}'_{11}\bar{\alpha}_{22}+\bar{\gamma}'_1-\bar{\delta}^2)\tilde{q}^2+\bar{\alpha}'_{11}\bar{\gamma}'_1=0, \quad (24)$$

where $\bar{\gamma}_1' = \gamma_1'/\gamma_2 = \bar{\gamma}_1 - \xi$ in the notation of (6). The roots of the quadratic (24) can be written as

$$\tilde{q}_1^2, \tilde{q}_2^2 = \frac{1}{2\bar{\alpha}_{22}}\left[\bar{\alpha}_{11}'\bar{\alpha}_{22} + \bar{\gamma}_1' - \bar{\delta}^2 \mp \sqrt{(\bar{\alpha}_{11}'\bar{\alpha}_{22} + \bar{\gamma}_1' - \bar{\delta}^2)^2 - 4(\bar{\alpha}_{22}\bar{\alpha}_{11}'\bar{\gamma}_1')}\right]. \tag{25}$$

For symmetric waves, the common factor $\gamma_2\tilde{q}_1\tilde{q}_2\bar{\delta}^2[\tilde{g}(\tilde{q}_1) - \tilde{g}(\tilde{q}_2)]$ taken out from the denominator of (15) leads to spurious roots of (17) given by

$$\xi = \xi_{S1}, \qquad \xi_{S2} = \bar{\alpha}_{11}, \bar{\gamma}_1, \text{ when } \tilde{q}_1 = 0 \text{ or } \tilde{q}_2 = 0, \qquad \xi = \xi_{S3},$$

$$\xi_{S4} = \frac{(\bar{\alpha}_{22}-1)(\bar{\alpha}_{11}\bar{\alpha}_{22}-\bar{\gamma}_1) - \bar{\delta}^2(1+\bar{\alpha}_{22})}{(\bar{\alpha}_{22}-1)^2} \pm \frac{2\sqrt{\bar{\alpha}_{22}\bar{\delta}^2\left[\bar{\delta}^2 - (\bar{\alpha}_{22}-1)(\bar{\alpha}_{11}-\bar{\gamma}_1)\right]}}{(\bar{\alpha}_{22}-1)^2}, \text{ when } \tilde{g}(\tilde{q}_1) = \tilde{g}(\tilde{q}_2).$$

Similarly for antisymmetric waves the common factor $\gamma_2\tilde{f}(\tilde{q}_1)\tilde{f}(\tilde{q}_2)[\tilde{g}(\tilde{q}_1) - \tilde{g}(\tilde{q}_2)]$ taken out from the denominator of (16) leads to spurious roots of (18) given by

$$\xi = \xi_{S1}, \text{ when } \tilde{f}(\tilde{q}_1)\tilde{f}(\tilde{q}_2) = 0, \qquad \xi = \xi_{S3}, \xi_{S4}, \text{ when } \tilde{g}(\tilde{q}_1) = \tilde{g}(\tilde{q}_2).$$

Low wave number limit $kh \to 0$. When $kh \to 0$ the thickness of the layer is very small compared to the wavelength. By considering small argument expansions of the hyperbolic functions, the squared phase speeds for symmetric waves for the slipping boundary ($k_1 = 0$) are obtained from (17) as

$$\xi^S = \bar{\alpha}_{11} - \frac{\bar{\alpha}_{12}^2}{k_2 + \bar{\alpha}_{22}}, \tag{26a}$$

and when $k_2 = 0$ or $k_2 = \infty$ the squared phase speeds for (26a) can be obtained as

$$\xi_{00}^S = \bar{\alpha}_{11} - \frac{\bar{\alpha}_{12}^2}{\bar{\alpha}_{22}}, \qquad \xi_{0\infty}^S = \bar{\alpha}_{11}. \tag{26b}$$

For antisymmetric waves for the vertically unconstrained boundary ($k_2 = 0$) the limiting squared phase speed as $kh \to 0$ can be obtained from (18) as

$$\xi^A = \bar{\gamma}_1 - \frac{(1-\bar{\sigma}_2)^2}{k_1 + 1}, \tag{27a}$$

and when $k_1 = 0$ or $k_1 = \infty$ the squared phase speeds for (27a) can be obtained as

$$\xi_{00}^A = \bar{\gamma}_1 - (1-\bar{\sigma}_2)^2, \qquad \xi_{\infty 0}^A = \bar{\gamma}_1. \tag{27b}$$

