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New results are presented for the stress conjugate to arbitrary Eulerian strain measures. The conjugate stress depends on two arbitrary quantities: the strain measure $f(\mathbf{V})$ and the corotational rate defined by the spin $\mathbf{\Omega}$. It is shown that for every choice of f there is a unique spin, called the f-spin, which makes the conjugate stress as close as possible to the Cauchy stress. The f-spin reduces to the logarithmic spin when the strain measure is the Hencky strain $\ln \mathbf{V}$. The formulation and the results emphasize the similarities in form of the Eulerian and Lagrangian stresses conjugate to the strains $f(\mathbf{V})$ and $f(\mathbf{U})$, respectively. Many of the results involve the solution to the equation $\mathbf{AX} - \mathbf{XA} = \mathbf{Y}$, which is presented in a succinct format.

1. Introduction

The notion of stress and strain are interlinked, regardless of the existence of a strain energy function. At the most basic level they are related by mechanical power, the rate of work per unit current volume of material,

$$\operatorname{tr}(\boldsymbol{\sigma}\mathbf{D}) = \dot{w}. \tag{1–1}$$

Here $\boldsymbol{\sigma}$ is the Cauchy stress and $\mathbf{D}$ the stretching tensor. This work-conjugate relation is independent of any notion of a reference configuration, although it is useful to introduce one. Let $\mathbf{F}$ be the deformation gradient between the current and reference states, and let $\mathbf{T}$ and $\mathbf{E}$ be the stress and strain associated with the reference state, respectively. $\mathbf{T}$ and $\mathbf{E}$ are mutually conjugate if they satisfy

$$\operatorname{tr}(\mathbf{T}\dot{\mathbf{E}}) = \dot{w}\ \det \mathbf{F}, \tag{1–2}$$

where the factor $\det \mathbf{F}$ arises from the change in volume between the current and reference descriptions. In fact, Equation (1–2) is usually taken as the starting point for determining stress. The choice of the strain $\mathbf{E}$ is not unique, but once chosen it fixes the definition of $\mathbf{T}$ through the work conjugacy of Equation (1–2). It is strange but true that the same simple connection does not apply to the relation between current or Eulerian strain and the Cauchy stress. The difficulty is in the definition of strain, say $\mathbf{e}$. What $\mathbf{e}$ is such that $\dot{\mathbf{e}} = \mathbf{D}$? It turns out that this question is incomplete and that we must broaden it and seek the strain for which $\overset{\circ}{\mathbf{e}} = \mathbf{D}$, where $\circ$ signifies a corotational rate. Actually, the corotational rate itself also has to be found. Fortunately, both the strain and the rate have been determined: Xiao et al. [1998a] showed that the unique solution is obtained by using the Hencky strain $\ln \mathbf{V}$ in combination with the logarithmic rate. But we are getting ahead of ourselves.

It is evident that work-conjugacy is simpler for reference or Lagrangian stress and strain than for their counterparts in the current or Eulerian configuration. Note that the distinction between Lagrangian and

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Eulerian is made explicit by the polar decomposition $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$; quantities associated with or defined by $\mathbf{U}$ and $\mathbf{V}$ will be called Lagrangian and Eulerian, respectively.

It is instructive to review work-conjugacy for Lagrangian stress and strain. The starting point is the fact that the stretching tensor $\mathbf{D}$ is the symmetric part of $\dot{\mathbf{F}}\mathbf{F}^{-1}$. Let the strain be chosen, quite generally, as $\mathbf{E} = f(\mathbf{U})$, where the function f is sufficiently smooth, then Equations (1–1) and (1–2) imply

$$\operatorname{tr}\left(\mathbf{T}\left[\nabla f(\mathbf{U})\right]\dot{\mathbf{U}}\right) = \operatorname{tr}\left(\boldsymbol{\sigma}\mathbf{D}\right)\ \det\mathbf{F}. \tag{1–3}$$

The gradient $\nabla f(\mathbf{U})$ is a fourth order tensor function which will be described later. At the same time the kinematic quantities, strain rate $\dot{\mathbf{U}}$ and stretching $\mathbf{D}$, may be related quite easily (see Appendix A)

$$\dot{\mathbf{U}} = 2(\mathbf{U}\boxtimes\mathbf{I} + \mathbf{I}\boxtimes\mathbf{U})^{-1}(\mathbf{U}\boxtimes\mathbf{U})\,\mathbf{R}^t\mathbf{D}\mathbf{R}. \tag{1–4}$$

Using the independence of $\mathbf{D}$, Equations (1–3) and (1–4) imply, formally at least, that the stress conjugate to the Lagrangian strain $f(\mathbf{U})$ is

$$\mathbf{T} = \left(\nabla f(\mathbf{U})\right)^{-1}\mathbf{T}^{(1)}, \tag{1–5}$$

where $\mathbf{T}^{(1)}$, sometimes called the Biot stress or the Jaumann stress, and $\mathbf{S}$, the second Piola–Kirchhoff stress tensor, are

$$\begin{aligned}\mathbf{T}^{(1)} &= \tfrac{1}{2}(\mathbf{U}\boxtimes\mathbf{I} + \mathbf{I}\boxtimes\mathbf{U})\,\mathbf{S},\\ \mathbf{S} &= \mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-t}\det\mathbf{F}.\end{aligned} \tag{1–6}$$

We have used the symmetry of $\mathbf{T}$ and certain commutative properties to express the stress in Equation (1–5) as a fourth order tensor acting on $\mathbf{T}^{(1)}$. The tensor product notation, $\boxtimes$, explained in the next section, is used throughout as we find it makes results more transparent. Equations (1–5) and (1–6) embody work-conjugacy for arbitrary Lagrangian strain $f(\mathbf{U})$.

Although the notation in Equation (1–5) might be unfamiliar, the result is not (see [Ogden 1984, eq. (3.5.31)]). The fourth order gradient tensor $\nabla f(\mathbf{U})$ is discussed in detail by Norris [2007]. In particular, it is positive definite, symmetric, and invertible for any strain measure function [Hill 1978]. Examples will be presented for the Seth–Hill strain measure functions,

$$f^{(m)}(x) = m^{-1}(x^m - 1).$$

For instance, the stress $\mathbf{T}^{(m)}$ associated with $f^{(m)}(\mathbf{U})$ is

$$\mathbf{T}^{(0)} = \left(\nabla\ln\mathbf{U}\right)^{-1}\mathbf{T}^{(1)} = \int_0^1 \mathrm{d}x\ \mathbf{U}^x\boxtimes\mathbf{U}^{1-x}\,\mathbf{T}^{(1)}, \tag{1–7a}$$

$$\mathbf{T}^{(\frac{1}{3})} = \tfrac{1}{3}(\mathbf{U}^{2/3}\boxtimes\mathbf{I} + \mathbf{U}^{1/3}\boxtimes\mathbf{U}^{1/3} + \mathbf{I}\boxtimes\mathbf{U}^{2/3})\,\mathbf{T}^{(1)}, \tag{1–7b}$$

$$\mathbf{T}^{(\frac{1}{2})} = \tfrac{1}{2}(\sqrt{\mathbf{U}}\boxtimes\mathbf{I} + \mathbf{I}\boxtimes\sqrt{\mathbf{U}})\,\mathbf{T}^{(1)}, \tag{1–7c}$$

$$\mathbf{T}^{(2)} = \mathbf{S}, \tag{1–7d}$$

$$\mathbf{T}^{(-m)} = (\mathbf{U}^m\boxtimes\mathbf{U}^m)\,\mathbf{T}^{(m)}. \tag{1–7e}$$

Some of the conjugate stresses listed are well known, for instance $m = 1, 2, -2$ [Hill 1978; Ogden 1984], and Equation (1–7e) follows from [Ogden 1984, p. 158]. Identities Equation (1–7a)–(1–7c) and the others will become evident later. The second identity in Equation (1–7a) follows from [Norris 2007]. We note that the Piola–Kirchhoff stress is conjugate to $\mathbf{E} = \frac{1}{2}(\mathbf{U}^2 - 1)$, the Green strain, which is typically used in applications.

A principal objective of this paper is to find analogous expressions for the Eulerian stress $\boldsymbol{\tau}$ conjugate to the strain $\mathbf{e} = f(\mathbf{V})$, where $\mathbf{V}$ is the right stretch tensor, and the function f is again arbitrary. We also require that the Cauchy stress be included among the Eulerian stresses, just as the Piola–Kirchhoff stress appears naturally for the Green strain.

Unlike the Lagrangian strains, $\dot{f}(\mathbf{V})$ is not an objective tensor [Dill 2006], and it is known that this strain rate does not, in general, possess a conjugate stress [MacVean 1968]. This difficulty can be avoided by defining conjugacy in terms of corotational strain rates. The corotational rate of a symmetric second order tensor $\mathbf{A}(t)$ is

$$\overset{\circ}{\mathbf{A}} \equiv \dot{\mathbf{A}} + \mathbf{A}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{A},$$

where the skew symmetric tensor $\boldsymbol{\Omega}$ is called the spin. Xiao et al. [1998c; 1998b] showed that an objective spin has the general form

$$\boldsymbol{\Omega} = \mathbf{W} + \mathbb{P}(\mathbf{V})\,\mathbf{D}, \tag{1–8}$$

where $\mathbf{W}$ is the skew symmetric part of $\dot{\mathbf{F}}\,\mathbf{F}^{-1}$ and $\mathbb{P}$ is an isotropic fourth order tensor-valued function of $\mathbf{V}$. Lehmann and Liang [1993] showed that using the rate associated with $\mathbf{R}$, a corotational spin equal to the "twirl" $\boldsymbol{\Omega}^R = \dot{\mathbf{R}}\,\mathbf{R}^t$, the Eulerian and Lagrangian stresses conjugate to $f(\mathbf{V})$ and $f(\mathbf{U})$ are related by $\boldsymbol{\tau} = \mathbf{R}\,\mathbf{T}\,\mathbf{R}^t$. This relationship simply rotates the Lagrangian stress, but does not reproduce the Cauchy stress for any choice of f.

