Journal of
Mechanics of
Materials and Structures

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Volume 3, Nº 5

May 2008

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 3, No. 5, 2008

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We show how some inconsistencies and infinities arising in solid and fracture mechanics can be avoided by renormalization techniques widely used in theoretical physics. Some examples, already known in solid mechanics or recently discovered in fracture mechanics, are given as illustrations.

1. Introduction

There are at least two kinds of renormalization that can be considered in the mechanics of continuous media. The first (type I) corresponds to models described by a given level of observation or scale for which an improper mathematical analysis leads to the divergence of a physical quantity. Very often, a suitable choice of a small parameter and then a correct limit procedure avoids the inconsistencies. Renormalization of type I appears to be a purely *mathematical* procedure. The second type of renormalization (type II) is more often considered in physics, for example when one changes the description of the physical system by using an appropriate scale of observation. Examples are known in quantum field theory [Gell-Mann and Low 1964; Bogolioubov and Chirkov 1960], phase transitions [Wilson 1971a; 1971b; Goldenfeld 1993], scaling [Kadanov 1966], and turbulence [Adzhemyan et al. 1999]. Here renormalization procedures consist in introducing physical arguments for treating the new scale and removing the singularities. Renormalization of type II is viewed as a *physical* justification of the singularity removal. Both types of renormalization will be considered in this paper and adapted to solid and fracture mechanics.

We recall briefly the purpose of renormalization in physics. Suppose that, for some level of description, a physical quantity, for example the (unit) mass of a naked electron in the absence of electromagnetic field, is described by a wave function, the norm of which is related to the mass m_0. Such a fictional naked electron does not exist. At a finer observation, there is an interaction between the naked electron and electromagnetic fields. One finds that the true mass m is given by the self-energy equation $mc^2 = (m_0 + am_1)c^2$, where $a = 1/127$ is the fine structure constant and am_1 is the first-order radiative correction. To recalculate or renorm the mass (the word renormalization comes from this purpose) one has to evaluate m_1, which is provided by a divergent integral. Therefore, to give a sense to the first-order solution, one needs a regularization procedure that consists in taking the finite part of a divergent integral. In this procedure, one may consider that the infinite part of am_1 will be further compensated by singularities of higher-order radiative corrections am_2 etc, in a hierarchical manner, like Russian dolls.

In such a procedure, the words renormalization and regularization become synonymous. In this paper, we still make use of the terminology 'renormalization' since it relates, for type II, to a physical change of the primary model. We shall adapt the renormalization procedure to solid and fracture mechanics. For

Keywords: fracture mechanics, divergence, renormalization, regularization.

example, in linear fracture mechanics, the best known singularity, associated with the stress intensity factor at the crack tip, indicates that some modified form of physics (and hence mathematics) must take over at a sufficiently *fine* scale.

To the author's best knowledge, the concept of renormalization is seldom fully recognized in solid and fracture mechanics, even if many examples of regularization procedures can be found in the literature. This paper focuses on some examples; some classical, others recently discovered in fracture mechanics. Type I renormalization will be illustrated by many examples: as the first one, considered in Section 2, we shall look at the divergence arising in boundary integral formulations employed in elastoplasticity, which has been removed by techniques based on the convected differentiation of integrals proposed in [Bui 1978]. The second example of renormalization of type I, given in Section 3, is related to the fundamental solution of the Helmholtz equation in acoustics, which must verify the 'perturbed' equation, in the sense that it corresponds to the continuous solution of the Helmholtz equation when the frequency or wave number k tends towards zero [Bonnet 1995]. One knows that such a fundamental solution is not unique and the fundamental Green's function usually selected in textbooks does not converge towards the static solution as k tends toward zero. To obtain a continuous solution, subtraction of a constant from the classic Green's function will be made.

Renormalization of type II is discussed in Section 4 for some models of fracture mechanics, such as Barenblatt's cohesive model [1959a; 1959b], or hierarchical models of fracture [Bui and Ehrlacher 1980]. In Section 5 we discuss a new example of type II renormalization that takes place in the differentiation of the energy release rate for point load. We propose to adequately handle some inconsistencies encountered in this process by a new renormalization procedure.

2. Renormalization of type I

How to deal with divergent integrals? We consider a simple example of a divergent integral, namely, the function $F(x)$ defined by the line integral

$$F(x) = \int_{-1}^{+1} \frac{dt}{t-x} = \ln\left|\frac{1-x}{1+x}\right|. \tag{1}$$

Its derivative is obviously given by $F'(x) = 1/(x-1) - 1/(x+1)$, but the formal integral expression is divergent for $-1 < x < +1$, since

$$F'(x) = \int_{-1}^{+1} \frac{dt}{(t-x)^2} = \infty. \tag{2}$$

To deal with such a divergent integral, many regularization procedures exist, one of them being Hadamard's definition [1932] of the finite part of a divergent integral. One first has to single out the singularity of the integral $F'_h(x)$ depending on a small parameter h, which is the infinite part $S_h = 2/h$, as $h \to 0$:

$$F'(x) := \left(\int_{-1}^{x-h} + \int_{x+h}^{+1}\right) \frac{dt}{(t-x)^2} = \frac{2}{h} + \text{regular terms} \tag{3}$$

