Appendix

Derivative quantities of p^{eq} are

$$\frac{\partial p^{eq}}{\partial\ \sigma_{11}} = \frac{(1+v)}{3} + \frac{3\left[\sigma_{11}^2 - 2\sigma_{22}^2 + 2\sigma_{11}\sigma_{22} + v(v-1)\ (\sigma_{11}+\sigma_{22})^2 - 3\sigma_{12}^2\right]}{M^2(1+v)\ (\sigma_{11}+\sigma_{22})^2}, \qquad \text{(A-1)}$$

$$\frac{\partial p^{eq}}{\partial\ \sigma_{22}} = \frac{(1+v)}{3} + \frac{3\left[\sigma_{22}^2 - 2\sigma_{11}^2 + 2\sigma_{11}\sigma_{22} + v(v-1)\ (\sigma_{11}+\sigma_{22})^2 - 3\sigma_{12}^2\right]}{M^2(1+v)\ (\sigma_{11}+\sigma_{22})^2}, \qquad \text{(A-2)}$$

$$\frac{\partial p^{eq}}{\partial\ \sigma_{12}} = \frac{18\sigma_{12}}{M^2(1+v)\ (\sigma_{11}+\sigma_{22})}. \qquad \text{(A-3)}$$

Due to the sensitivity of some of the parameters, the power term in equations (13) and (20) had to be set as positive, irrespective of the sign achieved by the computations. Accordingly, pertinent strain rates for vector $\dot{\boldsymbol{\varepsilon}}^c$ in plane strain conditions are

$$\dot{\varepsilon}_{11}^c = -\frac{1}{\alpha} \cdot \frac{\mu^*}{\tau} \cdot \left(\frac{p^{eq}}{p_p^{eq}}\right)^{\frac{\lambda^*-k^*}{\mu^*}} \cdot \frac{\partial p^{eq}}{\partial \sigma_{11}}, \qquad \text{(A-4)}$$

$$\dot{\varepsilon}_{22}^c = -\frac{1}{\alpha} \cdot \frac{\mu^*}{\tau} \cdot \left(\frac{p^{eq}}{p_p^{eq}}\right)^{\frac{\lambda^*-k^*}{\mu^*}} \cdot \frac{\partial p^{eq}}{\partial \sigma_{22}}, \qquad \text{(A-5)}$$

$$\dot{\varepsilon}_{12}^c = -\frac{1}{\alpha} \cdot \frac{\mu^*}{\tau} \cdot \left(\frac{p^{eq}}{p_p^{eq}}\right)^{\frac{\lambda^*-k^*}{\mu^*}} \cdot \frac{\partial p^{eq}}{\partial \sigma_{12}}. \qquad \text{(A-6)}$$

Matrix $\mathbf{S}^n$, is defined as

$$\mathbf{S}^n = \left(\frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\sigma}}\right) \text{ where } \boldsymbol{\beta} = \begin{Bmatrix} \dot{\varepsilon}_{11}^{c} \\ \dot{\varepsilon}_{22}^{c} \\ \dot{\varepsilon}_{12}^{c} \end{Bmatrix}, \tag{A-7}$$

$$\mathbf{S}^n = \begin{bmatrix} \dfrac{\partial \dot{\varepsilon}_{11}^{c}}{\partial \sigma_{11}} & \dfrac{\partial \dot{\varepsilon}_{11}^{c}}{\partial \sigma_{22}} & \dfrac{\partial \dot{\varepsilon}_{11}^{c}}{\partial \sigma_{12}} \\ \dfrac{\partial \dot{\varepsilon}_{22}^{c}}{\partial \sigma_{11}} & \dfrac{\partial \dot{\varepsilon}_{22}^{c}}{\partial \sigma_{22}} & \dfrac{\partial \dot{\varepsilon}_{22}^{c}}{\partial \sigma_{12}} \\ \dfrac{\partial \dot{\varepsilon}_{12}^{c}}{\partial \sigma_{11}} & \dfrac{\partial \dot{\varepsilon}_{12}^{c}}{\partial \sigma_{22}} & \dfrac{\partial \dot{\varepsilon}_{12}^{c}}{\partial \sigma_{12}} \end{bmatrix}. \tag{A-8}$$

