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CONCENTRATED FORCE ACTING ON A POWER LAW CREEP HALF-PLANE

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The problem of concentrated force acting on a half-plane made of a power-law creep material is solved analytically. In our approach, the constitutive equation that describes the process of dilatational deformation is omitted. The incomplete material description is used for constructing a solution by bringing the dilatational deformation to zero and, in this manner, making the material incompressible. We find solutions for two cases; one solution is for a linear viscous material, while the second is for a power-law material where the power exponent is equal to three. Solutions of the two problems are found to be very different. While the linear viscous solution is found to be the same as the linear elastic solution, the nonlinear solution is found to be significantly different. This result may give rise to a new experimental technique for characterization of materials with a nonlinear creep behavior.

1. Introduction

The problem of concentrated force acting on a linear elastic and isotropic half-space was solved by Boussinesq [1885] in three dimensions and by Flamant in 1892 (see [Love 1944; Malvern 1969; Timoshenko and Goodier 1970] for details) in two dimensions. Some recent solutions addressed a concentrated force acting on a linear elastic half-space [Jager 1997; Levy 2002; Unger 2002; Marzocchi and Musesti 2004], a transversely isotropic elastic half-space [Liao and Wang 1999], an inhomogeneous transversely isotropic elastic half-space [Wang et al. 2003], an elastic nonlocal half-plane [Artan 1996], a gradient elasticity half-space [Zhou and Jin 2003; Li et al. 2004; Lazar and Maugin 2006], an elastic linear hardening half-plane [Gao 1999], and a piezoelectric half-plane [Sosa and Castro 1994].

In this paper we focus on the problem of concentrated force acting on a half-plane made of a power-law creep material in the context of plane strain (see Figure 1). In our approach, following [Zubelewicz 2005], we omit the constitutive equation that describes dilatancy. In this manner we are able to examine various kinematically admissible solutions, from which we narrow our search to the solution for an incompressible material.

Recall that in the Flamant solution the only nonzero stress term is

$$\sigma_{rr} = -\frac{2P}{\pi}\frac{\cos\theta}{r}.$$

For the two dimensional case, the equivalent Tresca stress (the maximum shear stress) is

$$\sigma_{eq} = \left|\frac{\sigma_{rr}}{2}\right| = \frac{P}{\pi}\frac{\cos\theta}{r}.$$

Keywords: concentrated force, power-law material, creep, viscous material, incompressible material.
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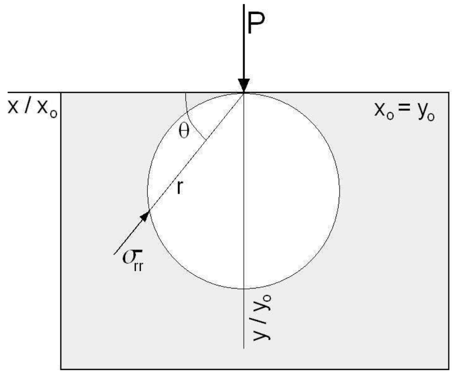

Figure 1. Force acting on a half-plane.

In this paper, the radial stress σ_{rr} for a linear elastic case will be compared to the radial stress for a power-law material, where the power is assumed to be one and three.

2. Problem statement

We study the problem involving a concentrated force acting on a half-plane made of a power-law creep incompressible material in the context of plane strain (see Figure 1). Thus, the applied load P has units of load per unit thickness. Following the technique presented in [Zubelewicz 2005], we convert all functions from a real into a complex domain, where the complex variables are defined, as usual, by $z = x + iy$ and $z^* = x - iy$.

The complex shear strains can be defined as

$$\eta = (\varepsilon_{xx} - \varepsilon_{yy}) + 2i\varepsilon_{xy}, \qquad \eta^* = (\varepsilon_{xx} - \varepsilon_{yy}) - 2i\varepsilon_{xy}.$$

The complex displacements, $v = (u_x + iu_y)$ and $v^* = (u_x - iu_y)$, and the strains are coupled and satisfy the relations

$$\eta = 2\frac{\partial v}{\partial z^*}, \qquad \eta^* = 2\frac{\partial v^*}{\partial z}.$$