The finite part of $F'(x)$ is precisely given by the regular terms, $(\mathrm{fp})F'(x) := F'(x) - S_h(x)$ [Schwartz 1978; Sellier 1996]:

$$(\mathrm{fp})F'(x) := F'(x) - \frac{2}{h} \quad (h \to 0). \tag{4}$$

What is the mathematical justification for this subtraction procedure? Looking at the origin of the divergence, we observe that Equation (2) is an improper definition of the derivative $F'(x)$ because the integral $F(x)$ itself must be understood as a principal value (pv) integral, defined here with the symmetric excluded zone $x - h < t < x + h$:

$$F(x) \equiv (\mathrm{pv})F(x) := \lim_{h\to 0}\left(\int_{-1}^{x-h} + \int_{x+h}^{+1}\right)\frac{dt}{(t-x)} = \ln\left|\frac{1-x}{1+x}\right|. \tag{5}$$

Therefore, the derivative of the pv integral $F(x)$ consists of two terms — the first, $1/(t-x)^2$, given by the pv integral of the derivative, and the second given by the *convected* differentiation at the endpoints $t = x \pm h$:

$$F'(x) = \lim_{h\to 0}\left(\int_{-1}^{x-h} + \int_{x+h}^{+1}\right)\frac{dt}{(t-x)^2} + \left[\frac{1}{t-x}\right]_{t=x-h} - \left[\frac{1}{t-x}\right]_{t=x+h}. \tag{6}$$

It follows that convected terms cancel the singularity of the hypersingular integral. We obtain the correct result

$$F'(x) = \lim_{h\to 0}\left(\frac{2}{h} + \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{h}\right) = \frac{1}{x-1} - \frac{1}{x+1}. \tag{7}$$

Thus Hadamard's finite part of a divergent integral and convected differentiation are just two different ways to renormalize the same divergent integral.

So far in the discussion, the excluded region $x - h < t < x + h$ has been symmetric — an assumption justified by the use of Cauchy's pv integrals. If we take an asymmetric segment $x - h < t < x + \lambda h$, where λ is a positive number not equal to 1, a different infinite part of the divergent integral (2) is obtained, $S_h = (\lambda + 1)/(\lambda h)$. Let the divergent integral (2) be written, for h tending towards zero, as

$$F'_\lambda(x) = \lim_{h\to 0}\left(\int_{-1}^{x-h} + \int_{x+\lambda h}^{+1}\right)\frac{dt}{(t-x)^2} = \frac{1}{x-1} - \frac{1}{x+1} + S_h.$$

We obtain the same result (7) given by the finite part integral (fp) $F'_\lambda(x) := F'_\lambda(x) - S_h = 1/(x-1) - 1/(x+1)$. In the asymmetric excluded zone case, for consistency, function $F_\lambda(x)$ must be written as

$$F_\lambda(x) = \lim_{h\to 0}\left(\int_{-1}^{x-h} + \int_{x+\lambda h}^{+1}\right)\frac{dt}{(t-x)} = \ln\left|\frac{1-x}{1+x}\right| - \ln\lambda.$$

After differentiation with respect to x, the constant $\ln\lambda$ does not contribute to $F'_\lambda(x)$.

Boundary integral equation formulations of three-dimensional elastoplasticity. An important application of the convected differentiation procedure has been given for the formulation of boundary integral equations in elastoplasticity, which does not involve hyper-singular integrals [Bui 1978]. Let us recall the boundary integral equations in elastoplasticity, for domain Ω with a smooth boundary $\partial\Omega$. The displacement rate $\dot{\boldsymbol{u}}$ and the traction rate $\dot{\boldsymbol{T}}$ on the boundary are related to the plastic strain rate $\dot{\epsilon}^p$, with unit outward normal $\boldsymbol{n}$ as follows (see [Mendelson 1973]):

$$\alpha(\boldsymbol{x})\,\dot{u}_k(\boldsymbol{x})+\int_{\partial\Omega}\Sigma_{kij}(\boldsymbol{x},\boldsymbol{y})\,\dot{u}_k(\boldsymbol{y})n_j(\boldsymbol{y})\,dS_y-\int_{\partial\Omega}G_{ki}(\boldsymbol{x},\boldsymbol{y})\dot{T}_i(\boldsymbol{y})\,dS_y=\int_P\Sigma_{kij}(\boldsymbol{x},\boldsymbol{y})\,\dot{\epsilon}^p_{ij}(\boldsymbol{y})\,dV_y, \quad (8)$$

where P is the active plastic zone, $G(\boldsymbol{x},\boldsymbol{y})$ is the fundamental Green's tensor in linear and isotropic elasticity and $\Sigma(\boldsymbol{x},\boldsymbol{y})=\sigma[G(\boldsymbol{x},\boldsymbol{y})]$ is the associated fundamental stress tensor. The notation $\sigma[\boldsymbol{u}]$ means $\sigma=L:\nabla\boldsymbol{u}$, where L is the isotropic symmetric moduli tensor. The Green's tensor and the fundamental stress tensor have the singularities $O(\|\boldsymbol{x}-\boldsymbol{y}\|^{-1})$ and $O(\|\boldsymbol{x}-\boldsymbol{y}\|^{-2})$ respectively. Finally, in Equation (8) the coefficient $\alpha(\boldsymbol{x})$ satisfies, for a domain with smooth boundary,

$$\alpha(\boldsymbol{x})=\begin{cases}1 & \text{for } \boldsymbol{x}\in\Omega,\\ 0 & \text{for } \boldsymbol{x}\notin\overline{\Omega},\\ \frac{1}{2} & \text{for } \boldsymbol{x}\in\partial\Omega.\end{cases}$$