Above derivates are derived by taking generalized preconsolidation pressure p_p^{eq} as a constant within the considered time step and are given as

$$S_{11}^n = \frac{\partial \dot{\varepsilon}_{11}^{c}}{\partial \sigma_{11}} = \frac{\mu^*}{\alpha\tau} \left(\frac{p^{eq}}{p_p^{eq}}\right)^{\frac{\lambda^*-k^*}{\mu^*}} . \left[- \frac{\partial^2 p^{eq}}{\partial \sigma_{11}^2} - \frac{\left(\lambda^* - k^*\right)}{\mu^* p^{eq}} . \left(\frac{\partial p^{eq}}{\partial \sigma_{11}}\right)^2 + \frac{1}{\alpha}.\ \frac{\partial \alpha}{\partial \sigma_{11}} . \frac{\partial p^{eq}}{\partial \sigma_{11}} \right], \tag{A-8-a}$$

$$S_{12}^n = \frac{\partial \dot{\varepsilon}_{11}^{c}}{\partial \sigma_{22}} = \frac{\mu^*}{\alpha\tau} \left(\frac{p^{eq}}{p_p^{eq}}\right)^{\frac{\lambda^*-k^*}{\mu^*}} . \left[- \frac{\partial^2 p^{eq}}{\partial \sigma_{22} - \partial \sigma_{11}} - \frac{\left(\lambda^* - k^*\right)}{\mu^* p^{eq}} . \left(\frac{\partial p^{eq}}{\partial \sigma_{11}} . \frac{\partial p^{eq}}{\partial \sigma_{22}}\right) + \frac{1}{\alpha}.\ \frac{\partial \alpha}{\partial \sigma_{22}} . \frac{\partial p^{eq}}{\partial \sigma_{11}} \right], \tag{A-8-b}$$

$$S_{13}^n = \frac{\partial \dot{\varepsilon}_{11}^{c}}{\partial \sigma_{12}} = \frac{\mu^*}{\alpha\tau} \left(\frac{p^{eq}}{p_p^{eq}}\right)^{\frac{\lambda^*-k^*}{\mu^*}} . \left[- \frac{\partial^2 p^{eq}}{\partial \sigma_{12} \partial \sigma_{11}} - \frac{\left(\lambda^* - k^*\right)}{\mu^* p^{eq}} . \left(\frac{\partial p^{eq}}{\partial \sigma_{11}} . \frac{\partial p^{eq}}{\partial \sigma_{12}}\right) + \frac{1}{\alpha}.\ \frac{\partial \alpha}{\partial \sigma_{12}} . \frac{\partial p^{eq}}{\partial \sigma_{11}} \right], \tag{A-8-c}$$

$$S_{21}^{n}=\frac{\partial\dot{\varepsilon}_{22}^{c}}{\partial\sigma_{11}}=\ \frac{\mu^{*}}{\alpha\tau}\ \left(\frac{p^{eq}}{p_{p}^{eq}}\right)^{\frac{\lambda^{*}-k^{*}}{\mu^{*}}}.\left[-\ \frac{\partial^{2}p^{eq}}{\partial\sigma_{11}\partial\sigma_{22}}\ -\ \frac{\left(\lambda^{*}-k^{*}\right)}{\mu^{*}p^{eq}}\ .\ \left(\frac{\partial p^{eq}}{\partial\sigma_{11}}.\frac{\partial p^{eq}}{\partial\sigma_{22}}\right)\right.$$
$$\left.+\ \frac{1}{\alpha}.\ \frac{\partial\alpha}{\partial\sigma_{11}}\ .\ \frac{\partial p^{eq}}{\partial\sigma_{22}}\right],\qquad\text{(A-8-d)}$$

