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SPLINE-BASED INVESTIGATION OF NATURAL VIBRATIONS OF ORTHOTROPIC RECTANGULAR PLATES OF VARIABLE THICKNESS WITHIN CLASSICAL AND REFINED THEORIES

YAROSLAV M. GRIGORENKO, ALEXANDER YA. GRIGORENKO AND TATYANA L. EFIMOVA

A spline-collocation approach is proposed for studying the natural vibrations of orthotropic rectangular plates of variable thickness. The approach is based on the spline-approximation method and the method of discrete orthogonalization coupled with the step-by-step search method. The study is carried out within the framework of the classical and refined theories of plates. The dynamic response of the plates is studied depending on the variation of the plate thickness, the mechanical parameters, and the type of boundary conditions.

1. Introduction

Plates, as rational structural elements, are widely used in some fields using modern techniques (mainly aircraft construction and shipbuilding). In connection with the wide application of composite materials and structural peculiarities, studying the mechanical behavior of anisotropic plates of varying thickness is currently of great importance. The essential point in securing the reliability of plate-shaped elements is the determination of the natural frequencies and modes with high accuracy. Such knowledge is needed in order to describe the response of the plates to operating conditions. For plates with constant thickness and hinged opposite edges, the solution can be constructed in closed form [Graff 1991; Varvak and Ryabov 1971]. The natural vibrations of orthotropic plates with other boundary conditions have been studied quite actively and are the subject of a number of publications [Leissa 1969; 1981; 1987].

Solutions for forced and natural vibrations of orhotropic plates were obtained in [Sakata and Hosokawa 1988] in the form of double trigonometric series. Lagrangian multipliers were used in [Ramkumar et al. 1987] to solve a similar problem, with allowance being made for shear strains in the first several modes. The superposition method was used in [Gorman 1990] to tabulate natural frequencies for a certain range of stiffness ratios. In [Yu and Cleghorn 1993], the superposition method and affine transformations were used to determine the natural frequencies of partially clamped and partially simply supported orthotropic plates. The Kantorovich method was used in [Bercin 1996] to study the natural vibrations of clamped plates. The natural vibrations of complex anisotropic plates were studied in [Bhat 1985; Kurpa and Chistilina 2003] using variational methods and the R-function method. The natural vibrations of rectangular plates of varying thickness were addressed by many authors. For example, the papers [Chen 1976; 1977] are concerned with the general natural-vibration problem for plates of varying thickness. The transverse vibrations of plates with exponentially varying thickness are studied in [Bhat 1987] and inhomogeneous rectangular plates with parabolically varying thickness in [Tomar et al. 1982]. The natural vibrations

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of simply supported plates with linearly varying thickness were investigated in [Appl and Bayers 1965; Bhat 1985; Bhat et al. 1990; Nog and Araar 1989].

The natural vibrations of rectangular plates with varying thickness were studied using Mindlin's theory with lesser activity than in similar investigations fulfilled within the framework of the classical theory of plates. Let us note some works, such as [Mindlin 1951; Mizusava 1993; Mizusava and Condo 2001; Roufacil and Dawe 1980], dedicated to this scientific trend. The collocation method based on orthogonal polynomials was used in [Mikami and Yoshimura 1984] to analyze vibrations of a plate with linearly varying thickness. In [Al-Kaabi and Aksu 1958; Bercin 1996], a method based on the variational procedure in combination with the finite-difference method was used to solve the problems for plates with linearly and parabolically varying thickness. To study the natural vibrations of wedge-like plates with varying thickness, some variants of the spline-element method were used. Note that the above-mentioned publications are devoted to isotropic plates.

The analysis presented allows us to conclude that there is a variety of different approximate approaches to the study of natural vibrations of rectangular plates with boundary conditions, which do not allow us to obtain the solutions in closed form. Recently, computational mathematics, mathematical physics, and mechanics have employed spline widely functions to solve such problems. This is due to the following advantages of the spline-approximation method over other ones: stability of splines against local perturbations, that is, the behavior of a spline in the neighborhood of a point does not affect the overall behavior of the spline (as do, for example, polynomial approximations); better convergence of spline-interpolation compared with polynomial interpolation; and simple and convenient computer implementation of spline algorithms. The use of spline functions in variational, projective, and other discrete-continuous methods allows us to obtain appreciable results compared to the use of classical polynomials and substantially simplify their numerical implementation, leading to highly accurate solutions.

In [Mizusava 1993; Mizusava and Condo 2001], in order to solve one-dimensional boundary-value problems or those reduced to them, which describe bending, stability, and vibrations of plates and shells, the solution is approximated by splines of the third or fifth power and the problem is reduced to a system of algebraic equations. This is more advantageous than other methods from the viewpoint of calculation time and accuracy. In a number of two-dimensional problems concerning the stress-strain state and vibrations of plates and shells under certain boundary conditions, the problem is reduced to a one-dimensional one by using some variational or projective method. Such a problem can be solved by the spline-approximation method.

To solve a two-dimensional linear boundary-value problem and boundary-value problem for eigenvalues, the approach, based on a reduction of a two-dimensional problem to a one-dimensional one by the spline-collocation method in one coordinate direction or by other stable numerical method, has an effective application, along with the above-mentioned approaches to solve the problems in the theory of plates and shells. Here we extend the spline-collocation method proposed in [Grigorenko and Trigubenko 1990] to study the natural vibrations of rectangular orthotropic plates of varying thickness with complex boundary conditions within the framework of different models. The spline-collocation method was used previously in [Grigorenko and Yaremchenko 2004; Grigorenko and Zakhariichenko 2003; 2004].

2. Basic relations and constitutive equations

We will solve the natural-vibration problem for a rectangular orthotropic plate with thickness $h(x, y)$, varying along two coordinate directions, in a rectangular coordinate system (the coordinate plane xOy is the mid-surface of the plate, $0 \le x \le a$, $0 \le y \le b$, $-h/2 \le z \le h/2$).

2A. ***Formulation of the problem within the framework of Kirchhoff theory.*** We assume that normals to the plate mid-surface under deformation remain straight and perpendicular to this surface and that the normal stresses on elemental areas, which are parallel to the mid-surface, are small and can be neglected. Then, the equations of motion can be written [Lekhnitskii 1957; Varvak and Ryabov 1971] as

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = \rho h \frac{\partial^2 w}{\partial t^2}, \qquad \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x, \qquad \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = Q_y, \tag{2-1}$$

where x, y are the Cartesian coordinates ($0 \le x \le a$, $0 \le y \le b$), t is time, w is the deflection of the plate, ρ is the density of a material, and Q_x and Q_y are shear forces. The moments M_x, M_y, M_{xy} satisfy the relations

$$M_x = -\Big(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2}\Big), \qquad M_y = -\Big(D_{12}\frac{\partial^2 w}{\partial x^2} + D_{22}\frac{\partial^2 w}{\partial y^2}\Big), \qquad M_{xy} = -2D_{66}\frac{\partial^2 w}{\partial x \partial y}, \tag{2-2}$$

where the stiffness characteristics D_{ij} of the plate are defined by $D_{ij} = B_{ij}h^3(x, y)/12$. Here $B_{11} = E_1/(1-\nu_1\nu_2)$, $B_{12} = \nu_2 E_1/(1-\nu_1\nu_2) = \nu_1 E_2/(1-\nu_1\nu_2)$, $B_{22} = E_2/(1-\nu_1\nu_2)$, $B_{66} = G_{12}$, where E_1, E_2, ν_1, and ν_2 are the elastic and shear moduli and Poisson's ratios.