The fundamental relation for Eulerian conjugate stress is based on the finding of Xiao et al. [1997] that

$$\overset{\circ_{\log}}{\ln(\mathbf{V})} = \mathbf{D}, \tag{1–9}$$

where $\circ_{\log}$ denotes an objective corotational rate defined by the logarithmic spin $\boldsymbol{\Omega}^{\log}$ [Xiao et al. 1997]. We will discuss $\boldsymbol{\Omega}^{\log}$ in detail, providing a new derivation and representation, and comparison with $\boldsymbol{\Omega}^R$. Equation (1–9) allows us to define a class of work-conjugate Eulerian stress-strain pairs for all $f(\mathbf{V})$ that includes the Cauchy stress. However, it should be borne in mind that the logarithmic rate is but one of a continuum of possibilities.

A second objective of this paper is a generalization of Equation (1–9) to arbitrary strain measure $f(\mathbf{V})$. Xiao et al. [1997] proved that $\mathbf{D}$ is recovered only from the Hencky strain $\ln(\mathbf{V})$ combined with the logarithmic spin; no other strain measure can yield $\mathbf{D}$, no matter what spin is used. Here we will show that for a given strain measure $f(\mathbf{V})$, there is a unique spin which provides the best *approximation* of $\mathbf{D}$, and the corresponding conjugate stress is the best approximation of the Cauchy stress.

1.1. ***Summary of principal results.*** Our first main result is:

Theorem 1. The stress conjugate to the Eulerian strain $f(\mathbf{V})$ is

$$\boldsymbol{\tau} = \big(\nabla f(\mathbf{V})\big)^{-1}\,\boldsymbol{\tau}^{(1)}(\boldsymbol{\Omega}), \tag{1–10}$$

where $\boldsymbol{\tau}^{(1)}$ depends on the corotational rate used as

$$\boldsymbol{\tau}^{(1)}(\boldsymbol{\Omega}) = \big[\mathbf{V}^2 \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{V}^2 + (\mathbf{V}^2 \boxtimes \mathbf{I} - \mathbf{I} \boxtimes \mathbf{V}^2)\mathbb{P}(\mathbf{V})\big]^{-1}(\mathbf{V} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{V})\,\boldsymbol{\sigma}.$$

We will explain this result in detail, and provide alternative representations for Equation (1–10). We note at this stage the similarity in form between Equations (1–5) and (1–10). In particular, the stress $\boldsymbol{\tau}^{(m)}$ conjugate to the strain $f^{(m)}(\mathbf{V})$ is

$$\begin{aligned}
\boldsymbol{\tau}^{(0)} &= \big(\nabla \ln \mathbf{V}\big)^{-1}\,\boldsymbol{\tau}^{(1)} = \int_0^1 \mathrm{d}x\, \mathbf{V}^x \boxtimes \mathbf{V}^{1-x}\,\boldsymbol{\tau}^{(1)},\\
\boldsymbol{\tau}^{(\frac{1}{3})} &= \tfrac{1}{3}(\mathbf{V}^{2/3}\boxtimes\mathbf{I}+\mathbf{V}^{1/3}\boxtimes\mathbf{V}^{1/3}+\mathbf{I}\boxtimes\mathbf{V}^{2/3})\,\boldsymbol{\tau}^{(1)},\\
\boldsymbol{\tau}^{(\frac{1}{2})} &= \tfrac{1}{2}(\sqrt{\mathbf{V}}\boxtimes\mathbf{I}+\mathbf{I}\boxtimes\sqrt{\mathbf{V}})\,\boldsymbol{\tau}^{(1)},\\
\boldsymbol{\tau}^{(2)} &= 2(\mathbf{V}\boxtimes\mathbf{I}+\mathbf{I}\boxtimes\mathbf{V})^{-1}\boldsymbol{\tau}^{(1)},\\
\boldsymbol{\tau}^{(-m)} &= (\mathbf{V}^m\boxtimes\mathbf{V}^m)\,\boldsymbol{\tau}^{(m)}.
\end{aligned}$$

The second principal result introduces a new spin defined by the Eulerian strain measure.

Theorem 2. For every Eulerian strain measure $f(\mathbf{V})$ there is a unique corotational rate which minimizes the difference between the conjugate stress and the Cauchy stress. The rate is defined by the f-spin $\boldsymbol{\Omega}^f = \mathbf{W} + \mathbb{P}^f\mathbf{D}$ which depends upon the function f via the fourth order projection tensor

$$\mathbb{P}^f = (\mathbf{V}\boxtimes\mathbf{I}-\mathbf{I}\boxtimes\mathbf{V})^*\big[\big(\nabla f(\mathbf{V})\big)^{-1} - (\mathbf{V}\boxtimes\mathbf{I}+\mathbf{I}\boxtimes\mathbf{V})^{-1}(\mathbf{V}^2\boxtimes\mathbf{I}+\mathbf{I}\boxtimes\mathbf{V}^2)\big],$$

and $\mathbb{A}^*$ denotes the pseudo-inverse (or Moore–Penrose inverse) of the tensor $\mathbb{A}$. The conjugate stress using the f-spin is

$$\boldsymbol{\tau} = \boldsymbol{\sigma}^f \equiv \boldsymbol{\sigma} + \sum_{i=1}^{n}\Big(\frac{1}{\lambda_i f'(\lambda_i)} - 1\Big)\mathbf{V}_i\boxtimes\mathbf{V}_i\boldsymbol{\sigma}, \tag{1–11}$$

where λ_i are the principal stretches, $\mathbf{V}_i$ the principal dyads, that is, the eigenvalues and eigentensors of $\mathbf{V}$, and the eigen-index $n \in \{1, 2, 3\}$ is the number of distinct eigenvalues. The conjugate stress is minimal in the sense that $|\boldsymbol{\tau} - \boldsymbol{\sigma}| > |\boldsymbol{\sigma}^f - \boldsymbol{\sigma}|$ for any other corotational rate.

The pseudo-inverse is a unique quantity and will be defined in detail later.

The logarithmic spin [Xiao et al. 1998a] is a very special case of the f-spin. It is clear from Equation (1–11) that when $f(x) = \ln x$ *and* the f-spin is used then the conjugate stress is simply the Cauchy stress, so $\boldsymbol{\sigma}^{\ln} = \boldsymbol{\sigma}$. Note that $\boldsymbol{\sigma}$ is recovered as $\boldsymbol{\tau}^{(0)}$, the stress conjugate to the Hencky strain $\ln(\mathbf{V})$. No other spin reproduces the Cauchy stress as the conjugate of any strain [Xiao et al. 1997]. This emphasizes the mutual relation between the Hencky strain and the logarithmic spin.

1.2. ***Review and plan of the paper.*** No attempt is made to summarize the considerable literature on work-conjugacy, strain measures and associated stresses, although two introductory reviews are worthy of mention. Curnier and Rakotomanana [1991] provide an instructive overview of strain measures and conjugate stresses, with extensive references to the literature prior to 1990. A more concise but in-depth description of work conjugacy and its implications is given by Ogden [1984]. These reviews and most of the work prior to 1991 dealt with stress conjugate to Lagrangian strain measures, although there had been some relevant work on quantities related to $\ln(\mathbf{V})$. For instance, Fitzgerald [1980] considered the stress conjugate to $\ln(\mathbf{V})$ in the context of hyperelasticity. Hoger [1987] derived expressions for the rate of change of $\mathbf{U}$, which subsequently proved useful for Lehmann et al. [1991; 1993] when they considered $\mathbf{V}$ specifically. The focus here is on Eulerian strain and its work-conjugate stress, and builds upon developments in the 1990s. Lehmann and Liang [1993] introduced a clear procedure to extend the idea of work-conjugacy to strains whose rates are not objective in a fixed frame (see also [Lehmann and Guo 1991]). The idea, reviewed in Section 4, permits the use of corotational rates. This is especially important for Eulerian strain measures since Xiao et al. [1997] proved that the only way to obtain $\mathbf{D}$ with Eulerian strain is as the logarithmic rate of $\ln(\mathbf{V})$. We emphasize that the pointwise rate of working $\dot{w}$ is the focus as we consider its implications for pointwise stress based on different definitions of strain. No assumptions of material homogeneity, isotropy, or otherwise is assumed or required.

In a series of groundbreaking papers, Xiao et al. [1997; 1998b; 1998c] provide the most complete and thorough analysis of Eulerian conjugate stress and strain. Culminating with [Xiao et al. 1998a], these authors showed that the notion of conjugate stress is just as relevant to Eulerian strain as it is for Lagrangian strain. Because the role of the rate, or spin, is central to the Eulerian problem but is absent from Lagrangian work-conjugacy, it is essential to have a thorough understanding of the possible spin tensors and their dependence on quantities such as $\mathbf{D}$, $\mathbf{W}$, and $\mathbf{V}$. Once this is understood then the form of the conjugate stress becomes apparent. Xiao et al. [1998a] derived expressions for the Eulerian conjugate stress for arbitrary strain measures f and for arbitrary permissible corotational strain rate. Their subsequent work has highlighted the role of the logarithmic rate and the Hencky strain in applications to hyperelasticity and other constitutive theories; see [Xiao et al. 2006] for a thorough review.