Equations (26) and (27) are the finite squared phase speeds of the lowest branches of the dispersion curves, which have frequencies that tend to zero as $kh \to 0$. The frequencies of higher modes which have infinite squared phase speeds ($\xi \to \infty$) when $kh \to 0$ are considered next. When $\xi \to \infty$ expressions for $\tilde{q}_1^2$ and $\tilde{q}_2^2$ can be obtained from (25) as

$$\tilde{q}_1^2 = -1 + \frac{-\bar{\delta}^2 + \bar{\alpha}_{11}(\bar{\alpha}_{22}-1)}{(\bar{\alpha}_{22}-1)\xi} + O(\xi^{-2}), \qquad \tilde{q}_2^2 = -\frac{1}{\bar{\alpha}_{22}} + \frac{\bar{\delta}^2 + \bar{\gamma}_1(\bar{\alpha}_{22}-1)}{(\bar{\alpha}_{22}-1)\bar{\alpha}_{22}\xi} + O(\xi^{-2}). \tag{28}$$

It can be seen that $\tilde{q}_1$ and $\tilde{q}_2$ are imaginary when $\xi \to \infty$. By substituting (28) into (17) and (19), introducing the nondimensional parameter $\Omega = kh\sqrt{\xi}$ and considering small argument expansions of the hyperbolic functions, the equation for cut-off frequencies of symmetric waves is obtained as

$$\omega_1^S\big(\Omega_C^{(S)}\big)\omega_2^S\big(\Omega_C^{(S)}\big)=0, \tag{29}$$

where

$$\begin{aligned}\omega_1^S\big(\Omega_C^{(S)}\big)&=\Big(k_1\cos\big(\Omega_C^{(S)}\big)-\Omega_C^{(S)}\sin\big(\Omega_C^{(S)}\big)\Big),\\ \omega_2^S\big(\Omega_C^{(S)}\big)&=\Big(k_2\sin\big(\Omega_C^{(S)}/\sqrt{\bar\alpha_{22}}\big)+\sqrt{\bar\alpha_{22}}\Omega_C^{(S)}\cos\big(\Omega_C^{(S)}/\sqrt{\bar\alpha_{22}}\big)\Big).\end{aligned} \tag{30}$$

Similarly for antisymmetric waves, the equation for cut-off frequencies is obtained as,

$$\omega_1^A\big(\Omega_C^{(A)}\big)\omega_2^A\big(\Omega_C^{(A)}\big)=0, \tag{31}$$

where

$$\begin{aligned}\omega_1^A\big(\Omega_C^{(A)}\big)&=\Big(k_1\sin\big(\Omega_C^{(A)}\big)+\Omega_C^{(A)}\cos\big(\Omega_C^{(A)}\big)\Big),\\ \omega_2^A\big(\Omega_C^{(A)}\big)&=\Big(-k_2\cos\big(\Omega_C^{(A)}/\sqrt{\bar\alpha_{22}}\big)+\sqrt{\bar\alpha_{22}}\Omega_C^{(A)}\sin\big(\Omega_C^{(A)}/\sqrt{\bar\alpha_{22}}\big)\Big).\end{aligned} \tag{32}$$

It can be seen that term $\omega_1^S(\Omega_C^{(S)})$ and $\omega_1^A(\Omega_C^{(A)})$ depend only on nondimensional spring parameter k_1 while $\omega_2^S(\Omega_C^{(S)})$ and $\omega_2^A(\Omega_C^{(A)})$ depend on nondimensional parameters $\bar\alpha_{22}$ and k_2.

5. Numerical results

The effects of constrained boundaries on wave propagation in a prestressed compressible elastic layer discussed in the previous sections are illustrated by considering a numerical example in this section. Here compressible material with a Blatz–Ko strain energy function is considered.