The system of equations in (2-1) and (2-2) yields an equivalent differential equation for the deflection:

$$\begin{aligned} D_{11}\frac{\partial^4 w}{\partial x^4} + D_{22}\frac{\partial^4 w}{\partial y^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 2\frac{\partial D_{11}}{\partial x}\frac{\partial^3 w}{\partial x^3} + 2\frac{\partial D_{22}}{\partial y}\frac{\partial^3 w}{\partial y^3} \\ + 2\frac{\partial}{\partial y}(D_{12}+2D_{66})\frac{\partial^3 w}{\partial x^2 \partial y} + 2\frac{\partial}{\partial x}(D_{12}+2D_{66})\frac{\partial^3 w}{\partial x \partial y^2} \\ + \Big(\frac{\partial^2 D_{11}}{\partial x^2} + \frac{\partial^2 D_{12}}{\partial y^2}\Big)\frac{\partial^2 w}{\partial x^2} + \Big(\frac{\partial^2 D_{12}}{\partial x^2} + \frac{\partial^2 D_{22}}{\partial y^2}\Big)\frac{\partial^2 w}{\partial y^2} + 4\frac{\partial^2 D_{66}}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \rho h \frac{\partial^2 w}{\partial t^2} = 0. \end{aligned} \tag{2-3}$$

It is assumed that all points of the plate vibrate harmonically with a frequency ω, that is, $w(x, y, t) = \hat{w}(x, y)e^{i\omega t}$ (the symbol ˆ is omitted hereafter).

Let us rearrange Equation (2-3) to the form

$$\begin{aligned} \frac{\partial^4 w}{\partial x^4} = a_1\frac{\partial^3 w}{\partial x^3} + a_2\frac{\partial^4 w}{\partial x^2 \partial y^2} + a_3\frac{\partial^3 w}{\partial x^2 \partial y} + a_4\frac{\partial^2 w}{\partial x^2} + a_5\frac{\partial^3 w}{\partial x \partial y^2} \\ + a_6\frac{\partial^2 w}{\partial x \partial y} + a_7\frac{\partial^4 w}{\partial y^4} + a_8\frac{\partial^3 w}{\partial y^3} + a_9\frac{\partial^2 w}{\partial y^2} + a_{10}w, \end{aligned} \tag{2-4}$$

where the coefficients $a_i = a_i(x, y)$, $i = 1, \dots, 9$, $a_{10} = a_{10}(x, y, \omega)$ are defined by

$$a_1 = -\frac{2}{D_{11}}\frac{\partial D_{11}}{\partial x}, \quad a_2 = -\frac{2}{D_{11}}(D_{12}+2D_{66}), \quad a_3 = -\frac{2}{D_{11}}\Big(\frac{\partial D_{12}}{\partial y}+2\frac{\partial D_{66}}{\partial y}\Big),$$
$$a_4 = -\frac{2}{D_{11}}\Big(\frac{\partial^2 D_{11}}{\partial x^2}+\frac{\partial^2 D_{12}}{\partial y^2}\Big), \quad a_5 = -\frac{2}{D_{11}}\Big(\frac{\partial D_{12}}{\partial x}+2\frac{\partial D_{66}}{\partial x}\Big), \quad a_6 = -\frac{4}{D_{11}}\frac{\partial^2 D_{66}}{\partial x\partial y},$$
$$a_7 = -\frac{D_{22}}{D_{11}}, \quad a_8 = -\frac{2}{D_{11}}\frac{\partial D_{22}}{\partial y}, \quad a_9 = -\frac{2}{D_{11}}\Big(\frac{\partial^2 D_{12}}{\partial x^2}+\frac{\partial^2 D_{22}}{\partial y^2}\Big), \quad a_{10} = \frac{\rho}{D_{11}}h(x, y)\omega^2.$$

Let us specify the boundary conditions expressed in terms of the deflection at the edges as $x = 0$, $x = a$, $y = 0$, $y = b$. We will consider the following boundary conditions:

i) All edges are clamped (boundary conditions of type A):

$$w = 0,\ \frac{\partial w}{\partial y} = 0 \text{ at } y = 0,\ y = b, \quad w = 0,\ \frac{\partial w}{\partial x} = 0 \text{ at } x = 0,\ x = a. \tag{2-5}$$

ii) Three edges are clamped and the fourth one is simply supported (boundary conditions of type B):

$$w = 0,\ \frac{\partial w}{\partial y} = 0 \text{ at } y = b, \quad w = 0,\ \frac{\partial w}{\partial x} = 0 \text{ at } x = 0,\ x = a, \quad w = 0,\ \frac{\partial^2 w}{\partial y^2} = 0 \text{ at } y = 0,$$

or boundary conditions of type C:

$$w = 0,\ \frac{\partial w}{\partial y} = 0 \text{ at } y = 0,\ y = b, \quad w = 0,\ \frac{\partial w}{\partial x} = 0 \text{ at } x = 0, \quad w = 0,\ \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = a.$$

iii) Two edges are clamped and two are simply supported (boundary conditions of type D):

$$\begin{aligned} &w = 0,\ \frac{\partial w}{\partial y} = 0 \text{ at } y = 0, \quad w = 0,\ \frac{\partial^2 w}{\partial y^2} = 0 \text{ at } y = b,\\ &w = 0,\ \frac{\partial w}{\partial x} = 0 \text{ at } x = 0, \quad w = 0,\ \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = a, \end{aligned} \tag{2-6}$$

or boundary conditions of type E:

$$w = 0,\ \frac{\partial w}{\partial y} = 0 \text{ at } y = 0,\ y = b, \quad w = 0,\ \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = 0,\ x = a,$$

or boundary conditions of type G:

$$w = 0,\ \frac{\partial w}{\partial x} = 0 \text{ on } x = 0,\ x = a, \quad w = 0,\ \frac{\partial^2 w}{\partial x^2} = 0 \text{ on } y = 0,\ y = b. \tag{2-7}$$