This paper presents new results which extend the work of Xiao et al. in several directions. The introduction and discovery of the role of the f-spin shows that there is a certain unique conjugate stress associated with every Eulerian strain measure. The dual formulation for the Eulerian and Lagrangian conjugate stresses in Equations (1–5) and (1–10) further emphasizes the similarities in the two descriptions. The formulation throughout is in direct tensor notation, which we believe makes the results more transparent.

The plan of the paper is as follows. The notation is introduced in Section 2, where the gradient and the pseudo-inverse of a tensor are defined. Corotational strain rates are discussed in Section 3 and some basic results for Eulerian strain measures are derived. The f-spin is introduced and discussed in Section 4. It is shown that the corotational rate defined by the f-spin, or f-rate, has certain unique and desirable properties. The main results for conjugate stress-strain pairs are deduced in Section 5.

2. Tensors functions and the pseudo-inverse.

2.1. ***Preliminaries.*** We will be dealing with tensors of second and fourth order. Second order tensors act on vectors in a three dimensional inner product space, $\mathbf{x} \to \mathbf{A}\mathbf{x}$ with transpose $\mathbf{A}^t$ such that $\mathbf{y} \cdot \mathbf{A}\mathbf{x} = \mathbf{x} \cdot \mathbf{A}^t\mathbf{y}$. Spaces of symmetric and skew-symmetric tensors are distinguished, Lin = Sym $\oplus$ Skw where $\mathbf{A} \in$ Sym (Skw) if and only if $\mathbf{A}^t = \mathbf{A}$ ($\mathbf{A}^t = -\mathbf{A}$). The inner product on Lin is defined by $\mathbf{A} \cdot \mathbf{B} = \operatorname{tr}(\mathbf{A}\mathbf{B}^t)$. The product $\mathbf{A}\mathbf{B} \in$ Lin is defined by $\mathbf{y} \cdot \mathbf{A}\mathbf{B}\mathbf{x} = (\mathbf{A}^t\mathbf{y}) \cdot \mathbf{B}\mathbf{x}$.

Psym is the space of positive definite second order tensors. When dealing with functions of a symmetric tensor it is often useful to rephrase the functional form in terms of the spectral decomposition:

$$\mathbf{A} = \sum_{i=1}^{n} \alpha_i \mathbf{A}_i, \qquad \mathbf{I} = \sum_{i=1}^{n} \mathbf{A}_i, \qquad \mathbf{A}_i \mathbf{A}_j = \begin{cases} \mathbf{A}_i, & i = j, \\ 0, & i \neq j, \end{cases}$$

where $\mathbf{A}_i \in$ Psym, and $n \leq 3$ is the eigen-index. Thus,

$$f(\mathbf{A}) = \sum_{i=1}^{n} f(\alpha_i)\mathbf{A}_i.$$

The Poisson bracket of two second order tensors is

$$\{\mathbf{A}, \mathbf{B}\} = \mathbf{AB} - \mathbf{BA}.$$

$\mathbb{L}$in is the space of fourth order tensors acting on Lin, $\mathbf{X} \to \mathbb{A}\mathbf{X}$ with transpose $\mathbb{A}^t$ such that

$$\mathbf{Y} \cdot \mathbb{A}\mathbf{X} = \mathbf{X} \cdot \mathbb{A}^t \mathbf{Y}$$

for all $\mathbf{X}$, $\mathbf{Y} \in$ Lin. The vector space may be decomposed $\mathbb{L}$in $=$ $\mathbb{S}$ym$\oplus$$\mathbb{S}$kw where $\mathbb{S}$ym and $\mathbb{S}$kw denote the spaces of symmetric ($\mathbb{A}^t = \mathbb{A}$) and skew-symmetric ($\mathbb{A}^t = -\mathbb{A}$) tensors, respectively. Any $\mathbb{A} \in \mathbb{L}$in can be uniquely partitioned into symmetric and skew parts: $\mathbb{A} = \mathbb{A}^{(+)} + \mathbb{A}^{(-)}$, where $\mathbb{A}^{(\pm)} = (\mathbb{A} \pm \mathbb{A}^t)/2$. The identity $\mathbb{I}$ satisfies $\mathbb{I}\mathbf{X} = \mathbf{X}$ for all $\mathbf{X} \in$ Lin. The product $\mathbb{AB} \in \mathbb{L}$in is defined by $\mathbf{Y} \cdot \mathbb{AB}\mathbf{X} = (\mathbb{A}^t\mathbf{Y}) \cdot \mathbb{B}\mathbf{X}$. $\mathbb{P}$sym is the space of positive definite fourth order tensors: $\mathbb{A} \in \mathbb{P}$sym if and only if $\mathbf{X} \cdot \mathbb{A}\mathbf{X} > 0$, for all nonzero $\mathbf{X} \in$ Sym.

The square tensor product $\mathbf{X} \boxtimes \mathbf{Y}$, Lin $\times$ Lin $\to \mathbb{L}$in, is defined by [Rosati 2000]

$$(\mathbf{X} \boxtimes \mathbf{Y})\, \mathbf{Z} = \mathbf{XZY}^t, \qquad \text{for all } \mathbf{Z} \in \text{Lin}.$$

In particular, we note the property $(\mathbf{A} \boxtimes \mathbf{B})(\mathbf{X} \boxtimes \mathbf{Y}) = (\mathbf{AX}) \boxtimes (\mathbf{BY})$.

2.1.1. *The tensor gradient function and its inverse.* The gradient of a tensor function $f(\mathbf{A})$ is a fourth order tensor $\nabla f \in \mathbb{L}$in defined by

$$\nabla f(\mathbf{A})\, \mathbf{X} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big[f(\mathbf{A} + \epsilon\mathbf{X}) - f(\mathbf{A})\big]. \tag{2–1}$$

We make extensive use of the following representation, which uses the spectral form of $\mathbf{A}$,

$$\nabla f(\mathbf{A}) = \sum_{i,j=1}^{n} \frac{f(\alpha_i) - f(\alpha_j)}{\alpha_i - \alpha_j}\, \mathbf{A}_i \boxtimes \mathbf{A}_j,$$

where the ratio becomes $f'(\alpha_i)$ for $i = j$. Equation (2–1) for the first derivative is well known[Ogden 1984; Xiao 1995]. Norris [2007] provides formulas for the n^{th} derivative of a tensor valued-function. We define the inverse tensor function $\triangle f(\mathbf{A}) \in \mathbb{L}$in by

$$\triangle f(\mathbf{A}) \equiv \big(\nabla f(\mathbf{A})\big)^{-1} = \sum_{i,j=1}^{n} \frac{\alpha_i - \alpha_j}{f(\alpha_i) - f(\alpha_j)}\, \mathbf{A}_i \boxtimes \mathbf{A}_j,$$

where the ratio is $1/f'(\alpha_i)$ for $i=j$. The definition of $\triangle f(\mathbf{A})$ is problematic if $f'(\alpha_i)$ vanishes, but we preclude this possibility next by restricting consideration to strictly monotonic functions: strain measure functions.

2.2. *Strain measure functions.* The function f is a *strain measure* [Hill 1978; Scheidler 1991] if it is a smooth function $f:\mathbb{R}^+\to\mathbb{R}$ which satisfies

$$f(1)=0, \qquad f'(1)=1, \qquad f'>0.$$

It may be shown [Norris 2007] that the gradient of a strain measure function and its inverse are positive definite fourth order tensors, for instance, $\nabla f(\mathbf{A}),\ \triangle f(\mathbf{A})\in\mathbb{P}\mathrm{sym}$. We restrict attention to strain measure functions for the remainder of the paper.

2.3. *The pseudo-inverse.* For $\mathbf{A}\in\mathrm{Psym}$ consider the equation

$$\big\{\mathbf{A},\mathbf{X}\big\}=\mathbf{Y}, \tag{2–2}$$

for the unknown $\mathbf{X}$ in terms of $\mathbf{Y}$. It is assumed that $\mathbf{Y}$ is either symmetric or skew and that $\mathbf{X}$ is of the opposite parity [Dui 2006]. The equation can be written $\mathbf{AX}-\mathbf{XA}=\mathbf{Y}$, or

$$\mathbb{J}(\mathbf{A})\mathbf{X}=\mathbf{Y}, \tag{2–3}$$

where $\mathbb{J}(\mathbf{A})\equiv\mathbf{A}\boxtimes\mathbf{I}-\mathbf{I}\boxtimes\mathbf{A}$. We will only consider $\mathbb{J}(\mathbf{A})$ for symmetric $\mathbf{A}$, implying $\mathbb{J}\in\mathbb{S}\mathrm{ym}$ and $\mathbb{J}$ maps Sym$\to$Skw and Skw$\to$Sym. Therefore, $\mathbb{J}$ does not possess eigenvalues, eigenvectors or an inverse in the usual sense.