$$S_{22}^{n}=\frac{\partial\dot{\varepsilon}_{22}^{c}}{\partial\sigma_{22}}=\ \frac{\mu^{*}}{\alpha\tau}\ \left(\frac{p^{eq}}{p_{p}^{eq}}\right)^{\frac{\lambda^{*}-k^{*}}{\mu^{*}}}.\left[-\ \frac{\partial^{2}p^{eq}}{\partial\sigma_{22}^{2}}\ -\ \frac{\left(\lambda^{*}-k^{*}\right)}{\mu^{*}p^{eq}}\ .\ \left(\frac{\partial p^{eq}}{\partial\sigma_{22}}\right)^{2}\right.$$
$$\left.+\ \frac{1}{\alpha}.\ \frac{\partial\alpha}{\partial\sigma_{22}}\ .\ \frac{\partial p^{eq}}{\partial\sigma_{22}}\right],\qquad\text{(A-8-e)}$$

$$S_{23}^{n}=\frac{\partial\dot{\varepsilon}_{22}^{c}}{\partial\sigma_{12}}=\ \frac{\mu^{*}}{\alpha\tau}\ \left(\frac{p^{eq}}{p_{p}^{eq}}\right)^{\frac{\lambda^{*}-k^{*}}{\mu^{*}}}.\left[-\ \frac{\partial^{2}p^{eq}}{\partial\sigma_{22}\partial\sigma_{22}}\ -\ \frac{\left(\lambda^{*}-k^{*}\right)}{\mu^{*}p^{eq}}\ .\ \left(\frac{\partial p^{eq}}{\partial\sigma_{22}}\right).\left(\frac{\partial p^{eq}}{\partial\sigma_{12}}\right)\right.$$
$$\left.+\ \frac{1}{\alpha}.\ \frac{\partial\alpha}{\partial\sigma_{12}}\ .\ \frac{\partial p^{eq}}{\partial\sigma_{22}}\right],\qquad\text{(A-8-f)}$$

$$S_{31}^{n}=\frac{\partial\dot{\varepsilon}_{12}^{c}}{\partial\sigma_{11}}=\ \frac{\mu^{*}}{\alpha\tau}\ \left(\frac{p^{eq}}{p_{p}^{eq}}\right)^{\frac{\lambda^{*}-k^{*}}{\mu^{*}}}.\left[-\ \frac{\partial^{2}p^{eq}}{\partial\sigma_{11}\partial\sigma_{12}}\ -\ \frac{\left(\lambda^{*}-k^{*}\right)}{\mu^{*}p^{eq}}\ .\ \left(\frac{\partial p^{eq}}{\partial\sigma_{12}}\right).\left(\frac{\partial p^{eq}}{\partial\sigma_{11}}\right)\right.$$
$$\left.+\ \frac{1}{\alpha}.\ \frac{\partial\alpha}{\partial\sigma_{11}}\ .\ \frac{\partial p^{eq}}{\partial\sigma_{12}}\right],\qquad\text{(A-8-g)}$$

$$S_{32}^{n}=\frac{\partial\dot{\varepsilon}_{12}^{c}}{\partial\sigma_{22}}=\ \frac{\mu^{*}}{\alpha\tau}\ \left(\frac{p^{eq}}{p_{p}^{eq}}\right)^{\frac{\lambda^{*}-k^{*}}{\mu^{*}}}\left[-\ \frac{\partial^{2}p^{eq}}{\partial\sigma_{22}\partial\sigma_{12}}\ -\ \frac{\left(\lambda^{*}-k^{*}\right)}{\mu^{*}p^{eq}}\ .\ \left(\frac{\partial p^{eq}}{\partial\sigma_{12}}\right).\left(\frac{\partial p^{eq}}{\partial\sigma_{22}}\right)\right.$$
$$\left.+\ \frac{1}{\alpha}.\ \frac{\partial\alpha}{\partial\sigma_{22}}\ .\ \frac{\partial p^{eq}}{\partial\sigma_{12}}\right],\qquad\text{(A-8-h)}$$