2B. ***Formulation of the problem within the framework of Mindlin's theory.*** We suppose that the element, which is initially normal to a coordinate surface in the undeformed state, remains rectilinear but perpendicular to the deformable surface of the plate and turns by some angle keeping its length unchanged. Also, the initial forces caused by the deflection of the element of the coordinate surface and by the turn of the normal element are taken into account. In keeping with this hypothesis, the displacements u_x, u_y, u_z can be written as

$$u_x(x, y, z) = u(x, y) + z\psi_x(x, y), \quad u_y(x, y, z) = v(x, y) + z\psi_y(x, y), \quad u_z(x, y, z) = w(x, y). \tag{2-8}$$

Here u, v, w are the displacements of the point of a median surface in the x, y, z-directions, respectively, and ψ_x, ψ_y are the full rotation angles of the rectangular element. Then the relations for strains will be defined by

$$\begin{gathered} e_x(x, y, z) = \varepsilon_x(x, y) + z\varkappa_x(x, y), \quad e_y(x, y, z) = \varepsilon_y(x, y) + z\varkappa_y(x, y), \\ e_{xy}(x, y, z) = \varepsilon_{xy}(x, y) + 2z\varkappa_{xy}(x, y), \quad e_{xz}(x, y, z) = \gamma_x(x, y), \quad e_{yz}(x, y, z) = \gamma_y(x, y), \end{gathered} \tag{2-9}$$

where

$$\begin{gathered} \varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \varkappa_x = \frac{\partial \psi_x}{\partial x}, \quad \varkappa_y = \frac{\partial \psi_y}{\partial y}, \quad 2\varkappa_{xy} = \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}, \\ \gamma_x = \psi_x - \theta_x, \quad \gamma_y = \psi_y - \theta_y, \quad \theta_x = -\frac{\partial w}{\partial x}, \quad \theta_y = -\frac{\partial w}{\partial y}. \end{gathered} \tag{2-10}$$

In equations (2-9) and (2-10), $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$ are tangential strains of a median surface, $\varkappa_x, \varkappa_y, \varkappa_z$ are bending strains, θ_x, θ_y are rotation angles of the normal without considering the transversal shears, and γ_x, γ_y are rotation angles of the normal caused by the transversal shears.

The equations describing natural bending vibrations of the plate can be written as follows:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = \rho h \frac{\partial^2 w}{\partial t^2}, \quad \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = \rho \frac{h^3}{12} \frac{\partial^2 \psi_x}{\partial t^2}, \quad \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = \rho \frac{h^3}{12} \frac{\partial^2 \psi_y}{\partial t^2}. \tag{2-11}$$

It is assumed that all points of the plate vibrate harmonically with a frequency ω that is $w(x, y, t) = \hat{w}(x, y)e^{i\omega t}$, $\psi_x(x, y, t) = \hat{\psi}_x(x, y)e^{i\omega t}$, $\psi_y(x, y, t) = \hat{\psi}_y(x, y)e^{i\omega t}$ (the symbol ˆ is omitted hereafter). in (2-11) ρ is the density of a material, $h = h(x, y)$ is the plate thickness. The moments M_x, M_y, M_z and shear forces Q_x and Q_y satisfy the elasticity relations

$$M_x = D_{11}\varkappa_x + D_{12}\varkappa_y, \;\; M_y = D_{22}\varkappa_y + D_{12}\varkappa_x, \;\; M_{xy} = 2D_{66}\varkappa_{xy}, \;\; Q_x = K_1\varkappa_x, \;\; Q_y = K_1\varkappa_y, \tag{2-12}$$

which are valid for the orthotropic plate whose orthotropy axes coincide with coordinate axes. In (2-12), the stiffness characteristics K_i and D_{ij} are defined by the formulas

$$\begin{gathered} K_1 = \tfrac{5}{6}h(x, y)G_{13}, \quad K_2 = \tfrac{5}{6}h(x, y)G_{23}, \quad D_{ij} = \frac{B_{ij}h^3(x, y)}{12}, \quad B_{11} = \frac{E_1}{1 - \nu_1\nu_2}, \\ B_{12} = \frac{\nu_2 E_1}{1 - \nu_1\nu_2} = \frac{\nu_1 E_2}{1 - \nu_1\nu_2}, \quad B_{22} = \frac{E_2}{1 - \nu_1\nu_2}, \quad B_{66} = G_{12}. \end{gathered}$$

Here E_i, G_{ij}, and ν_i are the Young's and shear moduli and Poisson's ratios, respectively. Having introduced the notation $\tilde{w} = \partial w/\partial x$, $\tilde{\psi}_x = \partial \psi_x/\partial x$, and $\tilde{\psi}_y = \partial \psi_y/\partial x$, we can write the resolving equations for functions w, $\tilde{w}$, ψ_x, $\tilde{\psi}_x$, ψ_y, and $\tilde{\psi}_y$ as

$$
\begin{aligned}
&\frac{\partial w}{\partial x}=\tilde{w}, \quad \frac{\partial \psi_x}{\partial x}=\tilde{\psi}_x, \quad \frac{\partial \psi_y}{\partial x}=\tilde{\psi}_y,\\
&\frac{\partial \tilde{w}}{\partial x}=a_{11}w+a_{12}\frac{\partial w}{\partial y}+a_{13}\frac{\partial^2 w}{\partial y^2}+a_{14}\tilde{w}+a_{15}\psi_x+a_{16}\tilde{\psi}_x+a_{17}\psi_y+a_{18}\frac{\partial \psi_y}{\partial y},\\
&\frac{\partial \tilde{\psi}_x}{\partial x}=a_{21}\tilde{w}+a_{22}\psi_x+a_{23}\frac{\partial \psi_x}{\partial y}+a_{24}\frac{\partial^2 \psi_x}{\partial y^2}+a_{25}\tilde{\psi}_x+a_{26}\frac{\partial \psi_y}{\partial y}+a_{27}\tilde{\psi}_y+a_{28}\frac{\partial \tilde{\psi}_y}{\partial y},\\
&\frac{\partial \tilde{\psi}_x}{\partial x}=a_{31}\frac{\partial w}{\partial y}+a_{32}\frac{\partial \psi_x}{\partial y}+a_{33}\tilde{\psi}_x+a_{34}\frac{\partial \tilde{\psi}_x}{\partial y}+a_{35}\psi_y+a_{36}\frac{\partial \psi_y}{\partial y}+a_{37}\frac{\partial^2 \psi_y}{\partial y^2}+a_{38}\tilde{\psi}_y.
\end{aligned}
\tag{2-13}
$$