The unique solution of the tensorial Equation (2–2) is [Norris 2007]

$$\mathbf{X}=\mathbb{J}^*(\mathbf{A})\mathbf{Y}=(\mathbf{A}\boxtimes\mathbf{I}-\mathbf{I}\boxtimes\mathbf{A})^*\mathbf{Y}. \tag{2–4}$$

The pseudo-inverse, or equivalently the Moore–Penrose inverse, $\mathbb{J}^*$, is defined such that

$$\mathbb{J}\mathbb{J}^*\mathbb{J}=\mathbb{J}, \qquad \mathbb{J}^*\mathbb{J}\mathbb{J}^*=\mathbb{J}^*. \tag{2–5}$$

The spectral forms of $\mathbb{J}(\mathbf{A})$ and its pseudo-inverse are

$$\mathbb{J}(\mathbf{A})=\sum_{i,j=1}^{n}(\alpha_i-\alpha_j)\,\mathbf{A}_i\boxtimes\mathbf{A}_j, \qquad \mathbb{J}^*(\mathbf{A})=\sum_{\substack{i,j=1\\ i\neq j}}^{n}(\alpha_i-\alpha_j)^{-1}\mathbf{A}_i\boxtimes\mathbf{A}_j,$$

which clearly satisfy Equation (2–5).

Further insight into the pseudo-inverse is gained by introducing the set of $N\le 6$ fourth order tensors associated with $\mathbf{A}\in\mathrm{Sym}$,

$$\mathbb{A}_I=\begin{cases}\mathbf{A}_I\boxtimes\mathbf{A}_I, & I=1,\dots,n,\\ \mathbf{A}_i\boxtimes\mathbf{A}_j+\mathbf{A}_j\boxtimes\mathbf{A}_i, & I=n+1,\dots N.\end{cases} \tag{2–6}$$

$N = 6$ for $n = 3$, and the indices $I = 4, 5, 6$ correspond to $(i, j) = (2, 3), (3, 1), (1, 2)$, respectively. Similarly $N = 3$ if $n = 2$ and $N = 1$ if $n = 1$. Note that

$$\mathbb{I} = \sum_{I=1}^{N} \mathbb{A}_I, \tag{2–7}$$

and

$$\mathbb{A}_I \mathbb{A}_J = \begin{cases} \mathbb{A}_I & I = J, \\ 0, & I \neq J. \end{cases} \tag{2–8}$$

The identity $\mathbb{I} = \mathbf{I} \boxtimes \mathbf{I}$ implies the partition of unity in Equation (2–7), and it may be readily checked that the $\mathbb{A}_I$ satisfy the orthogonality conditions of Equation (2–8).

The pseudo-inverse satisfies

$$\mathbb{J}^* \mathbb{J} = \mathbb{J}\mathbb{J}^* = \mathbb{I} - \sum_{I=1}^{n} \mathbb{A}_I = \sum_{I=n+1}^{N} \mathbb{A}_I. \tag{2–9}$$

This is never equal to the identity $\mathbb{I}$, which is the property that distinguishes the pseudo-inverse from the standard notion of inverse. Further properties of the pseudo-inverse are presented in [Norris 2007]. The explicit solution of Equation (2–3) can be expressed in a variety of ways without the use of fourth order tensors. Perhaps the simplest is the recently discovered solution of Dui et al. [2007]:

$$\mathbf{X} = \big(3\mathbf{A}'^2 - \tfrac{1}{2}(\operatorname{tr} \mathbf{A}'^2)\, \mathbf{I}\big)^{-1} (2\mathbf{A}'\mathbf{Y} + \mathbf{Y}\mathbf{A}'),$$

where $\mathbf{A}'$ is the deviatoric part of $\mathbf{A}$.

3. Kinematics

3.1. ***Basics.*** The polar decomposition of the deformation gradient is $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$ where $\mathbf{R} \in \mathrm{SO}(3)$ satisfies $\mathbf{R}\mathbf{R}^t = \mathbf{R}^t\mathbf{R} = \mathbf{I}$ and the right and left stretch tensors $\mathbf{U}$ and $\mathbf{V}$ are positive definite and related by $\mathbf{V} = (\mathbf{R} \boxtimes \mathbf{R})\mathbf{U}$. The fundamental Eulerian strain can be taken as either $\mathbf{V}$ or its square, $\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^t$. The spectral representations of $\mathbf{V}$ and $\mathbf{B}$ are

$$\mathbf{V} = \sum_{i=1}^{n} \lambda_i \mathbf{V}_i, \qquad \mathbf{B} = \sum_{i=1}^{n} \beta_i \mathbf{V}_i, \qquad \mathbf{V}_i \mathbf{V}_j = \begin{cases} \mathbf{V}_i, & i = j, \\ 0, & i \neq j, \end{cases} \tag{3–1}$$

where $\lambda_i > 0$ and $\beta_i = \lambda_i^2$.

The rate of change of $\mathbf{B}$ is $\dot{\mathbf{B}} = \mathbf{L}\mathbf{B} + \mathbf{B}\mathbf{L}^t$ where $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$. Let $\mathbf{D} \in \mathrm{Sym}$ and $\mathbf{W} \in \mathrm{Skw}$ be the symmetric and skew-symmetric parts of $\mathbf{L}$, respectively. Thus, $\mathbf{L} = \mathbf{D} + \mathbf{W}$ and $\dot{\mathbf{B}}$ can be expressed

$$\dot{\mathbf{B}} = (\mathbf{I} \boxtimes \mathbf{B} - \mathbf{B} \boxtimes \mathbf{I})\, \mathbf{W} + (\mathbf{I} \boxtimes \mathbf{B} + \mathbf{B} \boxtimes \mathbf{I})\, \mathbf{D}. \tag{3–2}$$

We will find this form useful for deriving more general strain rates.

3.2. ***Corotational rates.*** Let $\mathbf{A}(t)$ be a symmetric second order tensor, and $\mathbf{\Omega}$ is skew and arbitrary. Define the corotational rate

$$\overset{\circ}{\mathbf{A}} \equiv \dot{\mathbf{A}} + \{\mathbf{A}, \mathbf{\Omega}\}, \qquad \mathbf{\Omega} \in \mathrm{Skw}. \tag{3–3}$$

For any $\mathbf{\Omega}(t) \in \mathrm{Skw}$ we can identify a rotation $\mathbf{Q}(t) \in \mathrm{SO}(3)$ such that

$$\dot{\overline{\mathbf{Q}\mathbf{A}\mathbf{Q}^t}} = \mathbf{Q}\overset{\circ}{\mathbf{A}}\mathbf{Q}^t.$$

Differentiating the left member and using Equation (3–3) for the right member implies that $\mathbf{\Omega} = -\mathbf{Q}^t\dot{\mathbf{Q}}$. Hence, $\mathbf{Q}$ must satisfy $\dot{\mathbf{Q}} = -\mathbf{Q}\mathbf{\Omega}$, with solution unique up to a rigid body rotation. The corotational rate may therefore be interpreted as the Lie derivative with respect to spatial rotation defined by $\mathbf{Q}(t)$. Thus, let ϕ define the mapping (rotation) $\mathbf{x} \to \mathbf{Q}\mathbf{x}$, then the corotational rate is $\phi[\frac{\mathrm{d}}{\mathrm{d}t}\phi^{-1}(\cdot)]$.

The Jaumann rate $\overset{\circ J}{\mathbf{A}}$ defined by $\mathbf{\Omega} = \mathbf{W}$ corresponds to $\mathbb{P} = 0$ in Equation (1–8). Using the latter formula to parameterize the spin $\mathbf{\Omega}$ allows us to express the general corotational rate of $\mathbf{A}$ as

$$\overset{\circ}{\mathbf{A}} = \overset{\circ J}{\mathbf{A}} + (\mathbf{A} \boxtimes \mathbf{I} - \mathbf{I} \boxtimes \mathbf{A})\mathbb{P}(\mathbf{V})\,\mathbf{D}. \tag{3–4}$$

Equation (3–2) implies that the Jaumann rate of $\mathbf{B}$ is $\overset{\circ J}{\mathbf{B}} = (\mathbf{I} \boxtimes \mathbf{B} + \mathbf{B} \boxtimes \mathbf{I})\mathbf{D}$. The general rate $\overset{\circ}{\mathbf{B}}$ then follows from Equation (3–4), and $\overset{\circ}{\mathbf{V}}$ can be determined from the identity $\overset{\circ}{\mathbf{B}} = (\mathbf{V} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{V})\,\overset{\circ}{\mathbf{V}}$. In summary, the general form of the corotational rate of the fundamental Eulerian strains are

$$\overset{\circ}{\mathbf{B}} = \big[\mathbf{B} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{B} + (\mathbf{B} \boxtimes \mathbf{I} - \mathbf{I} \boxtimes \mathbf{B})\mathbb{P}(\mathbf{V})\big]\,\mathbf{D}, \tag{3–5a}$$

$$\overset{\circ}{\mathbf{V}} = \big[(\mathbf{V} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{V})^{-1}(\mathbf{V}^2 \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{V}^2) + (\mathbf{V} \boxtimes \mathbf{I} - \mathbf{I} \boxtimes \mathbf{V})\mathbb{P}(\mathbf{V})\big]\,\mathbf{D}. \tag{3–5b}$$