To complete the system of equations (8), one needs another relation between the displacement rate $\dot{\boldsymbol{u}}$ and the stress rate $\dot{\boldsymbol{T}}$, which requires the calculation of the gradient $\nabla\dot{\boldsymbol{u}}$ in the entire domain, by differentiating (8) term by term. For interior points $\boldsymbol{x}$, we have $\alpha=1$ and the term $\Sigma_{kij,h}(\boldsymbol{x},\boldsymbol{y})$ (where the subscript $_{,h}$ indicates partial differentiation with respect to x_h) is integrable over $\partial\Omega$ (because $\|\boldsymbol{x}-\boldsymbol{y}\|\neq 0$ for $\boldsymbol{y}$ on the boundary). By contrast, as soon as $\boldsymbol{x}$ lies inside the domain, there is a nonintegrable kernel with

$$\Sigma_{kij,h}(\boldsymbol{x},\boldsymbol{y})=O(\|\boldsymbol{x}-\boldsymbol{y}\|^{-3}). \quad (9)$$

For interior points $\alpha=1$, the gradient of the displacement rate is strictly given by

$$\dot{u}_{k,h}(\boldsymbol{x})+\int_{\partial\Omega}\frac{\partial}{\partial x_h}\Sigma_{kij}(\boldsymbol{x},\boldsymbol{y})\,\dot{u}_i(\boldsymbol{y})n_j(\boldsymbol{y})\,dS_y-\int_{\partial\Omega}\frac{\partial}{\partial x_h}G_{ki}(\boldsymbol{x},\boldsymbol{y})\dot{T}_i(\boldsymbol{y})\,dS_y \\ =\frac{\partial}{\partial x_h}\int_P\Sigma_{kij}(\boldsymbol{x},\boldsymbol{y})\,\dot{\epsilon}^p_{ij}(\boldsymbol{y})\,dV_y. \quad (10)$$

On the right-hand side of this equation, it would be incorrect to interchange the differentiation and integration operators. In the 1970s, most workers were not aware of this difficulty [Bui 1978]. Today one recognizes that the hypersingular integral (with kernel $\Sigma_{kij,h}(\boldsymbol{x},\boldsymbol{y})$) must be understood in the *finite part* sense and properly evaluate either by Hadamard's expansion at infinity or by a convected differentiation of a pv integral.

To calculate the hypersingular integral, it is simplest to use the notion of convected differentiation of integrals, similar to (6). First, the integral of $\Sigma_{kij}(\boldsymbol{x},\boldsymbol{y})\,\dot{\epsilon}^p_{ij}(\boldsymbol{y})$, which is of order $O(\|\boldsymbol{x}-\boldsymbol{y}\|^{-2})$ over the three-dimensional domain P, can be written as a pv integral, since the contribution of the spherical ball $B_e(\boldsymbol{x})$ of radius e and centered at $\boldsymbol{x}$, being of order $O(e)$, can be neglected:

$$\int_P\Sigma_{kij}(\boldsymbol{x},\boldsymbol{y})\,\dot{\epsilon}^p_{ij}(\boldsymbol{y})\,dV_y\equiv(\mathrm{pv})\int_P\Sigma_{kij}(\boldsymbol{x},\boldsymbol{y})\,\dot{\epsilon}^p_{ij}(\boldsymbol{y})\,dV_y:=\lim_{e\to 0}\int_{P-B_e(x)}\Sigma_{kij}(\boldsymbol{x},\boldsymbol{y})\,\dot{\epsilon}^p_{ij}(\boldsymbol{y})\,dV_y.$$

This relation holds due to the symmetry of the excluded subdomain. In differentiating the pv integral we have two terms: the principal value integral, with kernel $(\partial/\partial x_h)\Sigma_{kij}(\boldsymbol{x},\boldsymbol{y})$, and the second term, coming from convected differentiation along the sphere $\partial B_e(\boldsymbol{x})$ with unit outward normal $\nu(\boldsymbol{y})$ to the sphere

$$\frac{\partial}{\partial x_h}\int_P \Sigma_{kij}(\boldsymbol{x},\boldsymbol{y})\,\dot\epsilon^p_{ij}(\boldsymbol{y})\,dV_y = (\mathrm{pv})\int_P \frac{\partial}{\partial x_h}\Sigma_{kij}(\boldsymbol{x},\boldsymbol{y})\,\dot\epsilon^p_{ij}(\boldsymbol{y})\,dV_y - \lim_{e\to 0}\int_{\partial B_e(x)} \Sigma_{kij}(\boldsymbol{x},\boldsymbol{y})\,\dot\epsilon^p_{ij}(\boldsymbol{y})\nu_h(\boldsymbol{y})\,dV_y.$$

For linear and isotropic elasticity, and for deviatoric plastic strain rate, it can be shown that the convected differentiation of the above integral is [Bui 1978]

$$\frac{\partial}{\partial x_h}\int_P \Sigma_{kij}(\boldsymbol{x},\boldsymbol{y})\,\dot\epsilon^p_{ij}(\boldsymbol{y})\,dV_y = (\mathrm{pv})\int_P \frac{\partial}{\partial x_h}\Sigma_{kij}(\boldsymbol{x},\boldsymbol{y})\,\dot\epsilon^p_{ij}(\boldsymbol{y})\,dV_y + \frac{8-10\nu}{15(1-\nu)}\dot\epsilon^p_{ij}(\boldsymbol{x}). \tag{11}$$