The coefficients a_{ij} in system (2-13) are

$$
\begin{gathered}
a_{11}=-\frac{\rho h\omega^2}{K_1}, \quad a_{12}=a_{17}=-\frac{1}{K_1}\frac{\partial K_2}{\partial y}, \quad a_{13}=a_{18}=-\frac{K_2}{K_1}, a_{14}=a_{15}=-\frac{1}{K_1}\frac{\partial K_1}{\partial x}, \quad a_{16}=-1,\\
a_{21}=\frac{K_1}{D_{11}}, \quad a_{22}=\frac{1}{D_{11}}\big(K_1-\rho\frac{h^3}{12}\omega^2\big), \quad a_{23}=a_{27}=-\frac{1}{D_{11}}\frac{\partial D_{66}}{\partial y},\\
a_{24}=-\frac{D_{66}}{D_{11}}, \quad a_{25}=-\frac{1}{D_{11}}\frac{\partial D_{11}}{\partial x}, \quad a_{26}=-\frac{1}{D_{11}}\frac{\partial D_{12}}{\partial x}, \quad a_{28}=-\big(\frac{D_{12}+D_{66}}{D_{11}}\big),\\
a_{31}=\frac{K_2}{D_{66}}, \quad a_{32}=a_{38}=-\frac{1}{D_{66}}\frac{\partial D_{66}}{\partial x}, \quad a_{33}=-\frac{1}{D_{66}}\frac{\partial D_{12}}{\partial y}, \quad a_{34}=-\big(\frac{D_{12}+D_{66}}{D_{66}}\big),\\
a_{35}=\frac{1}{D_{66}}\big(K_2-\rho\frac{h^3}{12}\omega^2\big), \quad a_{36}=-\frac{1}{D_{66}}\frac{\partial D_{22}}{\partial y}, \quad a_{37}=-\frac{D_{22}}{D_{66}}.
\end{gathered}
$$

The resolving equations should be supplemented with boundary conditions at the plate edges $x=0$, $x=a$, $y=0$, and $y=b$. We will consider the following boundary conditions:

i) All edges are clamped (boundary conditions of type A):

$$w=0,\ \psi_x=0,\ \psi_y=0 \text{ at } y=0,\ y=b,\ x=0,\ x=a. \tag{2-14}$$

ii) Three edges are clamped, the fourth is simply supported (boundary conditions of type B):

$$w=0,\ \psi_x=0,\ \psi_y=0 \text{ at } y=0,\ y=b,\ x=0, \quad w=0,\ \frac{\partial \psi_x}{\partial x}=0,\ \psi_y=0 \text{ at } x=a. \tag{2-15}$$

iii) Two edges are clamped, the other two are simply supported (boundary conditions of type C):

$$w=0,\ \psi_x=0,\ \psi_y=0 \text{ at } x=0,\ x=a, \quad w=0,\ \psi_x=0,\ \frac{\partial \psi_y}{\partial y}=0 \text{ at } y=0,\ y=b. \tag{2-16}$$

iv) All edges are simply supported (boundary conditions of type D):

$$w=0,\ \frac{\partial \psi_x}{\partial x}=0,\ \psi_y=0 \text{ at } x=0,\ x=a, \quad w=0,\ \psi_x=0,\ \frac{\partial \psi_y}{\partial y}=0 \text{ at } y=0,\ y=b. \tag{2-17}$$

3. Solution method

To solve problems (2-4) and (2-13) with the corresponding boundary conditions, the spline-collocation, discrete-orthogonalization, and step-by-step search methods were used.

3A. ***Spline-approximation.*** We will search for the solution of Equation (2-4) by the Kirchhoff theory in the form

$$w = \sum_{i=0}^{N} w_i(x)\psi_i(y), \tag{3-1}$$

where $w_i(x)$ $(i = 1, \ldots, N)$ are unknown functions and $\psi_i(y)$ are the functions constructed using quintic B-splines $(N > 6)$.

The functions $\psi_i(y)$ are selected in order to satisfy the boundary conditions for $y = const$ using the linear combinations of B-splines

$$\begin{gathered}
\psi_0(y) = \alpha_{11} B_5^{-2}(y) + \alpha_{12} B_5^{-1}(y) + B_5^0(y),\\
\psi_1(y) = \alpha_{21} B_5^{-1}(y) + \alpha_{22} B_5^0(y) + B_5^1(y),\\
\psi_2(y) = \alpha_{31} B_5^{-2}(y) + \alpha_{32} B_5^0(y) + B_5^2(y),\\
\psi_i(y) = B_5^i(y),\ \ i = 3, 4, \ldots, N-3,\\
\psi_{N-2}(y) = \beta_{31} B_5^{N+2}(y) + \beta_{32} B_5^N(y) + B_5^{N+2}(y),\\
\psi_{N-1}(y) = \beta_{21} B_5^{N+1}(y) + \beta_{22} B_5^N(y) + B_5^{N-1}(y),\\
\psi_N(y) = \beta_{11} B_5^{N+2}(y) + \beta_{12} B_5^{N+1}(y) + B_5^N(y),
\end{gathered}$$

where $\psi_i(y) = B_5^i(y)$ $(i = -2, \ldots, N+2,$ i is the spline number) are splines constructed on a uniform mesh Δ with a spacing h_y: $y_{-5} < y_{-4} < \ldots < y_N < y_{N+5} < \ldots < y_{N+5}$, $y_0 = 0$, $y_N = b$,

$$B_5^i(y) = \frac{1}{120}\begin{cases}
0 & \text{at } -\infty < y < y_{i-3},\\
z^5 & \text{at } y_{i-3} \le y < x_{i-2},\\
-5z^5 + 5z^4 + 10z^3 + 10z^2 + 5z + 1 & \text{at } y_{i-2} \le y < y_{i-1},\\
10z^5 - 20z^4 - 20z^3 + 20z^2 + 50z + 26 & \text{at } y_{i-1} \le y < y_i,\\
-10z^5 + 30z^4 - 60z^2 + 66 & \text{at } y_i \le y < y_{i+1},\\
5z^5 - 20z^4 + 20z^3 + 20z^2 - 50z + 26 & \text{at } y_{i+1} \le y < y_{i+2},\\
(1-z)^5 & \text{at } y_{i+2} \le y < y_{i+3},\\
0 & \text{at } y_{i+3} \le y < \infty,
\end{cases}$$

where $z = (y - y_k)/h_y$ on the interval $[y_k, y_{k+1}]$, $k = \overline{i-3, i+2}$, $i = \overline{-3, N+2}$, $h_y = y_{k+1} - y_k = const$, α_{ij}, and β_{ij} $(i = 1, 2, 3,\ j = 1, 2)$ are constant coefficients that depend on the specified boundary conditions at $y = 0$ and $y = b$, respectively.