3.3. ***Spins.*** Many candidates have been considered from the infinity of possible spins [Dill 2006]. For instance, the polar spin

$$\mathbf{\Omega}^R = \dot{\mathbf{R}}\mathbf{R}^t, \tag{3–6}$$

corresponding to $\mathbf{Q} = \mathbf{R}^t$, is useful as a comparison spin. Other common spins [Xiao et al. 1998c] are Ω^E defined by the *twirl* of the Eulerian principal axes and $\mathbf{\Omega}^L$ related to the Lagrangian principal axes. It is shown in Appendix A that $\mathbf{\Omega}^\alpha = \mathbf{W} + \mathbb{P}^\alpha\mathbf{D}$, $\alpha = R, E, L$, where

$$\mathbb{P}^R = (\mathbf{I} \boxtimes \mathbf{V} - \mathbf{V} \boxtimes \mathbf{I})(\mathbf{I} \boxtimes \mathbf{V} + \mathbf{V} \boxtimes \mathbf{I})^{-1}, \tag{3–7a}$$

$$\mathbb{P}^E = (\mathbf{I} \boxtimes \mathbf{V}^2 - \mathbf{V}^2 \boxtimes \mathbf{I})^*(\mathbf{I} \boxtimes \mathbf{V}^2 + \mathbf{V}^2 \boxtimes \mathbf{I}), \tag{3–7b}$$

$$\mathbb{P}^L = (\mathbf{I} \boxtimes \mathbf{V}^2 - \mathbf{V}^2 \boxtimes \mathbf{I})^*\, 2\mathbf{V} \boxtimes \mathbf{V}. \tag{3–7c}$$

The three spins $\mathbf{\Omega}^R$, $\mathbf{\Omega}^E$ and $\mathbf{\Omega}^L$ are related by $\mathbf{\Omega}^E - \mathbf{\Omega}^R = \mathbf{\Omega}^L - \mathbf{W}$ (see Appendix A). The fourth order projection tensors are therefore connected by $\mathbb{P}^E - \mathbb{P}^L = \mathbb{P}^R$, and we note the additional relation $\mathbb{P}^E + \mathbb{P}^L = \mathbb{P}^{R*}$, which is readily verified.

The most general form of the isotropic tensor-valued function $\mathbb{P} \in \mathbb{S}\mathrm{ym}$ involves three isotropic scalar functions ν_1, ν_2, ν_3 [Xiao et al. 1998c],

$$\mathbb{P}(\mathbf{V}) = (\mathbf{V} \boxtimes \mathbf{I} - \mathbf{I} \boxtimes \mathbf{V})\big[\nu_1\mathbb{I} + \nu_2(\mathbf{V} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{V}) + \nu_3\mathbf{V} \boxtimes \mathbf{V}\big] = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} p_{ij}\,\mathbf{V}_i \boxtimes \mathbf{V}_j, \tag{3–8}$$

where

$$p_{ij} = (\lambda_i - \lambda_j)\left[\nu_1 + (\lambda_i + \lambda_j)\nu_2 + \lambda_i\lambda_j\nu_3\right], \qquad \nu_k = \nu_k(I_1, I_2, I_3), \qquad k = 1, 2, 3.$$

Here, I_1, I_2, I_3 are the invariants of $\mathbf{V}$: $I_1 = \operatorname{tr}(\mathbf{V})$, $I_2 = \frac{1}{2} I_1^2 - \frac{1}{2}\operatorname{tr}(\mathbf{V}^2)$, $I_3 = \det \mathbf{V}$.

The corotational rate of $\mathbf{V}$ can now be written

$$\overset{\circ}{\mathbf{V}} = \mathbb{Q}\mathbf{D}, \tag{3–9}$$

where the fourth order tensor $\mathbb{Q} \in \mathbb{S}\text{ym}$ follows from Equation (3–5b),

$$\mathbb{Q} = (\mathbf{V}\boxtimes\mathbf{I} + \mathbf{I}\boxtimes\mathbf{V})^{-1}(\mathbf{V}^2\boxtimes\mathbf{I} + \mathbf{I}\boxtimes\mathbf{V}^2) + (\mathbf{V}\boxtimes\mathbf{I} - \mathbf{I}\boxtimes\mathbf{V})\mathbb{P}(\mathbf{V}) \;=\; \sum_{i,j=1}^{n} q_{ij}\,\mathbf{V}_i \boxtimes \mathbf{V}_j,$$

and

$$q_{ij} = (\lambda_i - \lambda_j)p_{ij} + \frac{\lambda_i^2 + \lambda_j^2}{\lambda_i + \lambda_j}. \tag{3–10}$$

3.4. ***Eulerian strain measures.*** The Lagrangian Seth–Hill strain $\mathbf{E}^{(m)} = m^{-1}(\mathbf{U}^m - \mathbf{I})$ corresponds to $f(x) = f^{(m)}(x)$. We define the analogous Eulerian strain

$$\mathbf{e}^{(m)} = f^{(m)}(\mathbf{V}) = m^{-1}(\mathbf{V}^m - \mathbf{I}),$$

and note in particular the Hencky strain $\mathbf{e}^{(0)} = \ln \mathbf{V}$. Other examples include

$$\mathbf{e}^{(1)} = \mathbf{V} - \mathbf{I}, \quad \mathbf{e}^{(2)} = \tfrac{1}{2}(\mathbf{B} - \mathbf{I}), \quad \mathbf{e}^{(-1)} = \mathbf{I} - \mathbf{V}^{-1}, \quad \mathbf{e}^{(-2)} = \tfrac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}).$$

3.5. ***Eulerian strain rates.*** We now present some identities for the corotational rates of Eulerian strains. These will prove useful later in deriving conjugate Eulerian stresses. The first identity applies to arbitrary strain measures:

Lemma 1. The corotational rate of any Eulerian strain measure $f(\mathbf{V})$ is

$$f(\overset{\circ}{\mathbf{V}}) = [\nabla f(\mathbf{V})]\,\mathbb{Q}(\mathbf{V})\,\mathbf{D}.$$

The proof is a simple application of the chain rule, using Equation (3–9) for $\overset{\circ}{\mathbf{V}}$. This separates the dependence on the strain measure f from the dependence on the particular corotational rate used, which determines $\mathbb{Q}$.

The second identity connects the strain rate with the Hencky strain:

Lemma 2. The strain rate of any Eulerian strain measure $f(\mathbf{V})$ can be expressed in terms of the Hencky strain rate as

$$f(\overset{\circ}{\mathbf{V}}) = [\nabla f(\mathbf{V})](\triangle \ln \mathbf{V}) \ln \overset{\circ}{\mathbf{V}}.$$

The proof is a straightforward generalization of the chain rule of differentiation [Xiao et al. 1998c, Theorem 2]. Let $\mathbf{M} = \ln \mathbf{V}$ and $f(\mathbf{V}) = \hat{f}(\mathbf{M})$ then,

$$\hat{f}(\overset{\circ}{\mathbf{M}}) = \nabla_{\mathbf{M}} \hat{f}(\mathbf{M})\,\overset{\circ}{\mathbf{M}} = \nabla f(\mathbf{V})\,(\nabla_{\mathbf{M}}\mathbf{V})\,\overset{\circ}{\mathbf{M}}\,.$$

But the fourth order tensor $\nabla_{\mathbf{M}}\mathbf{V}$ is just the inverse of $\nabla_{\mathbf{V}}\mathbf{M}$ since $\nabla_{\mathbf{V}}\mathbf{V} = \mathbb{I}$.

4. The f-spin and the logarithmic spin

4.1. ***Strain rate and the stretching tensor.*** In order to make the connection between the kinematics and the power $\dot{w}$ we must relate some strain rate to the stretching tensor $\mathbf{D}$. A general connection can be found by starting with the rate of change of an arbitrary tensor valued function of $\mathbf{B}$. Thus,

$$\dot{\tilde{f}}(\mathbf{B}) = [\nabla \tilde{f}(\mathbf{B})]\,\dot{\mathbf{B}} = \sum_{i,j=1}^{n} \frac{\tilde{f}(\beta_i) - \tilde{f}(\beta_j)}{\beta_i - \beta_j}\,\mathbf{V}_i \boxtimes \mathbf{V}_j \dot{\mathbf{B}},$$

where the temporary definition $\tilde{f}(x) = f(x^2)$ is used, so that $f(\mathbf{V}) = \tilde{f}(\mathbf{B})$. Reverting to $f(\mathbf{V})$ and using $\beta_i = \lambda_i^2$, the rate of change of the associated function of $\mathbf{V}$ is

$$\dot{f}(\mathbf{V}) = \sum_{i,j=1}^{n} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i^2 - \lambda_j^2}\,\mathbf{V}_i \boxtimes \mathbf{V}_j \dot{\mathbf{B}}, \tag{4–1}$$

where the ratio becomes $f'(\lambda_i)/(2\lambda_i)$ for $i = j$. Substituting $\dot{\mathbf{B}}$ into Equation (4–1) and using the filtering properties of $\mathbf{V}_i$, such as $(\mathbf{V}_i \boxtimes \mathbf{V}_j)(\mathbf{I} \boxtimes \mathbf{B}) = \beta_j \mathbf{V}_i \boxtimes \mathbf{V}_j$, gives

$$\begin{aligned}\dot{f}(\mathbf{V}) &= \sum_{i,j=1}^{n} \big(f(\lambda_j) - f(\lambda_i)\big)\mathbf{V}_i \boxtimes \mathbf{V}_j \mathbf{W} + \sum_{i,j=1}^{n} \frac{f(\lambda_i) - f(\lambda_j)}{\beta_i - \beta_j}\,(\beta_i + \beta_j)\mathbf{V}_i \boxtimes \mathbf{V}_j\,\mathbf{D}\\ &= \big\{\mathbf{W},\, f(\mathbf{V})\big\} + \sum_{i,j=1}^{n} \frac{\lambda_i^2 + \lambda_j^2}{\lambda_i^2 - \lambda_j^2}\big(f(\lambda_i) - f(\lambda_j)\big)\,\mathbf{V}_i \boxtimes \mathbf{V}_j\,\mathbf{D}.\end{aligned}$$