Using the above equations it is possible to express the displacements u_x and u_y in terms of complex shear strains as

$$u_x = \frac{1}{4}\left[\int \eta\, dz^* + \int \eta^*\, dz\right] + \frac{1}{2}[\psi_v(z) + \psi_v^*(z^*)], \quad u_y = \frac{1}{4i}\left[\int \eta\, dz^* - \int \eta^*\, dz\right] + \frac{1}{2i}[\psi_v(z) - \psi_v^*(z^*)],$$

where the additional displacement functions ψ_v and ψ_v^* must satisfy kinematic boundary conditions. In order to find u_x and u_y, a path-dependent integration is chosen such that one of the complex variables is kept constant while integrating with respect to the other [Vekua 1962]. Then, the rate of volumetric change (dilatational deformation) can be defined as

$$\dot{I}_\varepsilon = \frac{1}{2}\left[\int \frac{\partial \dot{\eta}}{\partial z}\, dz^* + \int \frac{\partial \dot{\eta}^*}{\partial z^*}\, dz\right] + \dot{\psi}_v'(z) + \dot{\psi}_v^{*'}(z^*), \qquad \psi_v{}' = \frac{\partial \psi_v}{\partial z}. \tag{1}$$

Equation (1) is written in a rate form, where the dot indicates that it is a time derivative. It has been shown in [Zubelewicz 2005] that kinematical compatibility is always satisfied when the dilatant deformation is expressed through Equation (1).

Next, the complex shear stresses are defined as $\tau = \Delta\sigma + 2i\sigma_{xy}$ and $\tau^* = \Delta\sigma - 2i\sigma_{xy}$, where $\Delta\sigma = \sigma_{xx} - \sigma_{yy}$. The Tresca stress is equal to $\sigma_{eq} = \frac{1}{2}\sqrt{\tau\tau^*}$.

The power-law creep equation which couples the equivalent strain rate and equivalent stress is

$$\dot{e}_{eq} = \Lambda\left(\frac{\sigma_{eq}}{\sigma_o}\right)^p. \tag{2}$$

Assuming associated flow rules, we can express the constitutive Equation (2) in terms of $\dot{\eta}$ and τ such that

$$\dot{\eta} = \frac{\Lambda}{(2\sigma_o)^p}\tau^{\frac{p+1}{2}}(\tau^*)^{\frac{p-1}{2}}, \qquad \dot{\eta}^* = \frac{\Lambda}{(2\sigma_o)^p}\tau^{\frac{p-1}{2}}(\tau^*)^{\frac{p+1}{2}}. \tag{3}$$

The proposed formulation allows for a separation of the material response due to shear and dilatancy. As we notice, neither the hydrostatic stress ($I_\sigma = \sigma_{xx} + \sigma_{yy}$) nor the rate of volumetric change (1) is involved in the constitutive (3).

For completeness, the equilibrium equations are

$$\frac{\partial\tau}{\partial z} + \frac{\partial I_\sigma}{\partial z^*} = 0, \qquad \frac{\partial\tau^*}{\partial z^*} + \frac{\partial I_\sigma}{\partial z} = 0. \tag{4}$$

We solve the equilibrium equations (4) with the use of two stress functions, $\psi_\sigma(z)$ and $\Omega(z, z^*)$, where Ω is a real function of complex arguments. We satisfy these equations by representing the complex stresses as

$$\tau = \psi_\sigma^*(z^*) + \int\frac{\partial\Omega}{\partial z^*}\,dz, \qquad \tau^* = \psi_\sigma(z) + \int\frac{\partial\Omega}{\partial z}\,dz^*, \qquad I_\sigma = -\Omega. \tag{5}$$

Thus there are six variables to be determined: three complex stresses (τ, τ^*, I_σ) and three complex strain rates ($\dot{\eta}$, $\dot{\eta}^*$, $\dot{I}_\varepsilon$). On the other hand, there are five equations available: the kinematical compatibility equation (1), two constitutive equations (3), and two equilibrium equations (4). The sixth governing equation is the requirement of material incompressibility. We enforce the condition assuming that the dissipation energy due to volumetric change is equal to zero, while the dilatational deformation is non-negative in all points of the material. First, we calculate the total rate of dissipation

$$\dot{W}^t = \int_{\Delta V}\sigma_{eq}\dot{e}_{eq}dV + \frac{1}{2}\int_{\Delta V} I_\sigma \dot{I}_\varepsilon dV. \tag{6}$$