$$S_{33}^{n}=\frac{\partial\dot{\varepsilon}_{12}^{c}}{\partial\sigma_{12}}=\ \frac{\mu^{*}}{\alpha\tau}\ \left(\frac{p^{eq}}{p_{p}^{eq}}\right)^{\frac{\lambda^{*}-k^{*}}{\mu^{*}}}.\left[-\ \frac{\partial^{2}p^{eq}}{\partial\sigma_{12}^{2}}\ -\ \frac{\left(\lambda^{*}-k^{*}\right)}{\mu^{*}p^{eq}}\ .\ \left(\frac{\partial p^{eq}}{\partial\sigma_{12}}\right)^{2}\right.$$
$$\left.+\ \frac{1}{\alpha}.\ \frac{\partial\alpha}{\partial\sigma_{12}}\ .\ \frac{\partial p^{eq}}{\partial\sigma_{12}}\right].\qquad\text{(A-8-i)}$$

Required derivative quantities of p^{eq} for the matrix $\mathbf{S}^n$ are

$$\frac{\partial^2 p^{eq}}{\partial \sigma_{11}^2} = \frac{18 \left(\sigma_{22}^2 + \sigma_{12}^2\right)}{M^2 (1+v) \left(\sigma_{11} + \sigma_{22}\right)^3}, \tag{A-8-j}$$

$$\frac{\partial^2 p^{eq}}{\partial \sigma_{22}^2} = \frac{18 \left(\sigma_{11}^2 + \sigma_{12}^2\right)}{M^2 (1+v) \left(\sigma_{11} + \sigma_{22}\right)^3}, \tag{A-8-k}$$

$$\frac{\partial^2 p^{eq}}{\partial \sigma_{12}^2} = \frac{18}{M^2 (1+v) \left(\sigma_{11} + \sigma_{22}\right)}. \tag{A-8-l}$$

Derivative quantities of α for the matrix $\mathbf{S}^n$ are

$$\alpha = \frac{\partial p^{eq}}{\partial \sigma_{11}} + \frac{\partial p^{eq}}{\partial \sigma_{22}}, \tag{A-8-m}$$

$$\frac{\partial \alpha}{\partial \sigma_{11}} = \frac{\partial^2 p^{eq}}{\partial \sigma_{11}^2} + \frac{\partial^2 p^{eq}}{\partial \sigma_{11} \partial \sigma_{22}}, \tag{A-8-n}$$

$$\frac{\partial \alpha}{\partial \sigma_{22}} = \frac{\partial^2 p^{eq}}{\partial \sigma_{22}^2} + \frac{\partial^2 p^{eq}}{\partial \sigma_{22} \partial \sigma_{11}}, \tag{A-8-o}$$

$$\frac{\partial \alpha}{\partial \sigma_{12}} = 0, \tag{A-8-p}$$

$$\frac{\partial^2 p^{eq}}{\partial \sigma_{11} \partial \sigma_{22}} = \frac{\partial^2 p^{eq}}{\partial \sigma_{22} \partial \sigma_{11}} = \frac{18\left(\sigma_{12}^2 - \sigma_{11}\sigma_{22}\right)}{M^2 (1+v)\left(\sigma_{11} + \sigma_{22}\right)^3}, \tag{A-8-q}$$

$$\frac{\partial^2 p^{eq}}{\partial \sigma_{12} \partial \sigma_{11}} = \frac{\partial^2 p^{eq}}{\partial \sigma_{11} \partial \sigma_{12}} = \frac{-18\sigma_{12}}{M^2 (1+v)\left(\sigma_{11} + \sigma_{22}\right)^2}, \tag{A-8-r}$$

$$\frac{\partial^2 p^{eq}}{\partial\sigma_{12}\partial\sigma_{22}} = \frac{\partial^2 p^{eq}}{\partial\sigma_{22}\partial\sigma_{12}} = \frac{-18\sigma_{12}}{M^2(1+v)(\sigma_{11}+\sigma_{22})^2}. \tag{A-8-s}$$