Let

$$A_\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\\ \alpha_{31} & \alpha_{32} \end{bmatrix},\ A_\beta = \begin{bmatrix} \beta_{11} & \beta_{12}\\ \beta_{21} & \beta_{22}\\ \beta_{31} & \beta_{32} \end{bmatrix}.$$

Then, for edges $y=0$, $y=b$ we have

$$A_\alpha = A_\beta = \begin{bmatrix} \frac{165}{4} & -\frac{33}{8} \\ 1 & -\frac{26}{33} \\ 1 & -\frac{1}{33} \end{bmatrix}$$

if the edges $y=0$ and $y=b$ are clamped,

$$A_\alpha = A_\beta = \begin{bmatrix} 12 & -3 \\ -1 & 0 \\ -1 & 0 \end{bmatrix}$$

if those edges are simply supported, and

$$A_\alpha = \begin{bmatrix} \frac{165}{4} & -\frac{33}{8} \\ 1 & -\frac{26}{33} \\ 1 & -\frac{1}{33} \end{bmatrix}, A_\beta = \begin{bmatrix} 12 & -3 \\ -1 & 0 \\ -1 & 0 \end{bmatrix}$$

if the edge $y=0$ is clamped and the edge $y=b$ is simply supported. Now we rewrite Equation (2-4) as

$$\frac{\partial^4 w}{\partial x^4} = a_1 \frac{\partial^3 w}{\partial x^3} + a_2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + a_3 \frac{\partial^3 w}{\partial x^2 \partial y} + a_4 \frac{\partial^2 w}{\partial x^2} + a_5 \frac{\partial^3 w}{\partial x \partial y^2} + a_6 \frac{\partial^2 w}{\partial x \partial y} + a_7 \frac{\partial^4 w}{\partial y^4} + a_8 \frac{\partial^3 w}{\partial y^3} + a_9 \frac{\partial^2 w}{\partial y^2} + a_{10} w, \quad \text{(3-2)}$$

where $a_i = a_i(x, y)$, $i = 1, 2, \ldots, 9$, $a_{10} = a_{10}(x, y, \omega)$.

Substituting (3-1) into (3-2), we will require that the equation should be satisfied at given collocation points $\xi_k \in [0, b]$ for $k = 0, N$. Let us consider the case where the number of mesh nodes is even, that is, $N = 2n+1$, $(n \geq 3)$, and the collocation nodes satisfy the conditions $\xi_{2i} \in [y_{2i}, y_{2i+1}]$, $\xi_{2i+1} \in [y_{2i}, y_{2i+1}]$ with $i = 0, 1, \ldots, n$. The interval $[y_{2i}, y_{2i+1}]$ has two collocation points and the adjacent intervals $[y_{2i+1}, y_{2i+2}]$ do not have such points. Within the intervals $[y_{2i+1}, y_{2i+2}]$, the collocation points are selected as follows: $\xi_{2i} = x_{2i} + z_1 h_y$, $\xi_{2i+1} = y_{2i} + z_2 h_y$ with $i = 0, 1, 2, \ldots, n$, where z_1 and z_2 are the roots of the quadratic Legendre polynomial, equal to $z_1 = 1/2 - \sqrt{3}/6$, $z_2 = 1/2 + \sqrt{3}/6$ on the interval $[0, 1]$. Such collocation points are optimal and the accuracy of the approximation substantially increases. As a result, we obtain a system of $N+1$ linear differential equations for w_i. If $\Psi_j = [\psi_i^{(j)}(\xi_k)]$ with $k, i = 0, \ldots, N$, $j = 0, \ldots, 4$, $\bar{w} = \{w_0, w_1, \ldots, w_N\}^T$, $\bar{a}_r^T = \{a_r(x, \xi_0), a_r(x, \xi_1), \ldots, a_r(x, \xi_N)\}$ for $r = 1, \ldots, 9$, $\bar{a}_{10}^T = \{a_{10}(x, \xi_0, \omega), a_{10}(x, \xi_1, \omega), \ldots, a_{10}(x, \xi_N, \omega)\}$, and $\bar{c} * A$ denotes the matrix $[c_i a_{ij}]$, where vector $\bar{c} = \{c_0, c_1, \ldots, c_N\}^T$ and $A = [a_{ij}]$ with $i, j = 0, \ldots, N$, then the system of differential equations becomes

$$\bar{w}^{IV} = \Psi_0^{-1}(\bar{a}_7 \Psi_4 + \bar{a}_8 \Psi_3 + \bar{a}_9 \Psi_2 + \bar{a}_{10} \Psi)\bar{w} + \Psi_0^{-1}(\bar{a}_5 \Psi_2 + \bar{a}_6 \Psi_1)\bar{w}' + \Psi_0^{-1}(\bar{a}_2 \Psi_3 + \bar{a}_3 \Psi_1 + \bar{a}_4 \Psi_0)\bar{w}'' + \Psi_0^{-1}(\bar{a}_1 \Psi_0)\bar{w}'''.$$

This system can be normalized:

$$\frac{d\bar{Y}}{dx} = A(x, \omega)\bar{Y} \quad (0 \leq x \leq a), \quad \text{(3-3)}$$

where

$$\bar{Y} = \left\{w_1, w_2, \ldots, w_{N+1}, w_1', w_2', \ldots, w_{N+1}', w_1'', w_2'', \ldots, w_{N+1}'', w_1''', w_2''', \ldots, w_{N+1}'''\right\}^T,$$

where $w_K^{(I)} = w^{(I)}(x, \xi_K)$ with $K = 1, \ldots, N+1$, $I = 1, 2, 3$ and $A(x, \omega)$ is the square matrix of order $(N+1) \times (N+1)$.

The boundary conditions for this system can be expressed as

$$B_1 \bar{Y}(0) = \bar{0}, \quad B_2 \bar{Y}(a) = \bar{0}. \tag{3-4}$$

To solve the problem using Mindlin's theory, we will present the resolving functions in the form:

$$w(x, y) = \sum_{i=0}^{N} w_i(x)\varphi_{1i}(y), \quad \psi_x = \sum_{i=0}^{N} \psi_{xi}(x)\varphi_{1i}(y), \quad \psi_y = \sum_{i=0}^{N} \psi_{yi}(x)\varphi_{2i}(y), \tag{3-5}$$

where w_i, ψ_{xi}, ψ_{yi} are the searched functions of the variable X and $\varphi_{ij}(y)$ with $j = 1, 2$, $i = 0, 1, \ldots, N$ are the linear combinations of B-splines on the uniform mesh Δ: $0 = y_0 < y_1 < \ldots < y_N = b$ with allowance for boundary conditions at $y = 0$ and $y = b$. The system (2-13) includes derivatives of the resolving functions with respect to the coordinate y not greater than the second order. Thus, approximation by spline functions of the third power can be employed, leading to

$$B_3^i(y) = \frac{1}{6}\begin{cases} 0 & \text{at } -\infty < y < y_{i-2}, \\ z^3 & \text{at } y_{i-2} \le y < y_{i-1}, \\ -3z^3 + 3z^2 + 3z + 1 & \text{at } y_{i-1} \le y < y_i, \\ 3z^3 - 6z^2 + 4 & \text{at } y_i \le y < y_{i+1}, \\ (1-z)^3 & \text{at } y_{i+1} \le y < y_{i+2}, \\ 0 & \text{at } y_{i+2} \le y < \infty, \end{cases} \tag{3-6}$$