The strain energy function $W_c^{(BK)}$ of Blatz–Ko material [Roxburgh and Ogden 1994] is

$$W_c^{(BK)}=\frac{\mu}{2}(\lambda_1^{-2}+\lambda_2^{-2}+\lambda_3^{-2}+2\lambda_1\lambda_2\lambda_3-5),$$

and parameters α_{11}, α_{12}, α_{22}, γ_1, and γ_2 for this material are

$$\alpha_{11}=\frac{3\mu}{J\lambda_1^2},\qquad \alpha_{12}=\mu,\qquad \alpha_{22}=\frac{3\mu}{J\lambda_2^2},\qquad \gamma_1=\frac{\mu}{J\lambda_2^2},\qquad \gamma_2=\frac{\mu}{J\lambda_1^2},$$

which yields

$$\bar\alpha_{11}=3,\qquad \bar\alpha_{12}=J\lambda_1^2,\qquad \bar\alpha_{22}=\frac{3\lambda_1^2}{\lambda_2^2},\qquad \bar\gamma_1=\frac{\lambda_1^2}{\lambda_2^2}, \tag{33}$$

and the principal Cauchy stresses are

$$\sigma_i=\frac{\mu(\lambda_i^2J-1)}{\lambda_i^2J}\quad (i=1,2,3).$$

The compressible layer is equibiaxially deformed in the (x_1,x_2)-plane, that is, $\lambda_1=\lambda_2$. The nondimensional parameters prescribed are $\bar\alpha_{11}=3$ (from (33)), $\bar\alpha_{12}=1$, and $\bar\gamma_1=1$; the computed ones are $\bar\alpha_{22}=3$, $\bar\sigma_2=0$, and $\bar\delta=2$. The nondimensional squared phase speeds ξ of the fundamental mode and the next ten modes for symmetric and antisymmetric waves ($\xi_S^{(n)}$, $\xi_A^{(n)}$, $n=1,2,\dots,11$) are shown, using linear plots as well as log-log plots to clearly show the low wave number limits, in Figures 2 and 3 for

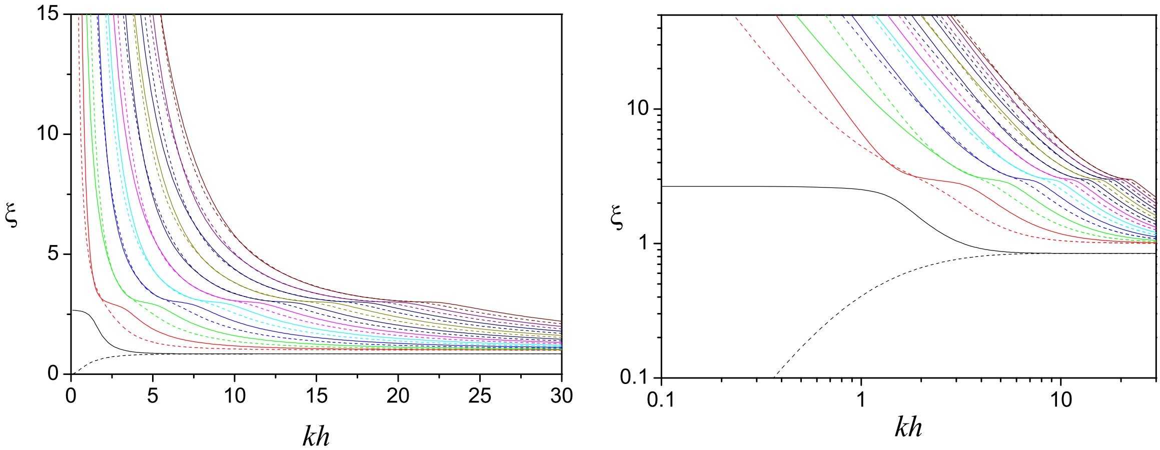

Figure 2. Dispersion curves of the fundamental mode and next ten modes for $k_1 = 0$, $k_2 = 0$ (traction free boundary). Linear scale plot (left) and logarithmic scale plot (right); solid lines for symmetric waves and dashed lines for antisymmetric waves.