Adding and subtracting terms, this becomes

$$\dot{f}(\mathbf{V}) = \widehat{\mathbf{D}} + \big\{\mathbf{W},\, f(\mathbf{V})\big\} + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \Big[\frac{\lambda_i^2 + \lambda_j^2}{\lambda_i^2 - \lambda_j^2} - \frac{1}{f(\lambda_i) - f(\lambda_j)}\Big]\big(f(\lambda_i) - f(\lambda_j)\big)\,\mathbf{V}_i \boxtimes \mathbf{V}_j\,\mathbf{D}, \tag{4–2}$$

where $\widehat{\mathbf{D}}$ is a modified version of the stretching tensor,

$$\widehat{\mathbf{D}} = \mathbf{D} + \sum_{i=1}^{n} \big[\lambda_i f'(\lambda_i) - 1\big]\,\mathbf{V}_i \boxtimes \mathbf{V}_i\,\mathbf{D}. \tag{4–3}$$

Note that the double sum in Equation (4–2) excludes the $i = j$ terms. We can therefore rewrite it in a form suggestive of a new corotational rate,

$$\dot{f}(\mathbf{V}) = \widehat{\mathbf{D}} + \big\{\mathbf{\Omega}^f,\, f(\mathbf{V})\big\}, \qquad \mathbf{\Omega}^f = \mathbf{W} + \mathbb{P}^f\mathbf{D}, \tag{4–4}$$

where $\mathbf{\Omega}^f \in \mathrm{Skw}$ is called the f-spin, and its fourth order projection tensor is

$$\mathbb{P}^f = (\mathbf{V} \boxtimes \mathbf{I} - \mathbf{I} \boxtimes \mathbf{V})^* \big[\triangle f(\mathbf{V}) - (\mathbf{V} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{V})^{-1}(\mathbf{V}^2 \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{V}^2)\big]. \tag{4–5}$$

Alternatively, $\mathbb{P}^f$ can be expressed in the form Equation (3–8) with matrix elements

$$p_{ij}^f = \frac{1}{f(\lambda_i) - f(\lambda_j)} - \frac{\lambda_i^2 + \lambda_j^2}{\lambda_i^2 - \lambda_j^2}.$$

Note that $\boldsymbol{\Omega}^f$ can blow up, but the action $\{\boldsymbol{\Omega}^f, g(\mathbf{V})\}$ remains finite for any differentiable function g, including f. In particular, the f-spin is an objective material spin in the sense defined by [Xiao et al. 1998c].

The corotational rate associated with the f-spin is defined in the usual manner as

$$\overset{\circ f}{g(\mathbf{V})} = \dot{g}(\mathbf{V}) + \{g(\mathbf{V}), \boldsymbol{\Omega}^f\}.$$

The reason for introducing this new rate is $\overset{\circ f}{f(\mathbf{V})} = \widehat{\mathbf{D}}$, which follows from Equation (4–4). This shows that for a particular choice of spin the corotational rate of an arbitrary strain measure $f(\mathbf{V})$ is related to the modified stretching tensor $\widehat{\mathbf{D}}$. The important point is that this is the closest, in a sense to be defined, the strain rate can get to the actual stretching tensor $\mathbf{D}$. These ideas are made concrete through:

Lemma 3. For any objective corotational rate

$$|\overset{\circ}{f(\mathbf{V})} - \mathbf{D}|^2 = |\overset{\circ}{f(\mathbf{V})} - \widehat{\mathbf{D}}|^2 + |\widehat{\mathbf{D}} - \mathbf{D}|^2, \tag{4–6}$$

where $\widehat{\mathbf{D}}$ is the modified stretching tensor defined by Equation (4–3).

The proof follows by writing

$$\begin{aligned} \overset{\circ}{f(\mathbf{V})} - \mathbf{D} &= \overset{\circ}{f(\mathbf{V})} - \widehat{\mathbf{D}} + (\widehat{\mathbf{D}} - \mathbf{D}) = \overset{\circ}{f(\mathbf{V})} - \overset{\circ f}{f(\mathbf{V})} + (\widehat{\mathbf{D}} - \mathbf{D}) \\ &= \sum_{\substack{i,j=1\\ i\neq j}}^{n} (p_{ij} - p_{ij}^f)\big(f(\lambda_i) - f(\lambda_j)\big)\mathbf{V}_i \boxtimes \mathbf{V}_j\, \mathbf{D} + \sum_{i=1}^{n} \big[\lambda_i f'(\lambda_i) - 1\big]\mathbf{V}_i \boxtimes \mathbf{V}_i\, \mathbf{D}. \end{aligned} \tag{4–7}$$

Hence,

$$|\overset{\circ}{f(\mathbf{V})} - \mathbf{D}|^2 = \sum_{\substack{i,j=1\\ i\neq j}}^{n} \big[(p_{ij} - p_{ij}^f)\big(f(\lambda_i) - f(\lambda_j)\big)\operatorname{tr}(\mathbf{V}_i\mathbf{D}\mathbf{V}_j)\big]^2 + \sum_{i=1}^{n} \big[\big(\lambda_i f'(\lambda_i) - 1\big)\operatorname{tr}(\mathbf{V}_i\mathbf{D})\big]^2,$$

where the two sums on the right hand side are the corresponding terms in Equation (4–6).

Therefore, we get:

Lemma 4. For every Eulerian strain measure f there is a unique spin which minimizes the difference between $\overset{\circ}{f(\mathbf{V})}$ and $\mathbf{D}$, and that spin is $\boldsymbol{\Omega}^f$. The minimal difference is

$$|\overset{\circ f}{f(\mathbf{V})} - \mathbf{D}|^2 = \sum_{i=1}^{n} \big[\big(\lambda_i f'(\lambda_i) - 1\big) \operatorname{tr}(\mathbf{V}_i\mathbf{D})\big]^2.$$

The proof follows using Lemma 3 in the form

$$|f(\overset{\circ}{\mathbf{V}}) - \mathbf{D}|^2 \geq |\widehat{\mathbf{D}} - \mathbf{D}|^2,$$

with equality if and only if $\mathbf{\Omega} = \mathbf{\Omega}^f$.

4.2. ***The logarithmic spin.*** Lemma 4 implies that the corotational rate of strain equals $\mathbf{D}$ if the strain measure has the property $xf'(x) - 1 = 0$. The only solution that satisfies the condition $f(1) = 0$ is $f(x) = \ln x$, and the associated spin follows from Equation (4–5) as $\mathbf{\Omega}^{\log} = \mathbf{W} + \mathbb{P}^{\log}\mathbf{D}$ where

$$p_{ij}^{\log} = \frac{1}{\ln\lambda_i - \ln\lambda_j} - \frac{\lambda_i^2 + \lambda_j^2}{\lambda_i^2 - \lambda_j^2}.$$

$\mathbf{\Omega}^{\log}$ is the well known logarithmic spin [Xiao et al. 1997]. Hence, of all possible rates and of all possible Eulerian strain measures only the combination of the Hencky strain and the rate defined by the logarithmic spin together yield the strain rate $\mathbf{D}$. This is the unique relationship between $\ln\mathbf{V}$, $\mathbf{D}$, and $\mathbf{\Omega}^{\log}$ which makes both the Hencky strain and the logarithmic spin special. This result was first derived by Xiao et al. [1997], and may be summarized as:

Lemma 5. The strain rate $\mathbf{D}$ is recovered only as the corotational rate of the Eulerian strain $\mathbf{e}^{(0)} = \ln\mathbf{V}$ with spin $\mathbf{\Omega}^{\log}$ where the fourth order projection tensor $\mathbb{P}^{\log}$ is given by Equation (4–5) with $f = \ln$. That is,

$$\ln\overset{\circ\log}{\mathbf{V}} = \mathbf{D}.$$

4.2.1. *Some properties of the logarithmic spin.* An instructive alternative form for $\mathbb{P}^{\log}$ is obtained by introducing

$$\mathbb{P}^{\ln} \equiv (\mathbf{V}\boxtimes\mathbf{I} - \mathbf{I}\boxtimes\mathbf{V})^* \triangle \ln(\mathbf{V}),$$

so that

$$\mathbb{P}^{\log} = \mathbb{P}^{\ln} + \mathbb{P}^{E} = \mathbb{P}^{\ln} + \mathbb{P}^{L} + \mathbb{P}^{R}.$$

Each of the projection tensors may be expressed in terms of matrix elements $p_{ij} = -p_{ji}$ according to Equation (3–8) as

$$p_{ij}^R = -\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}, \qquad p_{ij}^E = -\frac{\lambda_i^2 + \lambda_j^2}{\lambda_i^2 - \lambda_j^2}, \qquad p_{ij}^L = -\frac{2\lambda_i\lambda_j}{\lambda_i^2 - \lambda_j^2}, \qquad p_{ij}^{\ln} = \frac{1}{\ln\lambda_i - \ln\lambda_j}.$$

The form of $p_{ij}^{\log}$ agrees with the formula for $\mathbb{P}^{\log}$ derived by Xiao et al. [1997, Equation (41)]. Note that

$$\operatorname{sgn} p_{ij}^{\log} = \operatorname{sgn} p_{ij}^R = -\operatorname{sgn}(p_{ij}^{\log} - p_{ij}^R).$$

The implications are twofold. The first equalities indicate that the spin induced by both $\mathbf{\Omega}^{\log}$ and by $\mathbf{\Omega}^R$ are in the same sense relative to the underlying spin $\mathbf{W}$. The latter equalities imply that the relative spin induced by $\mathbf{\Omega}^{\log}$ is of smaller magnitude than that of $\mathbf{\Omega}^R$.