where $z = (y - y_k)/h_y$ on the interval $[y_k, y_{k+1}]$, $k = \overline{i-2, i+1}$, $i = \overline{-1, N+1}$, $h_y = y_{k+1} - y_k = \text{const}$. In this case, the functions $\varphi_{ji}(y)$ are formed as follows:

i) If the resolving function is equal to zero, then

$$\varphi_{j0}(y) = -4B_3^{-1}(y) + B_3^0(y), \quad \varphi_{j1}(y) = B_3^{-1}(y) - \tfrac{1}{2}B_3^0(y) + B_3^1(y),$$
$$\varphi_{ji}(y) = B_3^i(y), \quad (i = 2, 3, \ldots, N-2). \tag{3-7}$$

ii) If the derivative of the resolving function with respect to y is zero, then

$$\varphi_{j0}(y) = B_3^0(y), \quad \varphi_{j1}(y) = B_3^{-1}(y) - \tfrac{1}{2}B_3^0(y) + B_3^1(y), \quad \varphi_{ji}(y) = B_3^i(y), (i = 2, 3, \ldots, N-2). \tag{3-8}$$

Similar formulas hold for the functions $\varphi_{j,N-1}(y)$ and $\varphi_{j,N}(y)$.

Functions $\varphi_{1i}(y)$ (to define $w(x, y)$ and $\psi_x(x, y)$), as applied to the boundary conditions at the plate edges $y = 0$, $y = b$ being considered in the present paper, were selected in accordance with relation (3-6), because for hinge supporting and clamping $w = \psi_x = 0$. In this case, the function $\varphi_{2i}(y)$ was chosen depending on the type of specified boundary conditions or in the form of linear combination of B-splines (3-6) or (3-7).

Substituting (3-5) into equations (2-13), we will require that they are satisfied at the prescribed collocation points $\xi_k \in [0, b]$ for $k = 0, N$. The selection of the collocation points $\xi_{2i} \in [y_{2i}, y_{2i+1}]$,

$\xi_{2i+1} \in [y_{2i}, y_{2i+1}]$, $(i = 0, 1, \ldots, n)$ in the form $\xi_{2i} = y_{2i} + z_1 h_y$, $\xi_{2i+1} = y_{2i} + z_2 h_y$ with $i = 0, 1, 2, \ldots, n$, where $z_1 = 1/2 - \sqrt{3}/6$ and $z_2 = 1/2 + \sqrt{3}/6$ are the roots of the second-order Legendre polynomials in the segment [0, 1], is optimal. Due to this selection the accuracy of the approximation increases essentially. As a result, we obtain the system of $6(N+1)$ linear differential equations with respect to functions w_i, $\tilde{w}_i$, ψ_{xi}, $\tilde{\psi}_{xi}$, ψ_{yi}, $\tilde{\psi}_{yi}$ with $i = 0, \ldots, N$. Having adopted the notations

$$
\begin{gathered}
\Phi_j = [\varphi_{ji}(\xi_k)], \quad k, i = 0, \ldots, N, \ j = 1, 2, \\
\bar{w} = \{w_0, w_1, \ldots, w_N\}^T, \ \bar{\tilde{w}} = \{\tilde{w}_0, \tilde{w}_1, \ldots, \tilde{w}_N\}^T, \\
\bar{\psi}_x = \{\psi_{x0}, \psi_{x1}, \ldots, \psi_{xN}\}^T, \ \bar{\tilde{\psi}}_x = \{\tilde{\psi}_{x0}, \tilde{\psi}_{x1}, \ldots, \tilde{\psi}_{xN}\}^T, \\
\bar{\psi}_y = \{\psi_{y0}, \psi_{y1}, \ldots, \psi_{yN}\}^T, \ \bar{\tilde{\psi}}_y = \{\tilde{\psi}_{y0}, \tilde{\psi}_{y1}, \ldots, \tilde{\psi}_{yN}\}^T, \\
\bar{a}_{kl}^T = \{a_{kl}(x, \xi_0), a_{kl}(x, \xi_1), \ldots, a_{kl}(x, \xi_N)\}, \\
(k, l) \in \{(k, l) \mid k = 1, 2, 3; l = 1, \ldots, 8\} \backslash \{(1, 1), (2, 2), (3, 5)\}, \\
\bar{a}_{11}^T = \{a_{11}(x, \xi_0, \omega), a_{11}(x, \xi_1, \omega), \ldots, a_{11}(x, \xi_N, \omega)\}, \\
\bar{a}_{22}^T = \{a_{22}(x, \xi_0, \omega), a_{22}(x, \xi_1, \omega), \ldots, a_{22}(x, \xi_N, \omega)\}, \\
\bar{a}_{35}^T = \{a_{35}(x, \xi_0, \omega), a_{35}(x, \xi_1, \omega), \ldots, a_{35}(x, \xi_N, \omega)\},
\end{gathered}
$$

as well as $\bar{c} * A = [c_i a_{ij}]$ for an $N \times N$ matrix $A = [a_{ij}]$ and a vector $\bar{c} = \{c_0, c_1, \ldots, c_N\}^T$, we can express the system of ordinary differential equations with respect to $\bar{w}$, $\bar{\tilde{w}}$, $\bar{\psi}_x$, $\bar{\tilde{\psi}}_x$, $\bar{\psi}_y$, $\bar{\tilde{\psi}}_y$ as

$$
\frac{d\bar{w}}{dx} = \bar{\tilde{w}}, \quad \frac{d\bar{\psi}_x}{dx} = \bar{\tilde{\psi}}_x, \quad \frac{d\bar{\psi}_y}{dy} = \bar{\tilde{\psi}}_y,
$$