The components of the element stiffness matrix $\int_{V^e} \mathbf{B}^T \bar{\mathbf{D}}^0 \mathbf{B}\, dv$ are discussed below. For eight nodes per element and two DOFs per node, the matrix $\mathbf{B}$ is a 3×16 matrix; under numerical integration, the stiffness matrix is

$$\mathbf{B}^T \bar{\mathbf{D}}^0 \mathbf{B} = \begin{bmatrix} B_1^T \\ B_2^T \\ B_3^T \\ B_4^T \\ \vdots \\ B_8^T \end{bmatrix}_{16\times3} \left[\bar{\mathbf{D}}\right]^0_{3\times3} \begin{bmatrix} B_1 & B_2 & B_3 & \cdots & B_8 \end{bmatrix}_{3\times16}, \tag{A-9}$$

where each of B is a 3×2 matrix. Then equation (A-7) can be rewritten as

$$\mathbf{B}^T \bar{\mathbf{D}}^0 \mathbf{B} = \begin{bmatrix} B_1^T\left[\bar{\mathbf{D}}\right]^0 B_1 & B_1^T\left[\bar{\mathbf{D}}\right]^0 B_2 & \cdots & B_1^T\left[\bar{\mathbf{D}}\right]^0 B_8 \\ B_2^T\left[\bar{\mathbf{D}}\right]^0 B_1 & B_2^T\left[\bar{\mathbf{D}}\right]^0 B_2 & \cdots & B_2^T\left[\bar{\mathbf{D}}\right]^0 B_8 \\ \vdots & \vdots & & \vdots \\ B_7^T\left[\bar{\mathbf{D}}\right]^0 B_1 & B_7^T\left[\bar{\mathbf{D}}\right]^0 B_2 & \cdots & B_7^T\left[\bar{\mathbf{D}}\right]^0 B_8 \\ B_8^T\left[\bar{\mathbf{D}}\right]^0 B_1 & B_8^T\left[\bar{\mathbf{D}}\right]^0 B_2 & \cdots & B_8^T\left[\bar{\mathbf{D}}\right]^0 B_8 \end{bmatrix}. \tag{A-10}$$

Since matrix $\bar{\mathbf{D}}^0$ is non-symmetric, the stiffness matrix $\mathbf{B}^T \bar{\mathbf{D}}^0 \mathbf{B}$ will be a non-symmetric 16×16 matrix. The explicit expression for $\mathbf{B}_i^T \bar{\mathbf{D}}^0 \mathbf{B}_j$ will be

$$\mathbf{B}_i^T \bar{\mathbf{D}}^0 \mathbf{B}_j = \begin{bmatrix} \frac{\partial N_\iota}{\partial x} & 0 & \frac{\partial N_\iota}{\partial y} \\ 0 & \frac{\partial N_\iota}{\partial y} & \frac{\partial N_\iota}{\partial x} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} \frac{\partial N_j}{\partial x} & 0 \\ 0 & \frac{\partial Nj}{\partial y} \\ \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} \end{bmatrix}, \tag{A-11}$$

where $i,\ j = 1,\ 2,\ 3, ...8$ for eight nodes per element.

The stress increments can be expressed as

$$\Delta\underline{\boldsymbol{\sigma}}_{k+1}^{0} = \left[\bar{\mathbf{D}}\right]_{3\times3}^{0} \left(\left[\mathbf{B}\right]_{3\times6} (\Delta\underline{\mathbf{a}}_{k+1}^{0})_{16\times1} - \Delta t_k \boldsymbol{\beta}_{3\times1}\left(\underline{\boldsymbol{\sigma}}_k\right)\right), \qquad \text{(A-12)}$$

where the part $\left[\mathbf{B}\right]_{3\times6} (\Delta\underline{\mathbf{a}}_{k+1}^{0})_{16\times1}$ is evaluated as