The f-spin, which is uniquely defined by the strain measure f, defines the skew matrix elements p_{ij}^f. Consider the reverse problem: given some objective corotational rate defined by elements p_{ij}, is there a function f such that $p_{ij}^f = p_{ij}$? There is no such function for the spins $\mathbf{\Omega}^R$, $\mathbf{\Omega}^E$ and $\mathbf{\Omega}^L$, as the reader can readily verify. Obviously, $f = \ln$ for $\mathbf{\Omega} = \mathbf{\Omega}^{\log}$, but it remains an open question for general $\mathbf{\Omega}$ whether a strain measure function exists such that $\mathbf{\Omega} = \mathbf{\Omega}^{f}$.

5. Eulerian conjugate stress-strain pairs

5.1. ***Arbitrary strain and corotational rate.*** It was noted in Section 1 that the concept of work-conjugate stress-strain pairs is more complicated for Eulerian quantities owing to the fact that the connection between the strain rate and the stretching tensor is not evident a priori. This issue was resolved by Lehmann and Liang [1993], who introduced the notion that the Eulerian pair $\boldsymbol{\tau}$ and $\mathbf{e}$ are defined to be conjugate if

$$\dot{w} = \operatorname{tr}\left(\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^t\, \dot{\overline{\mathbf{Q}\mathbf{e}\mathbf{Q}^t}}\right), \tag{5–1}$$

for some rotation $\mathbf{Q}$. This clearly generalizes the Lagrangian work-conjugacy condition Equation (1–2), but it is necessary because of the fact that Eulerian rates are not as restricted. The definition in Equation (5–1) is equivalent to

$$\dot{w} = \operatorname{tr}(\boldsymbol{\tau}\overset{\circ}{\mathbf{e}}), \tag{5–2}$$

where $\overset{\circ}{\mathbf{e}} = \dot{\mathbf{e}} + \{\mathbf{e}, \mathbf{\Omega}\}$ and $\mathbf{\Omega} = -\mathbf{Q}^t\dot{\mathbf{Q}}$. Equation (5–2) is taken as the starting point, since it depends only on the corotational rate through the spin $\mathbf{\Omega}$, therefore $\mathbf{Q}$ is not required.

For a given strain measure $\mathbf{e} = f(\mathbf{V})$ and corotational rate $\mathbf{\Omega} = \mathbf{W} + \mathbb{P}\mathbf{D}$ the strain rate $\overset{\circ}{\mathbf{e}}$ follows from Equation (3–3). The stress $\boldsymbol{\tau}$ is therefore conjugate to $\mathbf{e}$ if the following holds for all stretching tensors $\mathbf{D}$:

$$\operatorname{tr}\left(\boldsymbol{\tau}[\nabla f(\mathbf{V})]\,\mathbb{Q}(\mathbf{V})\,\mathbf{D}\right) = \operatorname{tr}\left(\boldsymbol{\sigma}\,\mathbf{D}\right).$$

The fourth order tensor $\nabla f(\mathbf{V})$ is invertible for all strain measures. The necessary and sufficient condition required to determine $\boldsymbol{\tau}$ is therefore that the fourth order tensor $\mathbb{Q}$ is invertible. This requirement was obtained by Xiao et al. [1997] in a slightly different manner; basically, that the six elements q_{ij} of Equation (3–10) are all nonzero. Hence, q_{ij}^{-1} are bounded, and $\mathbb{Q}^{-1}$ exists. We refer the reader to [Xiao et al. 1997] for further details.

In summary, the conjugate stress is

$$\boldsymbol{\tau} = [\triangle f(\mathbf{V})]\mathbb{Q}^{-1}\boldsymbol{\sigma},$$

where the order of $[\triangle f(\mathbf{V})]$ and $\mathbb{Q}^{-1}$ are arbitrary since they commute. This is Theorem 1.

5.2. ***Conjugate stress and the f-rate.*** An alternative approach is suggested by Equation (4–7). Let $\overset{\circ}{\mathbf{e}} = \mathbb{F}\mathbf{D}$, then the fourth order tensor $\mathbb{F}$ is by assumption invertible and the conjugate stress is simply $\boldsymbol{\tau} = \mathbb{F}^{-1}\boldsymbol{\sigma}$. The tensor $\mathbb{F}$ can be obtained directly in spectral form from Equation (4–7) and easily inverted, to give

Lemma 6. For arbitrary strain measure and rate the conjugate stress can be expressed

$$\boldsymbol{\tau} = \boldsymbol{\sigma}^f - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{1}{1 + \left[(p_{ij} - p_{ij}^f)\left(f(\lambda_i) - f(\lambda_j)\right)\right]^{-1}}\, \mathbf{V}_i \boxtimes \mathbf{V}_j \boldsymbol{\sigma}. \tag{5–3}$$

The conjugate stress satisfies

$$|\boldsymbol{\tau}-\boldsymbol{\sigma}|^2 = |\boldsymbol{\tau}-\boldsymbol{\sigma}^f|^2 + |\boldsymbol{\sigma}^f-\boldsymbol{\sigma}|^2, \tag{5–4}$$

where the modified stress tensor $\boldsymbol{\sigma}^f$ is

$$\boldsymbol{\sigma}^f = \boldsymbol{\sigma} + \sum_{i=1}^{n} \Big[\frac{1}{\lambda_i f'(\lambda_i)} - 1\Big] \mathbf{V}_i \boxtimes \mathbf{V}_i \boldsymbol{\sigma}.$$

The proof follows from Equation (5–3) by analogy with the proof of Lemma 3. Hence,

$$|\boldsymbol{\tau}-\boldsymbol{\sigma}|^2 \geq |\boldsymbol{\sigma}^f-\boldsymbol{\sigma}|^2$$

with equality if and only if $\boldsymbol{\Omega} = \boldsymbol{\Omega}^f$, and we deduce the following.

Lemma 7. For every Eulerian strain measure $f(\mathbf{V})$ the corotational rate of the f-spin $\boldsymbol{\Omega}^f$ minimizes the difference between the conjugate stress and the Cauchy stress. The conjugate stress is then $\boldsymbol{\tau} = \boldsymbol{\sigma}^f$ and the minimal difference is

$$|\boldsymbol{\tau}-\boldsymbol{\sigma}|^2 = \sum_{i=1}^{n} \Big[\Big(\frac{1}{\lambda_i f'(\lambda_i)} - 1\Big) \operatorname{tr}(\mathbf{V}_i \boldsymbol{\sigma})\Big]^2.$$

This proves Theorem 2.

In general $\boldsymbol{\sigma}^f$ is not equal to the Cauchy stress for any strain measure, with the exception of $f = \ln$, discussed below. It is, however, possible for $\boldsymbol{\sigma}^f$ and $\boldsymbol{\sigma}$ to coincide under special circumstances: if the three elements $\operatorname{tr}(\mathbf{V}_i \boldsymbol{\sigma})$ simultaneously vanish. This is by definition a state of pure shear [Norris 2006]. Hence, we have the following statement.

Lemma 8. If the Cauchy stress is a state of pure shear with $\operatorname{diag}(\boldsymbol{\sigma}) = 0$ in the principal axes of $\mathbf{V}$, then

$$\boldsymbol{\sigma}^f = \boldsymbol{\sigma}.$$

The stress conjugate to $f(\mathbf{V})$ equals the Cauchy stress if the f-rate is used.

If the material is isotropic then the stress and strain share the same triad of principal axes. In that case $\operatorname{diag}(\boldsymbol{\sigma})$ expressed in the principal axes of $\mathbf{V}$ is simply the principle stresses, which vanishes only in the absence of stress. Hence the circumstances under which Lemma 8 applies cannot occur for isotropic materials. If the material is not isotropic, but we restrict attention to linear anisotropic elasticity, then $\operatorname{diag}(\boldsymbol{\sigma})$ expressed in the principal axes of strain $\mathbf{e}$ will vanish only if both stress and strain are zero. This follows from the assumed positive definite property of the strain energy, equal to $\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}\mathbf{e})$. In summary, the circumstances under which Lemma 8 apply require nonlinear and anisotropic elasticity. This does not eliminate its possibility but it makes it difficult to envision a situation when Lemma 8 would occur.