$$
\begin{aligned}
\frac{d\bar{\tilde{w}}}{dx} &= \Phi_1^{-1}(\bar{a}_{11}\Phi_1 + \bar{a}_{12}\Phi_1' + \bar{a}_{13}\Phi_1'' + \bar{a}_{14}\Phi_1)\bar{w} + \Phi_1^{-1}(\bar{a}_{15}\Phi_1)\bar{\psi}_x \\
&\qquad + \Phi_1^{-1}(\bar{a}_{16}\Phi_1)\bar{\tilde{\psi}}_x + \Phi_1^{-1}(\bar{a}_{17}\Phi_2 + \bar{a}_{18}\Phi_2')\bar{\psi}_y, \\
\frac{d\bar{\tilde{\psi}}_x}{dx} &= \Phi_1^{-1}(\bar{a}_{21}\Phi_1)\bar{\tilde{w}} + \Phi_1^{-1}(\bar{a}_{22}\Phi_1 + \bar{a}_{23}\Phi_1' + \bar{a}_{24}\Phi_1'')\bar{\psi}_x + \Phi_1^{-1}(\bar{a}_{25}\Phi_1)\bar{\tilde{\psi}}_x \\
&\qquad + \Phi_1^{-1}(\bar{a}_{26}\Phi_2')\bar{\psi}_y + \Phi_1^{-1}(\bar{a}_{27}\Phi_2 + \bar{a}_{28}\Phi_2')\bar{\tilde{\psi}}_y, \\
\frac{d\bar{\tilde{\psi}}_y}{dy} &= \Phi_2^{-1}(\bar{a}_{31}\Phi_1')\bar{w} + \Phi_2^{-1}(\bar{a}_{32}\Phi_1')\bar{\psi}_x + \Phi_2^{-1}(\bar{a}_{33}\Phi_1 + \bar{a}_{34}\Phi_1')\bar{\tilde{\psi}}_x \\
&\qquad + \Phi_2^{-1}(\bar{a}_{35}\Phi_2 + \bar{a}_{36}\Phi_2' + \bar{a}_{37}\Phi_2'')\bar{\psi}_y + \Phi_2^{-1}(\bar{a}_{38}\Phi_2)\bar{\tilde{\psi}}_y,
\end{aligned}
$$

and can be written in the form

$$
\frac{d\bar{Y}}{dx} = A(x, \omega)\bar{Y}, \tag{3-9}
$$

where $\bar{Y} = \{w_0, \ldots w_N, \tilde{w}_0, \ldots, \tilde{w}_N, \psi_{x0}, \ldots, \psi_{xN}, \tilde{\psi}_{xo}, \ldots, \tilde{\psi}_{xN}, \psi_{y0}, \ldots, \psi_{yN}, \tilde{\psi}_{y0}, \ldots, \tilde{\psi}_{yN}\}^T$ is a vector function of x and $A(x, \omega)$ is a $6(N+1) \times 6(N+1)$ quadratic matrix. The boundary conditions for this system can be written as

$$
B_1\bar{Y}(0) = \bar{0}, \quad B_2\bar{Y}(a) = \bar{0}. \tag{3-10}
$$

To solve the eigenvalue problems for the systems of ordinary differential equations with variable coefficients (3-3), (3-4), (3-9), and (3-10), we will use the stable numerical method of discrete-orthogonalization coupled with the step-by-step search method, which makes it possible to obtain the desired solutions with a high degree of accuracy [Grigorenko et al. 1986]. Let us consider in detail the basic principles of the method of discrete orthogonalization.

3B. ***Method of discrete orthogonalization.*** Let us consider the linear system of differential equations in the Cauchy normal form

$$\frac{d\bar{Y}}{dx} = A(x, \omega)\bar{Y}(x), \quad 0 \le x \le a \tag{3-11}$$

with boundary conditions

$$B_1\bar{Y}(0) = \bar{0}, \tag{3-12}$$

$$B_2\bar{Y}(a) = \bar{0}, \tag{3-13}$$

where $\bar{Y} = \{y_1, y_2, \dots, y_n\}^T$ is the vector-column, $A(x, \omega)$ is the n-order quadratic matrix, and B_1 and B_2 are matrixes of the order $k \times n$ and $(n-k) \times n$ $(k < n)$, respectively.

The boundary-value problem (3-11)–(3-13) was solved by the step-by-step search method coupled with the method of discrete orthogonalization. At the fixed frequency ω the solution of the problem is reduced to the form

$$\bar{Y}(x) = \sum_{j=1}^{m} C_j \bar{Y}_j(x), \tag{3-14}$$

where $m = \min\{k, n-k\}$ (for the sake of definiteness $m = n-k$), $\bar{Y}_j$ are the solutions of the Cauchy problems for the system of equations (3-11) with initial conditions which satisfy the boundary condition (3-12) on the left end of the interval $[0, a]$, and m is the number of boundary conditions on the right end of the interval of integration.

Let us present the boundary conditions (3-12) at the point $x = 0$ in detail:

$$\begin{aligned}
b_{11}y_1 + b_{12}y_2 + \cdots + b_{1k}y_k + b_{1,k+1}y_{k+1} + \cdots + b_{1n}y_n &= 0,\\
b_{21}y_1 + b_{22}y_2 + \cdots + b_{2k}y_k + b_{2,k+1}y_{k+1} + \cdots + b_{2n}y_n &= 0,\\
&\vdots\\
b_{k1}y_1 + b_{k2}y_2 + \cdots + b_{kk}y_k + b_{k,k+1}y_{k+1} + \cdots + b_{kn}y_n &= 0.
\end{aligned} \tag{3-15}$$

Assuming that the coefficients of the first k columns in (3-15) form a nonsingular matrix, we transfer the rest of the columns to the right-hand side. Then conditions (3-15) take the form

$$\begin{aligned}
b_{11}y_1 + b_{12}y_2 + \cdots + b_{1k}y_k &= -b_{1,k+1}y_{k+1} - \cdots - b_{1n}y_n,\\
b_{21}y_1 + b_{22}y_2 + \cdots + b_{2k}y_k &= -b_{2,k+1}y_{k+1} - \cdots - b_{2n}y_n,\\
&\vdots\\
b_{k1}y_1 + b_{k2}y_2 + \cdots + b_{kk}y_k &= -b_{k,k+1}y_{k+1} - \cdots - b_{kn}y_n.
\end{aligned} \tag{3-16}$$

Then, setting the components y_{k+1}, y_{k+2},…, y_n equal to the columns of a unit matrix, we obtain the initial conditions for $\bar{Y}_j$ with $j = 1, 2, \dots, m$. The Cauchy problems with corresponding initial conditions can be solved by the Runge–Kutta method, for example. The numerical integration is performed in combination with the orthonormalization of vectors $\bar{Y}_j$ $(j = 1, 2, \dots, m)$ at a finite number of points on the interval of change of argument that provides a stable calculation process.