There is an extra term[1] appearing in the regularization process of the hyper-singular integral. Thus the concept of renormalization is necessary to properly derive boundary-integral equations in elastoplasticity. Examples of renormalization in (7) and (11) show that, when dealing with singular integrals, differentiation and integration are noncommutative operators. Other examples of finite-part integrals in fracture mechanics can be found in [Bui 1978; Lazarus 1998],

3. The fundamental solution of the Helmholtz equation in acoustics and its regularization

Another example of type I renormalization is provided by the classical fundamental solution of the Helmholtz equation

$$\Delta_y G(\boldsymbol{y},\boldsymbol{x};k) + k^2 G(\boldsymbol{y},\boldsymbol{x};k) + \delta(\boldsymbol{y}-\boldsymbol{x}) = 0, \tag{12}$$

where δ is the Dirac delta pseudofunction. In this equation, $k=\omega/c$ is the wave number and $\Delta_{\boldsymbol{y}}$ is the Laplacian operator with respect to the observation point $\boldsymbol{y}$, while $\boldsymbol{x}$ is the so-called source point. Let us consider the solution of the equation for small k and examine the continuity of the solution as k tends to zero. If $G(\boldsymbol{y},\boldsymbol{x};k)$ tends towards the static fundamental solution $G(\boldsymbol{y},\boldsymbol{x})$, one says that $G(\boldsymbol{y},\boldsymbol{x};k)$ satisfies the regularly perturbed equation. Such a solution is also called a regularized one. Otherwise, the Green's function is not necessarily continuous at $k=0$. In three dimensions, the fundamental solution satisfies the continuity condition, because (with $r=\|\boldsymbol{y}-\boldsymbol{x}\|$)

$$G(\boldsymbol{y},\boldsymbol{x};k) = \frac{\exp(ikr)}{4\pi r}, \qquad G(\boldsymbol{y},\boldsymbol{x}) = \frac{1}{4\pi r}, \qquad G(\boldsymbol{y},\boldsymbol{x};k) - G(\boldsymbol{y},\boldsymbol{x}) = O(k). \tag{13}$$

But this is not true anymore in two dimensions. Indeed, the two-dimensional Green's function given in standard textbooks reads

$$G(\boldsymbol{y},\boldsymbol{x};k) = \frac{i}{4}H^1_0(kr), \tag{14}$$

where $H^1_0(kr)$ is the Hankel function of the first kind and order zero. The Hankel function $H^1_0(z)$ can be expanded as [Abramovitz and Stegun 1972; Gradshteyn and Ryzhik 1965]

$$H^1_0(z) = J_0(z) + iN_0(z), \tag{15}$$

[1]The extra term $E(B)$ given by the second term in the right-hand side of (11) depends on the geometry of the symmetric ball B. With another symmetric excluded domain B' (ellipsoid, cube) the extra term $E(B')$ is different. The difference $E(B)-E(B')$ is hidden in the pv integral corresponding to B' [Sellier 1996]. Therefore the numerical calculation of the pv integral must be carefully undertaken with a correct excluded domain, which is a spherical ball in (11).

where

$$J_0(z) = 1 - \tfrac{1}{4}z^2 + O(z^4), \qquad N_0(z) = \frac{2}{\pi}\Big(\ln\frac{z}{2} + \gamma\Big) - \frac{1}{2\pi}\Big(\ln\frac{z}{2} + \gamma - 1\Big) + O(z^4) \tag{16}$$

with $\gamma = 0.577215\ldots$ Euler's constant.

This yields

$$G(\boldsymbol{y}, \boldsymbol{x}; k) = -\frac{1}{2\pi}\ln r + \frac{i}{4} - \frac{1}{2\pi}(\ln\frac{k}{2} + \gamma) + O(k^2 \ln k). \tag{17}$$

The first term on the right is the static two-dimensional solution $G = -(1/2\pi)\ln r$. Thus

$$G(\boldsymbol{y}, \boldsymbol{x}; k) - G(\boldsymbol{y}, \boldsymbol{x}) = \frac{i}{4} - \frac{1}{2\pi}(\ln\frac{k}{2} + \gamma) + O(k^2 \ln k)$$

The classical Green's function is not continuous at $k = 0$, where it diverges as $O(\ln k)$ as $k \to 0$. Therefore, to obtain the fundamental solution G^{reg} of the perturbed Helmholtz equation, which is continuous at $k = 0$, one subtracts the constant revealed by the expansion (which does not change the integral equation). Thus the regularized fundamental solution is [Bonnet 1995]

$$G^{\text{reg}}(\boldsymbol{y}, \boldsymbol{x}; k) = G(\boldsymbol{y}, \boldsymbol{x}; k) - \frac{i}{4} + \frac{1}{2\pi}\Big(\ln\frac{k}{2} + \gamma\Big). \tag{18}$$