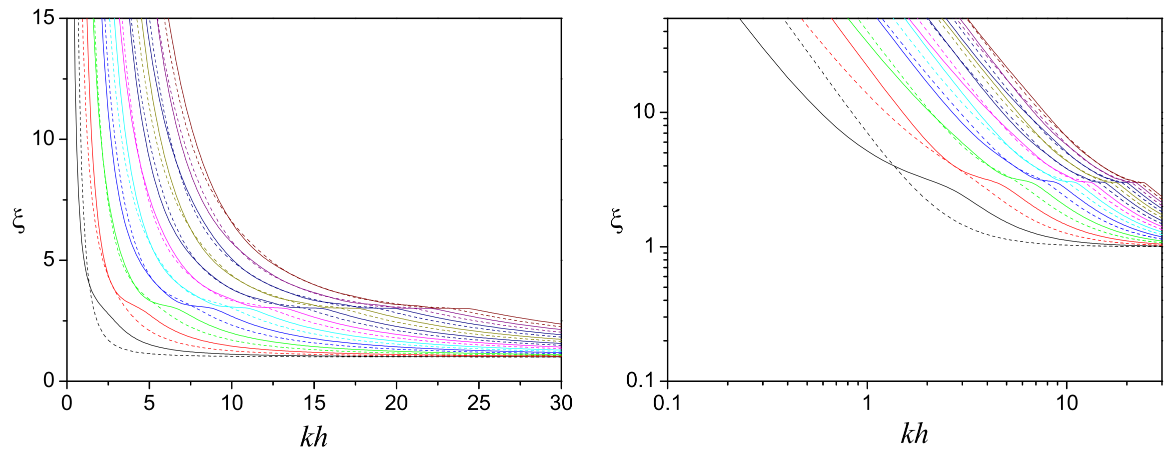

Figure 3. Dispersion curves of the fundamental mode and next ten modes for $k_1 = \infty$, $k_2 = \infty$ (fixed boundary). Linear scale plot (left) and logarithmic scale plot (right); solid lines for symmetric waves and dashed lines for antisymmetric waves.

the two extreme cases $k_1 = 0$, $k_2 = 0$ (corresponding to traction free boundaries) and $k_1 = \infty$, $k_2 = \infty$ (corresponding to fixed boundaries). When $kh \to 0$, it can be seen from Figure 2 for the traction free case ($k_1 = 0$, $k_2 = 0$) that the squared phase speed of the fundamental mode for symmetric waves tends to a finite limit, that is, $\xi_S^{(1)} \to \xi_{00}^S = 2.667$ while the squared phase speed of the fundamental mode of antisymmetric waves $\xi_A^{(1)}$ tends to zero and the other higher modes have infinite squared phase speeds, that is, $\xi_S^{(n)}, \xi_A^{(n)} \to \infty$ $(n = 2, 3, \ldots)$. For the fixed boundary case ($k_1 = \infty$, $k_2 = \infty$) when $kh \to 0$, it can be seen from Figure 3 that both symmetric and antisymmetric modes have infinite squared phase speeds, that is, $\xi_S^{(n)}, \xi_A^{(n)} \to \infty$ $(n = 1, 2, \ldots)$.