Then, we bring the second term in (6) to zero. This assures that there is no volumetric change and, therefore, the material is incompressible. Our chosen stress functions are

$$\psi_\sigma(z) = \sum_m C_m z^{\lambda_m}, \qquad \Omega = \sum_{n,m}\left[D_{nm} z^{\alpha_m^n}(z^*)^{\beta_m^n} + D_{nm}^* z^{\beta_m^n}(z^*)^{\alpha_m^n}\right], \tag{7}$$

where λ_m satisfies the condition $\lambda_m = \alpha_m + \beta_m$ for any $n, m = 1, 2, \ldots \infty$. For an asymptotic analysis, n is equal to 1, and can be omitted in further equations.

Now, substituting Equation (7) into (5) leads to

$$\begin{aligned}\tau &= C^*(z^*)^\lambda + \sum_m \Big(\frac{\beta_m D_m}{\alpha_m+1} z^{\alpha_m+1}(z^*)^{\beta_m-1} + \frac{\alpha_m D_m^*}{\beta_m+1} z^{\beta_m+1}(z^*)^{\alpha_m-1}\Big),\\ \tau^* &= C z^\lambda + \sum_m \Big(\frac{\beta_m D_m^*}{\alpha_m+1}(z^*)^{\alpha_m+1} z^{\beta_m-1} + \frac{\alpha_m D_m}{\beta_m+1}(z^*)^{\beta_m+1} z^{\alpha_m-1}\Big),\\ I_\sigma &= -\Omega = -\sum_m \Big(D_m z^{\alpha_m}(z^*)^{\beta_m} + D_m^* z^{\beta_m}(z^*)^{\alpha_m}\Big),\end{aligned} \tag{8}$$

where C is a real constant, $C^* = C$, $D_m = D_m^0 e^{i\phi_m}$, and $D_m^* = D_m^0 e^{-i\phi_m}$, where ϕ_m need to be defined together with D_m^0. Thus, in summary, we will find these constants through the process of minimization of the plastic dissipation rate due to volumetric change (the second term in (6)) which results in solutions for incompressible materials.

3. General solution

In the case of concentrated force acting on a half plane (1), there are zero tractions along the free surfaces, thus $\sigma_{yy} \pm i\sigma_{xy} = 0$, at $\theta = 0, \pi$. In the complex domain, the conditions are $\sigma_{yy} + i\sigma_{xy} = \frac{1}{2}(I_\sigma - \tau^*) = 0$ and $\sigma_{yy} - i\sigma_{xy} = \frac{1}{2}(I_\sigma - \tau) = 0$, where the second equation is redundant. Using the definitions of the complex stresses given in (8), we find

$$\begin{aligned}-Cz^\lambda - \sum_m \Big[D_m^* \Big(\frac{\beta_m}{\alpha_m+1} z^{\beta_m-1}(z^*)^{\alpha_m+1} + z^{\beta_m}(z^*)^{\alpha_m}\Big)\\ + D_m\Big(\frac{\alpha_m}{\beta_m+1} z^{\alpha_m-1}(z^*)^{\beta_m+1} + z^{\alpha_m}(z^*)^{\beta_m}\Big)\Big] = 0.\end{aligned} \tag{9}$$

In a polar coordinate system $z = R(\cos\theta + i\sin\theta)$, and $z^* = R(\cos\theta - i\sin\theta)$. For $\theta = 0$, (9) reduces to

$$C + \sum_m (\lambda+1)\Big(\frac{D_m}{\beta_m+1} + \frac{D_m^*}{\alpha_m+1}\Big) = 0. \tag{10}$$

When $\theta = \pi$, the equation becomes

$$\begin{aligned}C(\cos\lambda\pi + i\sin\lambda\pi) + \sum_m \Big[\frac{D_m}{\beta_m+1}(\lambda+1)\Big\{\cos[(\alpha_m-\beta_m)\theta] + i\sin[(\alpha_m-\beta_m)\theta]\Big\}\\ + \frac{D_m^*}{\alpha_m+1}(\lambda+1)\Big\{\cos[(\alpha_m-\beta_m)\theta] + i\sin[(\alpha_m-\beta_m)\theta]\Big\}\Big] = 0.\end{aligned} \tag{11}$$