4. Renormalization of type II in fracture mechanics

Barenblatt's cohesive force model. In linear fracture mechanics, it is well known that the stress has a square root singularity at the crack tip in two dimensions, $\sigma \approx O(r^{-1/2})$ while the normal crack displacement discontinuity is of $O(r^{1/2})$. In order to remove the stress singularity, Barenblatt [1959a; 1959b] proposed a model of cohesive force in the vicinity of the process zone. We may interpret Barenblatt's model as a physical renormalization (type II) of the Griffith crack. In a narrow zone AB, cohesive forces are introduced so that the corresponding stress intensity factor is opposite to the one calculated at point B from external loads, thus $K_{\text{I}}(\text{external}) + K_{\text{I}}(\text{cohesive}) = 0$. As the consequence of cohesive force, the crack opening displacement varies in a smooth manner and differs from Griffith's solution in a narrow zone (Figure 1). Despite the change on the crack geometry and the stress near the crack tip zone, the crack opening displacement of both models are practically the same *outside* the cohesive zone AB.

More precisely, we consider two scales: the macroscopic scale of the structure (including singularities), which determines the SIF from boundary conditions, and the microscopic scale of the process zone. The macroscopic solution influences Barenblatt's fine-scale solution through the condition

$$K_{\text{I}}(\text{external}) + K_{\text{I}}(\text{cohesive}) = 0.$$

But the reverse is not true because, for the same external K_{I}, there are different fine-scale solutions depending on the cohesive model considered, and also as long as there is a large ratio between the two scales, the finite-stress distribution in the narrow zone AB does not influence the macroscopic solution. In the process zone, the displacement is $O(r^{3/2})$; while immediately outside the process zone, where we are still in the dominant singular zone, the displacement is $O(r^{1/2})$. There is a smooth transition between the two asymptotics which depends on the model of the process zone considered.

The inner solution of the process zone is generally obtained by an asymptotic expansion matching. This means that (i) the fine-scale problem can be treated as a semi-infinite domain subjected to the stress

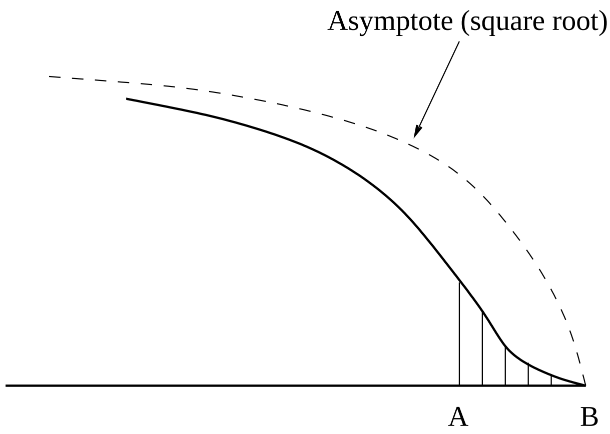

Figure 1. Barenblatt's model of an autonomous cohesive zone.

field at infinity specified by the remote singularities (parameterized by the SIF); (ii) the fine-scale field has only as many degrees of freedom as the singular terms in the macroscale solution. Solutions of the fine-scale problem have thus a great generality. An example of the fine-scale solution for a damage theory [Bui and Ehrlacher 1980], will be discussed further in the context of renormalization.

The asymptotic behaviour outside the cohesive zone governed by the SIF is important for applications since Barenblatt's model does not change the linear theory for global quantities: the macroscopic SIF is the same since it is the parameter which governs the *outer* field, the energy release rate remains unchanged. This renormalization of type II provides a better physical description of the process zone which avoids singularity of the stress in the fine-scale zone. To establish the link with classical linear fracture mechanics, for example the existence of a toughness K_{Ic}, Barenblatt assumed that the near tip region is *autonomous* in crack propagation. In other words, the asymptote of the crack opening profile $O(r^{1/2})$ is unchanged during crack propagation, or K_{I} is constant and identified to the toughness. A simpler model of the process zone, with constant normal stress equation, equal to the yield stress σ^o was proposed in [Leonov and Panasyuk 1959; Dugdale 1960].

Hierarchical models of fracture. The Griffith crack in an elastic medium has been extended to perfect plasticity in [Rice 1968]. Rice's model is an attempt to remove the deviatoric stress singularity of a crack, by considering perfect plasticity. In mode III, the solution in small scale yielding [2] is a circular disc. The plastic zone is governed by the stress singular field, and the fine-scale problem has been treated as a semi-infinite domain with the remote stress field $\sigma_{ij} = K_{\mathrm{III}}\, g_{ij}(\theta)/\sqrt{2\pi r}$. Because of the yield stress assumption, the deviatoric stress field is finite while the strain is singular as $\epsilon \approx O(1/r)$. Does a model exist in which both stress and strain are regular in the continuum? Suppose that the behavior law of the body involves a yield stress σ_0 and a limit strain ϵ_R beyond which local stresses vanish. This model of sudden damage has been studied for mode III loading in [Bui and Ehrlacher 1980]. In the stationary case (Figure 2), a solution satisfying the equilibrium equation, the compatibility equation, under the assumption of negligible elastic strain in the plastic zone, has the following features:

[2] The plastic zone size a_p is assumed to be very small in comparison with the crack size a; that is, $a_p \approx (K_{\mathrm{III}}/\sigma_0)^2/\pi << a$, where σ_0 is the shear yield stress.