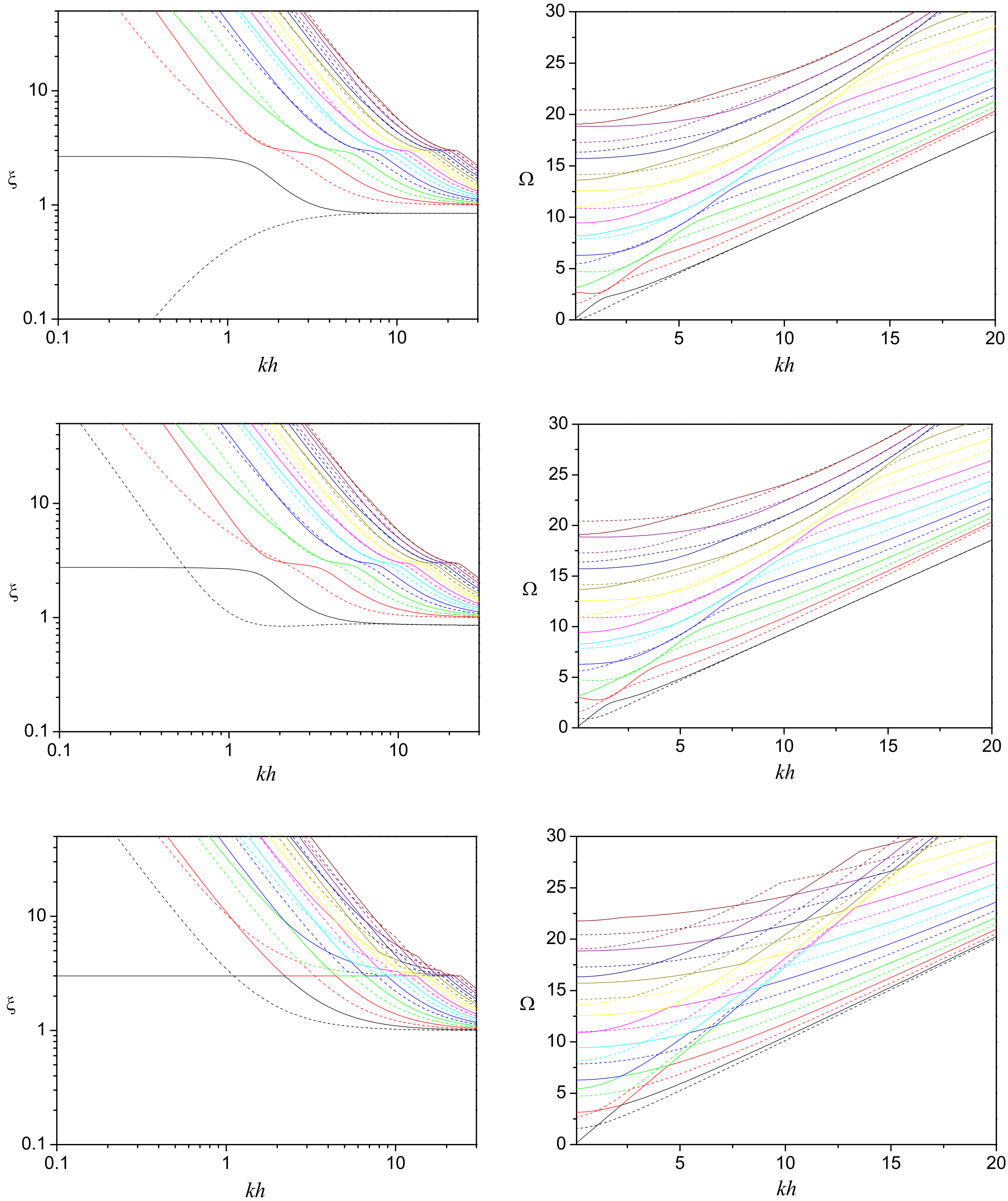

Figure 4. Dispersion curves of the fundamental mode and next ten modes for $k_1 = 0$, $k_2 = 0, 1, \infty$ (slipping boundary). Nondimensional squared phase speed ξ (left column) and nondimensional frequency Ω (right column); solid lines for symmetric waves and dashed lines for antisymmetric waves.