Equation (10) is used to solve for C, and then (11) becomes

$$\begin{aligned}\sum_m (\lambda+1)\Big[-\Big(\frac{D_m^*}{\alpha_m+1} + \frac{D_m}{\beta_m+1}\Big)(\cos\lambda\pi + i\sin\lambda\pi) + \frac{D_m}{\beta_m+1}\Big\{\cos[(\alpha_m-\beta_m)\theta] + i\sin[(\alpha_m-\beta_m)\theta]\Big\}\\ + \frac{D_m^*}{\alpha_m+1}\Big\{\cos[(\alpha_m-\beta_m)\theta] + i\sin[(\alpha_m-\beta_m)\theta]\Big\}\Big] = 0.\end{aligned} \tag{12}$$

It follows from (12) that $\lambda = -1$. Then, from (10), C is found to be equal to zero.

Applying these conditions to the complex stresses we find

$$\tau = -\sum_m D_m^0\Big[e^{i\phi_m} z^{\alpha_m+1}(z^*)^{\beta_m-1} + e^{-i\phi_m} z^{\beta_m+1}(z^*)^{\alpha_m-1}\Big],$$

$$\tau^* = -\sum_m D_m^0\Big[e^{i\phi_m} (z^*)^{\alpha_m+1} z^{\beta_m-1} + e^{-i\phi_m} (z^*)^{\beta_m+1} z^{\alpha_m-1}\Big],$$

$$I_\sigma = -\sum_m D_m^0\Big[e^{i\phi_m} z^{\alpha_m}(z^*)^{\beta_m} + e^{-i\phi_m} z^{\beta_m}(z^*)^{\alpha_m}\Big].$$

At this point it is convenient to convert the complex stresses into the real domain. The stress components are found as

$$\sigma_{xx} = \tfrac{1}{2}R^{-1}\sum_m -2D_m^0\left[e^{i\phi_m}\cos\left[(\alpha_m-\beta_m+2)\theta\right] - e^{i\phi_m}\sin\left[(\alpha_m-\beta_m-2)\theta\right]\right],$$

$$\sigma_{yy} = -R^{-1}\sum_m D_m^0\left[(-\cos 2\theta+1)(e^{i\phi_m}\cos\left[(\alpha_m-\beta_m)\theta\right] - e^{i\phi_m}\sin\left[(\alpha_m-\beta_m)\theta\right]\right],$$

$$\sigma_{xy} = -R^{-1}\sum_m D_m^0\left[\sin 2\theta(e^{i\phi_m}\cos\left[(\alpha_m-\beta_m)\theta\right] - e^{i\phi_m}\sin\left[(\alpha_m-\beta_m)\theta\right]\right],$$

and in the polar coordinate system, there is only one nonzero stress component, namely

$$\sigma_{rr} = -2R^{-1}\sum_m D_m^0[e^{i\phi_m}\cos[(\alpha_m-\beta_m)\theta] - e^{-i\phi_m}\sin[(\alpha_m-\beta_m)\theta]].$$

Recalling that $\alpha_n+\beta_n=\lambda_n=-1$ and knowing that σ_{rr} is symmetric with respect to y (at $\theta=\pi/2$), this allows us to determine the parameters α_n and β_n such that

$$(\alpha_n-\beta_n)+\frac{2\phi_n}{\pi}=2n, \qquad \alpha_n+\beta_n=-1.$$

In the final form we find

$$\alpha_n = n-\frac{\phi_n}{\pi}-\frac{1}{2}, \qquad \beta_n = -n+\frac{\phi_n}{\pi}-\frac{1}{2},$$

where $n=1,2,3,\ldots,\infty$.

3.1. ***Linear case.*** When $p=1$, the rate of volumetric change

$$\dot{I}_\varepsilon = \frac{\Lambda}{4\sigma_o}\Big(\int\frac{\partial\tau}{\partial z}dz^* + \int\frac{\partial\tau^*}{\partial z^*}dz\Big) + \dot{\psi}_v'(z) + \dot{\psi}_v^{*'}(z^*)$$

is simplified and takes the form

$$\dot{I}_\varepsilon = \frac{\Lambda}{\sigma_o}R^{-1}\sum_m D_m^0(\cos[\phi_m+\delta_m\theta]) + A_1\sin\lambda\theta,$$

where the constant A_1 comes from the functions $\psi_v'(z)\psi$ and $\psi_v^{*'}(z^*)$. The solution is obtained by selecting the parameters (A_1, D_m^0, and ϕ_m) such that the condition for material incompressibility is satisfied. These constants are listed for reference in Table 1.