5.3. ***Logarithmic rate.*** The logarithmic rate, as noted before, is a special case of the f-rate. We conclude by examining the conjugate stress for arbitrary strain measure using the logarithmic rate. Xiao et al. [1997] showed that the logarithmic rate is the only one with the property of Lemma 5, which is that among all strains and all rates, only $\ln \mathbf{V}$ and $\boldsymbol{\Omega}^{\log}$ correspond to the stretching tensor $\mathbf{D}$. This fundamental result for $\ln(\mathbf{V})$ is generalized to arbitrary Eulerian strain measure $\mathbf{e} = f(\mathbf{V})$ by

$$\overset{\circ\log}{\mathbf{e}} = \big(\nabla f(\mathbf{V})\big)\,(\triangle \ln \mathbf{V})\,\mathbf{D},$$

which follows from Lemmas 2 and 5. Now require that the work-conjugacy identity $\mathrm{tr}\,(\boldsymbol{\tau}\,\overset{\circ_{\log}}{\mathbf{e}}) = \mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D})$ holds for all $\mathbf{D}$, and use the invertibility of the fourth order tensors $\nabla f(\mathbf{V})$ and $\triangle \ln \mathbf{V}$ plus the property that they commute. This implies that the stress conjugate to the Eulerian strain $\mathbf{e} = f(\mathbf{V})$ is

$$\boldsymbol{\tau} = \big(\triangle f(\mathbf{V})\big)\,(\nabla \ln \mathbf{V})\,\boldsymbol{\sigma},$$

for $\boldsymbol{\Omega} = \boldsymbol{\Omega}^{\log}$. It is straightforward to show that this can be expressed in spectral form as

$$\boldsymbol{\tau} = \boldsymbol{\sigma}^f + \sum_{i,j=1}^{n} \Big(\frac{\ln\lambda_i - \ln\lambda_j}{f(\lambda_i) - f(\lambda_j)} - 1\Big)\mathbf{V}_i \boxtimes \mathbf{V}_j \boldsymbol{\sigma}, \tag{5–5}$$

again for $\boldsymbol{\Omega} = \boldsymbol{\Omega}^{\log}$. This identity, although valid only for the logarithmic rate, shows how the conjugate stress in that case is related to the modified stress $\boldsymbol{\sigma}^f$. The latter depends upon the strain measure f, and is optimal in the sense of best for all possible strain rates. Equation (5–5) shows that the logarithmic rate is not optimal since τ satisfies Equation (5–4) with both terms on the RHS of the latter nonzero. However, when the strain measure f reduces to ln then $\boldsymbol{\sigma}^f \to \boldsymbol{\sigma}$ and the sum in Equation (5–5) vanishes. This again shows the combined properties of the Hencky strain and the logarithmic rate as being doubly optimal for all strain measures and spins.

6. Conclusion

We have examined the implications of work-conjugacy with emphasis on Eulerian stress-strain pairs. There is, however, remarkable similarity in the form of the dual conjugate stresses for both Lagrangian and Eulerian strains. The similarity is evident from the identical format of Equations (1–5) and (1–10), which involve fundamental stresses $\mathbf{T}^{(1)}$ and $\boldsymbol{\tau}^{(1)}$ defined by the strains $f(\mathbf{U})$ and $f(\mathbf{V})$, respectively. The Lagrangian stress $\mathbf{T}^{(1)}$ is called Biot stress or Jaumann stress, but there does not appear to be a common term for its Eulerian counterpart $\boldsymbol{\tau}^{(1)}$.

The major distinction between Lagrangian and Eulerian work-conjugacy is that the latter requires the introduction of the corotational rate, which itself is quite arbitrary. We have shown that every permissible Eulerian strain measure $f(\mathbf{V})$ has associated with it a unique corotational rate, the f-rate. The conjugate stress obtained using the f-rate is optimal in the sense that it is the closest possible to the Cauchy stress $\boldsymbol{\sigma}$. The optimal stress, $\boldsymbol{\sigma}^f$, is defined by f and $\boldsymbol{\sigma}$ through Lemma 6, and it reduces to the Cauchy stress if and only if $f = \ln$. This reinforces the results of Xiao et al. [1998a] for the logarithmic rate and the Hencky strain, while generalizing the notion of the logarithmic rate to arbitrary strain functions through the strain dependent spin $\boldsymbol{\Omega}^f$.

Appendix A: The spins $\boldsymbol{\Omega}^R$, $\boldsymbol{\Omega}^E$ and $\boldsymbol{\Omega}^L$

From the definition of $\boldsymbol{\Omega}^R$ in Equation (3–6), and using $\mathbf{F} = \mathbf{R}\mathbf{U}$, we have

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \boldsymbol{\Omega}^R + \mathbf{R}\dot{\mathbf{U}}\mathbf{U}^{-1}\mathbf{R}^t.$$

The symmetric and skew parts of this relation yield [Truesdell and Noll 1965]

$$\mathbf{D} = \tfrac{1}{2}\,\mathbf{R}\,(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}})\,\mathbf{R}^t, \tag{A.1}$$

and

$$\mathbf{W} = \boldsymbol{\Omega}^R + \tfrac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^t. \tag{A.2}$$

Equation (A.1) may be solved for $\dot{\mathbf{U}}$ in the form given by Equation (1–4). Substituting $\dot{\mathbf{U}}$ in Equation (A.2) gives

$$\boldsymbol{\Omega}^R = \mathbf{W} + (\mathbf{I}\boxtimes\mathbf{V} - \mathbf{V}\boxtimes\mathbf{I})(\mathbf{I}\boxtimes\mathbf{V} + \mathbf{V}\boxtimes\mathbf{I})^{-1}\mathbf{D}.$$

Let $\mathbf{v}_i$, $i = 1, \ldots, n \le 3$, be the principal axes of $\mathbf{B}$ and $\mathbf{V}$. The twirl $\boldsymbol{\Omega}^E$ defines the rate of rotation of this triad by $\dot{\mathbf{v}}_i = \Omega^E\mathbf{v}_i$. The rate of change of the eigentensors of $\mathbf{B}$ follows from $\mathbf{V}_i = \mathbf{v}_i \otimes \mathbf{v}_i$ as $\dot{\mathbf{V}}_i = \{\Omega^E, \mathbf{V}_i\}$. The second portion of Equation (3–1) then gives

$$\dot{\mathbf{B}} = \sum_{i=1}^{n} \dot{\beta}_i\mathbf{V}_i + \{\Omega^E, \mathbf{B}\}, \tag{A.3}$$

which can be considered as an equation for Ω^E, similar to Equation (2–2). The solution follows from Equations (2–4) and (3–2) as

$$\boldsymbol{\Omega}^E = (\mathbf{I}\boxtimes\mathbf{B} - \mathbf{B}\boxtimes\mathbf{I})^*\big(\dot{\mathbf{B}} - \sum_{i=1}^{n}\dot{\beta}_i\mathbf{V}_i\big) = \mathbf{W} + (\mathbf{I}\boxtimes\mathbf{B} - \mathbf{B}\boxtimes\mathbf{I})^*(\mathbf{I}\boxtimes\mathbf{B} + \mathbf{B}\boxtimes\mathbf{I})\mathbf{D}.$$

Hence $\boldsymbol{\Omega}^E = \mathbf{W} + \mathbb{P}^E\mathbf{D}$ where $\mathbb{P}^E$ is given by Equation (3–7b). The rate of change of the principal stretches are obtained by substituting $\boldsymbol{\Omega}^E$ into Equation (A.3), as

$$\sum_{i=1}^{n}\dot{\beta}_i\mathbf{V}_i = \big[\mathbb{I} - (\mathbf{I}\boxtimes\mathbf{B} - \mathbf{B}\boxtimes\mathbf{I})^*(\mathbf{I}\boxtimes\mathbf{B} - \mathbf{B}\boxtimes\mathbf{I})\big](\mathbf{I}\boxtimes\mathbf{B} + \mathbf{B}\boxtimes\mathbf{I})\mathbf{D}.$$

Then using Equations (2–6) and (2–9), we obtain the well known result

$$\sum_{i=1}^{n}\dot{\beta}_i\mathbf{V}_i = 2\sum_{i=1}^{n}\beta_i\mathbf{V}_i\boxtimes\mathbf{V}_i\mathbf{D} \quad\Leftrightarrow\quad \dot{\lambda}_i = \lambda_i \operatorname{tr}(\mathbf{V}_i\mathbf{D}).$$

The twirl $\tilde{\Omega}^L$ defines the rate of rotation of the Lagrangian principal axes $\mathbf{u}_i$, $i = 1, \ldots, n$ as $\dot{\mathbf{u}}_i = \tilde{\Omega}^L\mathbf{u}_i$. Hence, $\dot{\mathbf{U}}_i = \{\tilde{\Omega}^L, \mathbf{U}_i\}$, where $\mathbf{U}_i = \mathbf{u}_i \otimes \mathbf{u}_i$, are the eigentensors of $\mathbf{U}$. Taking the rate of change of the identity $\mathbf{V}_i = (\mathbf{R}\boxtimes\mathbf{R})\mathbf{U}_i$, linking the Eulerian and Lagrangian eigentensors, gives

$$\dot{\mathbf{U}}_i = (\mathbf{R}\boxtimes\mathbf{R})^{-1}\{\Omega^E - \Omega^R, \mathbf{V}_i\}.$$

The Lagrangian twirl is therefore

$$\tilde{\Omega}^L = (\mathbf{R}\boxtimes\mathbf{R})^{-1}(\Omega^E - \Omega^R) = (\mathbf{I}\boxtimes\mathbf{U}^2 - \mathbf{U}^2\boxtimes\mathbf{I})^*2(\mathbf{U}\boxtimes\mathbf{U})(\mathbf{R}\boxtimes\mathbf{R})^{-1}\,\mathbf{D}.$$

This is related to the spin $\Omega^L = \mathbf{W} + \mathbb{P}^L\mathbf{D}$ defined via $\mathbb{P}^L$ of Equation (3–7c) by $\tilde{\Omega}^L = (\mathbf{R}\boxtimes\mathbf{R})^{-1}(\Omega^L - \mathbf{W})$.

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