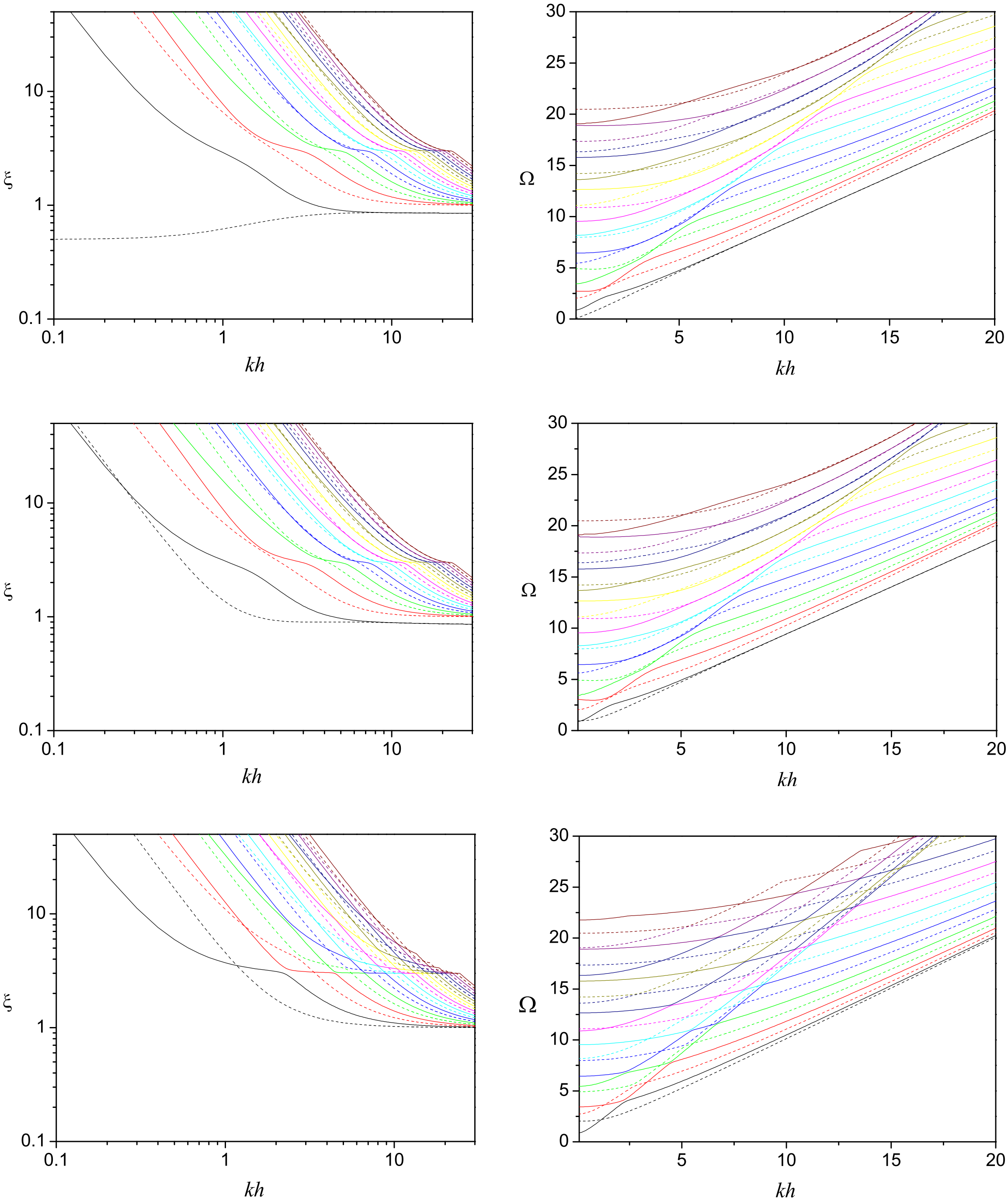

Figure 5. Dispersion curves of the fundamental mode and next ten modes for $k_1 = 1$, $k_2 = 0, 1, \infty$ (partially constrained slipping boundary). Nondimensional squared phase speed ξ (left column) and nondimensional frequency Ω (right column); solid lines for symmetric waves and dashed lines for antisymmetric waves.