1. An elastic zone with the remote stress field $\sigma_{ij} = K_{\mathrm{III}}\, g_{ij}(\theta)/\sqrt{2\pi r}$,
2. An active plastic zone with straight characteristics along α − lines, with strain varying between points N and M, $\epsilon_{3\beta} = \epsilon_R$, $\sigma_0 = 2\mu\epsilon_0$.
3. A damage zone with width $2h$, containing particles that undergo a strain greater than ϵ_R and where the stress tensor vanishes.

We recall the solution for perfect plasticity $\sigma_0 = \sigma_R$, without going into the details of the derivation [Bui and Ehrlacher 1980; Bui 2006]. The solution is characterized by the following equations, using complex variables $z = x_1 + ix_2$, $\tau = \sigma_{32} + i\sigma_{31}$. The field solution $\tau(z)$ in the physical plane is searched for in its inverted form $z(\tau)$.

Point B ($z = -ih$) is mapped to point $\tau = i\sigma_R$:

$$z(\tau) = \frac{K_{\mathrm{III}}^2}{2\pi\tau^2} - \frac{2h}{\pi}\ln\frac{\tau}{\sigma_R} + \text{constant}, \tag{19}$$

$$u_3 = \mathrm{Im}\left(K_{\mathrm{III}}^2 \frac{(\sigma_R - \tau)}{\pi\mu\tau\sigma_R} - \frac{2h(\tau - \sigma_R)}{\pi\mu}\right). \tag{20}$$

The real constant is chosen so that $x_1(B) = 0$, thus const $= -K_{\mathrm{III}}^2/(2\pi\sigma_R^2)$. Solution (20) exists for $\sigma_R/2\mu \le \epsilon_{3t} < \epsilon_R$. The latter conditions imply conditions on the SIF, $K_R^2 \le K_{\mathrm{III}}^2 < 2h\sigma_R(4\mu\epsilon_R - \sigma_R)$, where

$$K_R^2 = 2h\sigma_R^2. \tag{21}$$

The damage front BNB' (the locus of N) is the cusped cycloid $(-\pi < \theta < +\pi)$

$$N_1(\theta) = \frac{h}{\pi}(1 + \cos 2\theta), \qquad N_2(\theta) = -\frac{h}{\pi}(2\theta + \sin 2\theta) \tag{22}$$

The elastic-plastic boundary is defined by the circle $|\tau| = \sigma_R$ of the stress plane. Thus, the locus of point M is the curled cycloid

$$M_1(\theta) = K_{\mathrm{III}}^2 \frac{1}{2\pi\sigma_R^2}(1 + \cos 2\theta), \qquad M_2(\theta) = -K_{\mathrm{III}}^2 \frac{1}{2\pi\sigma_R^2}\sin 2\theta + \frac{2h\theta}{\pi}. \tag{23}$$

For unlimited ϵ_R, the cusped cycloid BNB' (damage front) reduces to the crack tip point, while the curled cycloid BMB' (elastic-plastic boundary) becomes Rice's circle, tangent to the x_2 axis. For finite ϵ_R, the thickness of the damage zone is $2h = K_{\mathrm{III}}^2/\sigma_R^2$. The elastic strain energy on the damage front is $W = \sigma_R^2/2\mu$. The J-integral with contour along the damage zone takes the value

$$J_{tip} = \frac{h}{\mu}\sigma_R^2 = \frac{1}{2\mu}K_{\mathrm{III}}^2. \tag{24}$$

Thus damage model does not change global quantities like the energy release rate, which can be calculated from the asymptotic outer field. But it gives more details on the inner process zone. Renormalization of type II as considered here serves successively to remove the stress singularity only [Rice 1968] and then to ensure that both stress and strain in the continuum are finite. These models, like

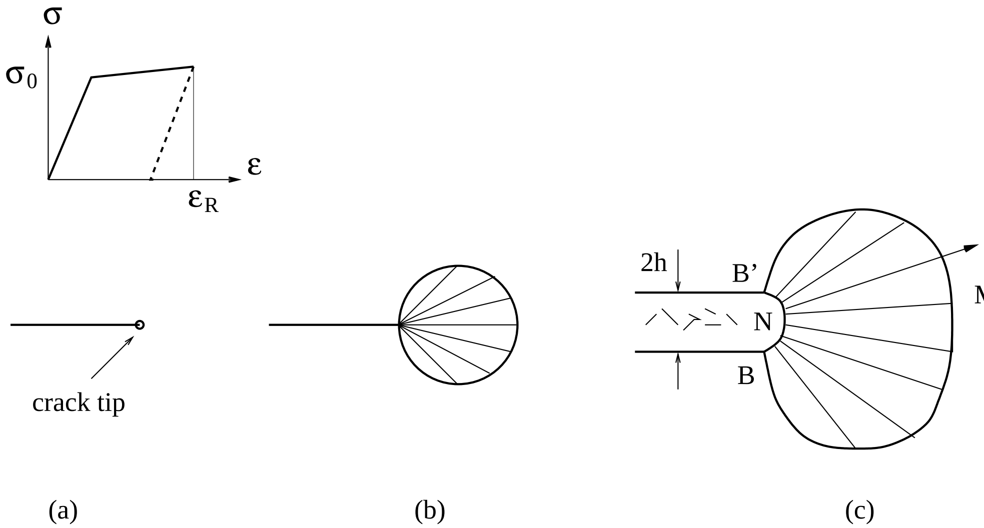

Figure 2. Hierarchical models in mode III: (a) crack tip in elastic medium; (b) Rice's plastic correction; (c) sudden damage model.