$$\begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \cdots & \frac{\partial N_8}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \cdots & 0 & \frac{\partial N_8}{\partial y} \\ \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_8}{\partial x} & \frac{\partial N_8}{\partial y} \end{bmatrix}_{\substack{8(3\times2)\\ \equiv(3\times16)}} \begin{Bmatrix} \Delta u_1 \\ \Delta v_1 \\ \Delta u_2 \\ \Delta v_2 \\ \vdots \\ \Delta u_8 \\ \Delta v_8 \end{Bmatrix}_{16\times1}, \qquad \text{(A-13)}$$

$$\equiv \begin{bmatrix} \frac{\partial N_1}{\partial x}.\Delta u_1 + \frac{\partial N_2}{\partial x}.\Delta u_2 + ... + \frac{\partial N_8}{\partial x}.\Delta u_8 \\ \frac{\partial N_1}{\partial y}.\Delta v_1 + \frac{\partial N_2}{\partial y}.\Delta v_2 + ... + \frac{\partial N_8}{\partial y}.\Delta v_8 \\ \left(\frac{\partial N_1}{\partial y}.\Delta u_1 + \frac{\partial N_1}{\partial x}.\Delta v_1\right) + \left(\frac{\partial N_1}{\partial y}.\Delta u_2 + \frac{\partial N_{21}}{\partial x}.\Delta v_2\right) + ... + \left(\frac{\partial N_8}{\partial y}.\Delta u_8 + \frac{\partial N_{81}}{\partial x}.\Delta v_8\right) \end{bmatrix}_{3\times1}. \qquad \text{(A-14)}$$

2-D Linear Elasticity for plane strain condition is discussed below. The stiffness matrix is given by $\int_{V^e} \mathbf{B}^T \mathbf{DB}\ dv$ where $\mathbf{D}$ is given by

$$\mathbf{D}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \nu/(1-\nu) & 0 \\ \nu/(1-\nu) & 1 & 0 \\ 0 & 0 & (1-2\nu)/2(1-\nu) \end{bmatrix}. \tag{A-15}$$

A typical 2 × 2 submatrix $\mathbf{B}_i^T\bar{\mathbf{D}}\mathbf{B}_j$ will be

$$\mathbf{B}_i^T\bar{\mathbf{D}}\mathbf{B}_j=\frac{E(1-v)}{(1+v)(1-2v)}\begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_i}{\partial y} \\ 0 & \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}\begin{bmatrix} 1 & \frac{v}{1-v} & 0 \\ \frac{v}{1-v} & 1 & 0 \\ 0 & 0 & \frac{(1-2v)}{2(1-v)} \end{bmatrix}\begin{bmatrix} \frac{\partial N_j}{\partial x} & 0 \\ 0 & \frac{\partial N_j}{\partial y} \\ \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} \end{bmatrix}$$

$$=\frac{E(1-v)}{(1+v)(1-2v)}\begin{bmatrix} \frac{\partial N_i}{\partial x}.\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}.\frac{\partial N_j}{\partial y}.\frac{(1-2v)}{2(1-v)} & \frac{\partial N_i}{\partial x}.\frac{v}{(1-v)}.\frac{\partial N_j}{\partial y}+\frac{\partial N_i}{\partial y}.\frac{(1-2v)}{2(1-v)}.\frac{\partial N_j}{\partial x} \\ \frac{\partial N_i}{\partial y}.\frac{\partial N_j}{(1-v)}.\frac{\partial N_i}{\partial x}.\frac{\partial N_j}{\partial x}.\frac{(1-2v)}{2(1-v)}.\frac{\partial N_j}{\partial y} & \frac{\partial N_i}{\partial y}.\frac{\partial N_j}{\partial y}+\frac{\partial N_i}{\partial x}.\frac{(1-2v)}{2(1-v)}.\frac{\partial N_j}{\partial x} \end{bmatrix}. \tag{A-16}$$

For eight nodes per element $\mathbf{B}_i^T\bar{\mathbf{D}}\mathbf{B}_j$ matrix will be 16×16 and $i,\ j=1,\ 2,\ 3,\ldots 8$.