3.2. ***Non-linear case* ($p = 3$).** In the nonlinear case, the expression for the rate of volumetric change is

$$\begin{aligned}\dot{I}_\varepsilon = -\frac{\Lambda}{8\sigma_o^3}R^{3\lambda}\sum_{n,m,k} D_m^0 D_m^0 D_m^0 \Big[\Big(\frac{\alpha_n+\alpha_m+\alpha_k+1}{\beta_n+\beta_m+\beta_k}+\frac{\beta_n+\beta_m+\beta_k+1}{\alpha_n+\alpha_m+\alpha_k}\Big)\times\cos\left[\phi_n+\phi_m+\phi_k+(\delta_n+\delta_m+\delta_k)\,\theta\right]\\ +2\Big(\frac{\beta_n+\alpha_m+\alpha_k+1}{\alpha_n+\beta_m+\beta_k}\Big)\times\cos\left[\phi_n-\phi_m-\phi_k+(\delta_n-\delta_m-\delta_k)\,\theta\right]\\ +2\Big(\frac{\beta_n+\alpha_m+\alpha_k+1}{\alpha_n+\beta_m+\beta_k}\Big)\times\cos\left[\phi_n+\phi_m-\phi_k+(\delta_n+\delta_m-\delta_k)\,\theta\right]\\ +\Big(\frac{\alpha_n+\beta_m+\alpha_k+1}{\beta_n+\alpha_m+\beta_k}+\frac{\beta_n+\alpha_m+\beta_k+1}{\alpha_n+\beta_m+\alpha_k}\Big)\times\cos\left[\phi_n+\phi_m-\phi_k+(\delta_n+\delta_m-\delta_k)\,\theta\right]\Big]\\ +A_1\sin\left[3\lambda\theta\right],\end{aligned}$$

where $\delta_m = \alpha_m - \beta_m$. Again, as in the linear case, the solution is obtained by determining A_1, D_m^0, and ϕ_m for a material that exhibits zero dissipation energy due to dilatation. These constants are again given in Table 1.

4. Discussion

Since incompressibility is our requirement, we evaluate the solutions at each point by comparing the rate of the volumetric change (an error of the solution) to the shear strain rates. We generated solutions assuming a three term series (D_m^0, ϕ_m) for $m = 1$, 2, and 3.

The calculated constants for cases $p = 1$ and $p = 3$ are given in Table 1. A contour plot of the maximum shear strain rate and the rate of volumetric change is presented in Figure 2 (left). The large curve is the strain rate, while the small curve near the origin is the rate of volumetric change. As can be seen (for example, in Figure 2, left), the rate of volumetric change is negligible compared with the shear strain rate, with an error less than 3.5%. In the case of $p = 3$, the contours are very different (compare Figure 2, left and right). As before, we are satisfied with an error less than 3.5%.

In our next step we examine stress σ_{rr}. The contour plot for the case of $p = 1$ is essentially the same as that for Flamant's solution (see Figure 3). The slight difference in the shapes is due to numerical error. Taking higher order terms in D_m^0 and ϕ_m would reduce the discrepancies even further. Thus the solution for a linear viscous case, having linear stress-strain rate relations, is the same, within numerical accuracy, as for the linear elastic case, having linear stress-strain constitutive relations. This is expected since the governing equations for these two cases are analogous, and thus can serve as a check of our method. Contours of the stress σ_{rr} for a power-law material, where the power is equal to one and

p	A_1	D_1^0	D_2^0	D_3^0	ϕ_1	ϕ_2	D_3
1	−0.80	−8.16	−5.16	3.83	−0.19	3.49	5.70
3	−6.70	−149.59	−158.42	−9.42	0.78	3.89	6.65

Table 1. Calculated constants used to solve cases $p = 1$ and $p = 3$.