Russian dolls, are hierarchical in the sense that finer and finer scales of description have been used for renormalization (Figure 2):

$$\text{Griffith crack} \ll \text{Plastic correction} \ll \text{Damage model}$$

5. Renormalization of the energy release rate for point loads

In linear fracture mechanics, it is well known that the formula for the energy release rate $G = -dP/da$, defined as the rate of the potential energy P with respect to the crack length, holds under the assumption of square integrable stress and strain fields. Irwin's formula then relates G to the SIF, in plane strain mode I, by

$$G = \frac{1-\nu^2}{E} K_{\mathrm{I}}^2. \tag{25}$$

What happens when the potential energy is infinite? This is the case where one deals with a pair of point loads acting on the crack faces, at a distance a from the crack tip. For a pair of point forces $\pm F\delta(\boldsymbol{x})$ acting on the crack at the distance $a > 0$ from the crack tip, the plane strain solution for an infinite plane is known in textbooks as $K_{\mathrm{I}} = 2F(\pi/a)^{1/2}$; see [Sih 1973; Tada 1973]. Now let us try to make use of Irwin's formula (25) for calculating the stress intensity factor. We realize that we are unable to calculate the potential energy P before differentiating it with respect to a, $G = -dP/da$. The reason is that, for point loads, stress and strain fields in two dimensions are not in $L^2(\Omega)$. The displacement under the point load is logarithmically singular in two dimensions, the strain energy density is singular as $O(r^{-2})$, so the potential energy is divergent $P = \infty$. How can we take a derivative with respect to a of an infinite quantity?

To avoid these inconsistencies, we shall introduce a renormalization of type II.

Before doing that, let us address some general remarks on point loads giving rise to unbounded energy. There is a strong connection of this section with the whole issue of existence and uniqueness theorems in linear quasistatic elasticity, which involve bounded energy solutions, as discussed in [Sternberg and Eubanks 1955]. It is known that Green's solution exists for point loads in an unbounded domain, as given explicitly by the Kelvin–Somigliana fundamental solution $G(\boldsymbol{x}, \boldsymbol{y})$ (see classical textbooks). The fundamental solution for a *bounded* solid Ω does not exist for a point load alone, because the point load alone is not in equilibrium. For the existence of a Green's function for bounded solid, one needs to impose either a distribution of appropriate surface loads on its boundary which equilibrates the point load [Bergman and Schiffer 1953] or a homogeneous Dirichlet boundary condition $u_i(x) = 0$ on some part of the boundary (not reduced to a single point). The fundamental solution is not unique since there are many types of boundary conditions.[3]

In what follows, we shall consider bounded energy solutions in elasticity, so we modify the physics of the problem. Instead of a point force, we consider a finite distribution of normal stress on a small zone of width h, with σ_{22} constant over a finite zone $0 < x_1 < h$, $x_2 = 0$. The mollified applied stress is

$$\sigma_{22}(x_1, x_2 = 0; h) = \begin{cases} -\dfrac{F}{h} & \text{for } 0 < x_1 < h, \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

Function (26) is *independent* of the crack tip location. Its limit as h tends to zero is the Dirac delta. For $h \neq 0$, stress, strain and displacement are finite and square integrable. The energy release rate of the original problem is therefore given by the limit

$$G = -\lim_{h \to 0+} \frac{dP_h}{da}, \tag{27}$$

where P_h is the potential energy. We will indicate below another expression for G.

To prove that the limit (27) exists we can, by virtue of the symmetry of the geometry, restrict ourselves to the lower half-plane and double the result for the potential energy. On the boundary of the lower half-plane, the normal applied stress consists of the mollified stress (26), denoted hereafter by $\sigma^{(1)}(\boldsymbol{x}; h)$, and the normal tensile stress on the ligament, denoted by $\sigma^{(2)}(\boldsymbol{x}; a, h)$. The stress solution decomposes as

$$\sigma(\boldsymbol{x}; a, h) = \sigma^{(1)}(\boldsymbol{x}; h) + \sigma^{(2)}(\boldsymbol{x}; a, h). \tag{28}$$

The normal loads for small h together with the corresponding normal displacements $u_2^{(1)}, u_2^{(2)}$ on the line $x_2 = 0$ are displayed in Figure 3. The potential energy of the cracked body is written in line integral form as

$$P_h = 2 \times \frac{1}{2} \int_0^h \sigma_{22}^{(1)} u_2 dx_1. \tag{29}$$

[3]The proofs of existence and uniqueness, for a specific boundary condition, are not based on bounded energy considerations, but rather on singular boundary integral equations with unknown C($\partial\Omega$) density. The two fundamental solutions for a bounded solid, of both the Neumann and the Dirichlet types, are provided in [Pham 1967].