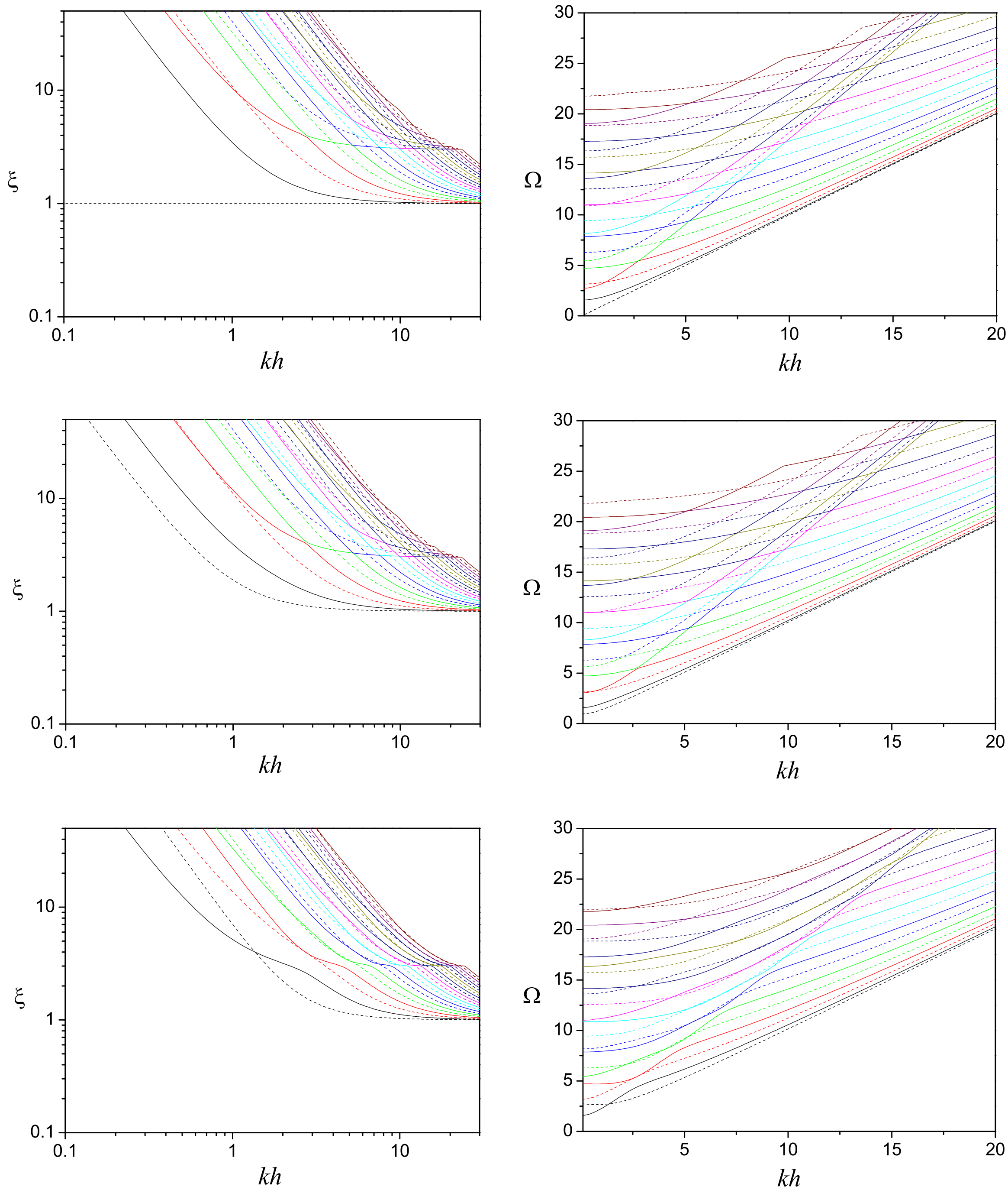

Figure 6. Dispersion curves of the fundamental mode and next ten modes for $k_1 = \infty$, $k_2 = 0, 1, \infty$ (no slip boundary). Nondimensional squared phase speed ξ (left column) and nondimensional frequency Ω (right column); solid lines for symmetric waves and dashed lines for antisymmetric waves.

The effect of the spring parameters k_1 and k_2 are seen in the next three figures. Figure 4 corresponds to a slipping boundary ($k_1 = 0$), Figure 5 to a partially constrained slipping boundary ($k_1 = 1$) and Figure 6 to a no slip boundary ($k_1 = \infty$), for $k_2 = 0, 1, \infty$. From the left column of Figure 4 it can be seen that, when $kh \to 0$, for the slipping boundary ($k_1 = 0$), the squared phase speed of the fundamental mode for symmetric waves tends to a finite limit, $\xi^S_{00} = 2.667$, $\xi^S_{01} = 2.75$, $\xi^S_{0\infty} = 3.0$, while the fundamental mode for antisymmetric waves $\xi^{(1)}_A$ tends to zero for $k_2 = 0$ but has infinite squared phase speeds for $k_2 = 1, \infty$. For the partially constrained slipping boundary ($k_1 = 1$), when $kh \to 0$, it can be seen from the left column of Figure 5 that the fundamental mode for antisymmetric waves $\xi^{(1)}_A$ tends to a finite limit, meaning $\xi^{(1)}_A \to \xi^A_{10} = 0.5$, for $k_2 = 0$, while the fundamental mode for symmetric waves $\xi^{(1)}_S$ and the other higher modes have infinite squared phase speeds, that is, $\xi^{(n)}_S, \xi^{(n)}_A \to \infty$ ($n = 2, 3, \ldots$). For the no slip boundary ($k_1 = \infty$), when $kh \to 0$, it can be seen from the left column of Figure 6 that the behavior of the different modes are similar to the partially constrained slipping boundary case, the fundamental mode for antisymmetric waves $\xi^{(1)}_A$ tends to a finite limit, that is, $\xi^{(1)}_A \to \xi^A_{\infty 0} = 1$ for $k_2 = 0$. From the right columns in Figures 4, 5, and 6, it can be seen that the modes that have finite limiting phase speeds when $kh \to 0$, have frequencies that tend to zero. The frequencies of the other modes tend to the cut-off frequencies calculated from (29)–(32).

6. Summary and conclusions

In the present analysis, the dispersive behavior of in-plane time harmonic waves in a prestressed compressible layer with constrained boundaries is considered. The dispersion relations for both symmetric and antisymmetric waves are obtained. From the asymptotic analysis of the dispersion relations the limiting squared phase speed at the low wave number limit is obtained. The equations for cut-off frequencies of the modes that have infinite phase speeds at the low wave number limit are also obtained.

The behavior of the dispersion curves for symmetric and antisymmetric waves are similar at the low wave number limit. At low wave number limit, depending on the prestress, at most only one finite limiting squared phase speed may exist.

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