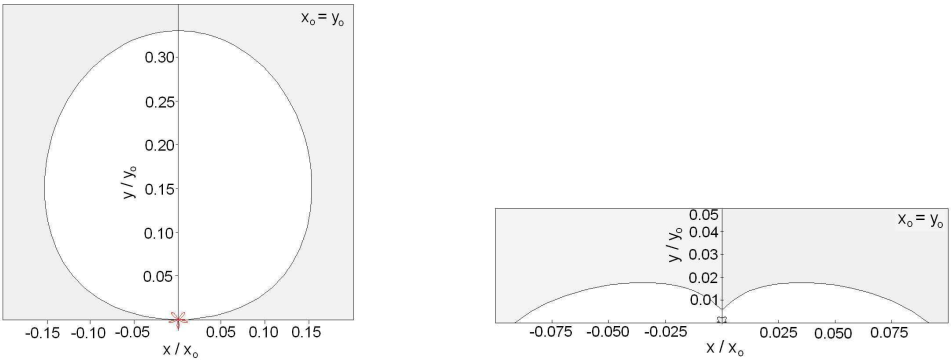

Figure 2. Contours of maximum strain rate (large curve) versus change of volume (small curve near origin) for $p = 1$ (left) and $p = 3$ (right).

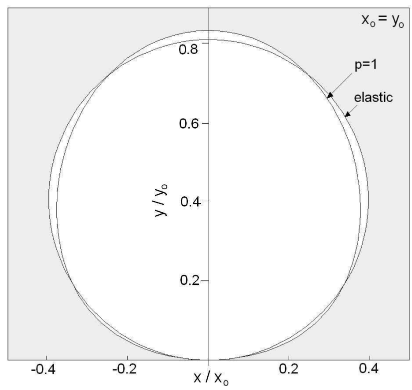

Figure 3. σ_{rr} for $p = 1$ versus elastic case.

three, are shown in Figure 4. Again, the contours for $p = 1$ and $p = 3$ are very different. Thus, the contours of the equivalent shear strain rate and stress for the two materials are surprisingly different (see Figure 2 and Figure 4). In summary, the new technique provides a new way to look at the problem of a concentrated force on a half-plane. Although the methods of finding the linear elastic and linear viscous solutions are completely different, the results are the same. That proves the validity of the proposed technique.

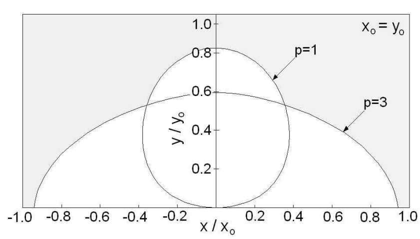

Figure 4. σ_{rr} for $p = 3$ versus $p = 1$.

The power-law material with power three ($p = 3$) represents materials which have viscous-like behavior, and thus are close to their melting point. One can construct a solution for higher powers such as five, which would represent a ductile material at room temperature. The solution would be obtained in the same manner but would involve more terms in the series and thus a more extensive algebra. The solution obtained in this paper for a power of three could be suitable for nonNewtonian fluids, solder pastes used in electronic packaging industry, and in geological applications involving sand or clay saturated in oil, or other oil-like fluids with suspensions, for example.

The problem solved in this paper could also be studied experimentally. The set-up would involve a material under a plane strain geometry constrained at the ends, with one side having a transparent frictionless plate. The loading would be a line load perpendicular to the plane of observation. Such a set-up would allow us to measure experimentally the rate of surface subsidence (surface velocity). In power-law materials, given by Equation (2), where the power p is greater than one, the profile of the out of plane surface velocity is proportional to $r^{-(p-1)}$. This relation between surface velocity and p could be used to obtain p for a given viscous material from surface velocity measurements. Thus, we conjecture that such an experimental set-up, if done successfully (meaning plane strain conditions and frictionless boundary conditions would be achieved) could serve as a new experimental method, a two dimensional analog to indentation techniques, to characterize a constitutive law of viscous (fluid-like) materials which are difficult to test otherwise. We plan to test this conjecture in the future.

5. Summary

We examined a new solving scheme for ductile materials that obeyed a power-law creep behavior. We constructed solutions for a viscous linear ($p = 1$) and power-law nonlinear ($p = 3$) material. We compared the linear elastic (Flamant) and linear viscous solutions and have shown that both solutions produce the same distribution of stresses, and that the strain rate corresponds one to one to strain in an elastic material. We used this comparison to validate our approach. We are intrigued with the solution for the nonlinear power-law material where the stress exponent p is equal to three. The two materials ($p = 1$ and $p = 3$) have very different stress contours. The theoretical result suggests the feasibility of a two dimensional indentation test which could be used for characterization of constitutive laws of viscous materials.

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