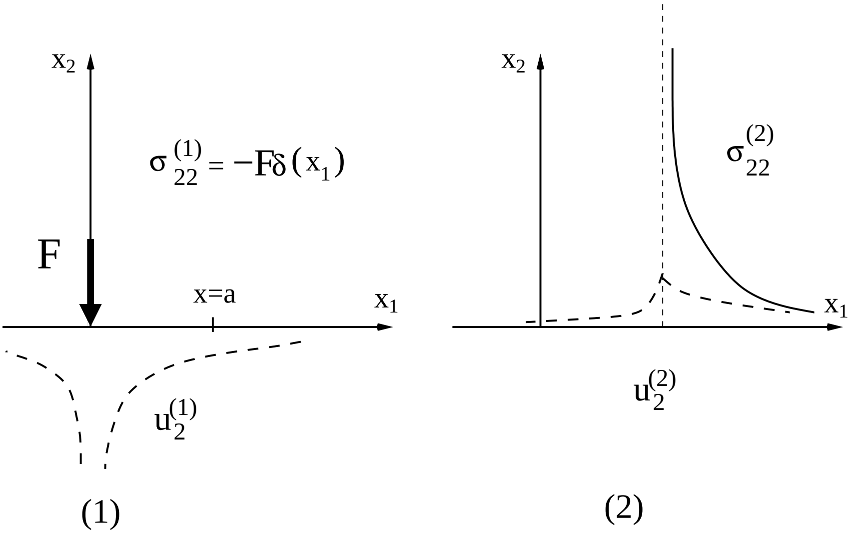

Figure 3. (1) Point load F on the boundary of a half-plane, defining the stress field $\sigma_{22}^{(1)}$ on $x_2 = 0$ and the logarithmically singular displacement $u_2^{(1)}$. (2) Distributed normal load $\sigma_{22}^{(2)}$ equal to the stress in the ligament, with continuous surface displacement $u_2^{(2)}$.

The displacement u_2 is equal to the sum $u_2^{(1)} + u_2^{(2)}$. The field $u_2^{(2)}$ satisfies the condition $u_2^{(2)} = -u_2^{(1)}$ on the ligament $x_1 > a$. The potential energy is

$$P_h = \int_0^h \sigma_{22}^{(1)}(x_1; h) u_2^{(1)}(x_1; h)\, dx_1 + \int_0^h \sigma_{22}^{(1)}(x_1; h) u_2^{(2)}(x_1; a, h)\, dx_1. \tag{30}$$

The first integral diverges logarithmically as h tends to zero. It is the singular part of the potential energy, denoted by S_h. We observe that the singular part S_h of P_h is *independent* of a, $(d/da)S_h = 0$ for any h. Therefore G is given by either of the formulas

$$G = -\lim_{h\to 0+} \frac{d}{da} \int_0^h \sigma_{22}^{(1)}(x_1; h) u_2^{(2)}(x_1; a, h)\, dx_1, \tag{31}$$

$$G = -\lim_{h\to 0+} \frac{d}{da} \big(P_h - S_h\big). \tag{32}$$

Equation (32) is nothing but Hadamard's finite part procedure, which has been considered in type I renormalization. It remains to prove that (27) or (32) provides the classical formula in mode I, $G = (1-\nu^2)K_{\mathrm{I}}^2/E$.

Proof. The solution for the mollified load (26) is continuous in h. Therefore, stress $\sigma^{(h)}$ converges to the stress solution σ for point loads and $K_{\mathrm{I}}^{(h)}$ converges to K_{I}. Since we have $G^{(h)} = -dP_h/da = (1-\nu^2)K_{\mathrm{I}}^{(h)2}/E$ for any h, upon taking the limit for vanishing h we obtain the expected result. □

6. Conclusions

By discussing some examples in previous sections, we forgot the primary purpose of renormalization in physics, more precisely in quantum electrodynamics, which deals with perturbation methods applied to solutions for some scale of observation. The calculation of the norm of higher-order solutions involves divergent integrals. Mathematical techniques to overcome some divergences appearing in the renormalization procedure seem to be useful in other contexts of solid mechanics. Some of these techniques have been considered here, but not all. For example, scaling laws in fatigue and fracture [Barenblatt 2006] and the cutoff method (by ignoring energies greater than mc^2 [Feymann 1948] have not been considered in this paper. The cutoff method was recently considered in [Dini et al. 2007] for contact problems with a sharp corner defined by a distribution of curvature that goes to infinity at the corner. A cutoff of the curvature distribution will round up the corner. The cutoff of curvature of a sharp punch profile will remove the singularity arising in frictional sliding contact and resolve a paradox pointed out in [Adams et al. 2005] in violation of the kinematic Coulomb friction condition, as done in [Hills et al. 2005].

Singularity removal is very often encountered in solid and fracture mechanics. It is shown that operators like differentiation and singular integration are not commutative, nor can we interchange differentiation and the limiting process. Examples of noncommutative limiting processes also exist in the literature of homogeneization; for example in models with a small parameter a of the microscopic cell and a small thickness h of the macrostructure, the successive limits $a \to 0$ and $h \to 0$ are not commutative [Geymonat et al. 1987]. But these noncommutative limits are not considered here since they are not related to renormalization procedures.

The use of different levels of observation is considered in hierarchical models of the process zone. However, it seems that no simple scaling laws are found for the latter models which can be compared to size dependent scaling laws in damage theory [Bazant 2002].

Acknowledgments. I thank Professors A. Constantinescu, A. Ehrlacher, A. Oueslati, M. Predeleanu and A. Sellier for helpful discussions. Many thanks also to one reviewer who made pertinent remarks that improved the original text.

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Received 4 Apr 2008. Accepted 26 May 2008.

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