Let us divide the interval $[0, a]$ by integration points x_s with $s = 0, 1, \dots, N$ into small segments so that $x_0 = 0$ and $x_N = a$. Among these points we choose the points of orthogonalization X_i with $i = 0, 1, \dots, M$. The choice of points of orthogonalization depends only on the necessary accuracy of the problem solution. Assume that solutions to the Cauchy problems, which we will designate as $\bar{Y}_r(X_i)$ with $r = 1, 2, \dots, m$, have been obtained at the points X_i using some numerical method. We perform the orthonormalization of the vectors $\bar{Y}_r(X_i)$ at the points X_i and denote the resulting vectors by $\bar{Z}_r(X_i)$. We have

$$\bar{Z}_r = \frac{1}{w_{rr}}\Bigl(\bar{Y}_r - \sum_{j=1}^{r-1} w_{rj}\,\bar{Z}_j\Bigr), \qquad r = 1, 2, \dots, m, \tag{3-17}$$

where

$$w_{rj} = (\bar{Y}_r, \bar{Z}_j)\ (j < r), \qquad w_{rr} = \sqrt{(\bar{Y}_r, \bar{Z}_r) - \sum_{j=1}^{r-1} w_{rj}^2}.$$

According to (3-17), at $x = X_i$ we have

$$w_{11}\bar{Z}_1 = \bar{Y}_1, \quad w_{22}\bar{Z}_2 = \bar{Y}_2 - w_{21}\bar{Z}_1, \quad \dots \quad w_{mm}\bar{Z}_m = \bar{Y}_m - w_{m1}\bar{Z}_1 - w_{m2}\bar{Z}_2 - \dots - w_{m,m-1}\bar{Z}_{m-1}. \tag{3-18}$$

Having transformed (3-18), we obtain the matrix equality

$$\begin{bmatrix} \bar{Y}_1(X_i) \\ \bar{Y}_2(X_i) \\ \vdots \\ \bar{Y}_m(X_i) \end{bmatrix} = \Omega_i \begin{bmatrix} \bar{Z}_1(X_i) \\ \bar{Z}_2(X_i) \\ \vdots \\ \bar{Z}_m(X_i) \end{bmatrix}, \tag{3-19}$$

where

$$\Omega_i = \Omega(X_i) = \begin{bmatrix} w_{11}(X_i) & 0 & 0 & 0 & \dots & 0 \\ w_{21}(X_i) & w_{22}(X_i) & 0 & 0 & \dots & 0 \\ w_{31}(X_i) & w_{32}(X_i) & w_{33}(X_i) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ w_{m-1,1}(X_i) & w_{m-1,2}(X_i) & w_{m-1,3}(X_i) & w_{m-1,4}(X_i) & \dots & 0 \\ w_{m1}(X_i) & w_{m2}(X_i) & w_{m3}(X_i) & w_{m4}(X_i) & \dots & w_{mm}(X_i) \end{bmatrix}. \tag{3-20}$$

The vectors $\bar{Z}_r(X_i)$, with $r = 1, 2, \dots, m$, are the initial values of the Cauchy problems for the system of differential equations (3-11) on the interval $X_i \le x \le X_{i+1}$.

Solutions of the system (3-11), which satisfy the boundary conditions on the left end of the interval (3-12), can be written at each point X of discrete orthogonalization in the form of two expressions: prior to orthogonalization

$$\bar{Y}(X_i) = \sum_{j=1}^{m} C_j^{(i-1)} \bar{Y}_j(X_i), \tag{3-21}$$

and upon orthogonalization

$$\bar{Y}(X_i) = \sum_{j=1}^{m} C_j^{(i)} \bar{Z}_j(X_i). \tag{3-22}$$

We represent the solution of system (3-11) on the interval $X_i \le X \le X_{i+1}$ as

$$\bar{Y}(x) = \sum_{j=1}^{m} C_j^{(i)} \bar{Z}_j(x). \tag{3-23}$$

Having integrated the initial system of equations on the last interval $X_{M-1} \le X \le X_M$ and having performed the orthogonalization at point X_M by formula (3-17), we obtain

$$\bar{Y}(X_M) = \sum_{j=1}^{m} C_j^{(M)} \bar{Z}_j(X_M). \tag{3-24}$$

Satisfying the boundary conditions on the right end of the integration interval, that is, substituting equation (3-24) into (3-13), we obtain the uniform system of m linear algebraic equations relative to $C_j^{(M)}$ with $j = 1, 2, \dots, m$. In order for the nontrivial solution of the boundary-value problem (3-11) and (3-12) to exist, it is necessary and sufficient to set the determinant $D(\omega)$ of the system,

$$B_2 \sum_{j=1}^{m} C_j^{(M)} \bar{Z}_j(X_M) = 0, \tag{3-25}$$

equal to zero, that is,

$$D(\omega) = 0. \tag{3-26}$$

In this case, the determinant may be calculated, for example, by the Gauss method. Condition (3-26) is nonlinear with respect to the parameter ω. Because the solution of the boundary-value problem is a continuous function, the dependency $D(\omega)$ is also a continuous function. To solve the nonlinear equation (3-26), we can use, for example, Newton's method or the method of chords. However, these methods may be inefficient, if the initial approximation is chosen improperly. In this case, the use of the step-by-step method for searching the interval of the change in sign of function $D(\omega)$ would be more advantageous. Having determined the interval where the sign changes, we can find the frequency with the necessary accuracy using, for instance, the method of chords or a binary search. To determine the eigenmodes, it is necessary to define the approximate value of C_j^M using the largest minor of the matrix of the system of linear equations.

In what follows, the values of C_j^{i-1} may be determined by values C_j^i with $j = 1, 2, \dots, m$ beginning with $i = M$. To this end, we equate the right-hand sides of (3-22) and (3-23):

$$\sum_{j=1}^{m} C_j^{(i-1)} \bar{Y}_j(X_i) = \sum_{j=1}^{m} C_j^{(i-1)} \bar{Z}_j(X_i). \tag{3-27}$$

Substituting $\bar{Y}_j$ from (3-19) for $x = X_i$ we find

$$C_1^{(i-1)} w_{11}\bar{z}_1 + C_2^{(i-1)}(w_{21}\bar{z}_1 + w_{22}\bar{z}_2) + C_3^{(i-1)}(w_{31}\bar{z}_1 + w_{32}\bar{z}_2 + w_{33}\bar{z}_3) + \cdots$$
$$\ldots + C_m^{(i-1)}(w_{m1}\bar{z}_1 + w_{m2}\bar{z}_2 + \cdots + w_{mm}\bar{z}_m) = C_1^{(i)}\bar{z}_1 + C_2^{(i)}\bar{z}_2 + \cdots + C_m^{(i)}\bar{z}_m. \quad (3\text{-}28)$$

Equating of the coefficients for vectors $\bar{Z}_j$ for $j = 1, 2, \ldots, m+1$ in (3-28), we obtain

$$\Omega_i' \bar{C}^{(i-1)} = \bar{C}^{(i)}, \quad (i = 1, 2, \ldots, M)$$

or

$$\bar{C}^{(i-1)} = [\Omega_i']^{-1} \bar{C}^{(i)}, \quad (3\text{-}29)$$

where Ω_i' is the transposed matrix (3-20), $\bar{C}^{(i)}$ is the vector-column with components $C_1^{(i)}, C_2^{(i)}, \ldots, C_m^{(i)}$. So, using Equation (3-29), we may find $C_j^{(i)}$ at all points beginning with $i = M$. The eigenshapes of $\bar{Y}(X_i)$ can be obtained using the formula (3-23) as a solution of the boundary-value problem.

4. Analysis of results

Based on the proposed techniques, the natural vibrations of square and rectangular plates of varying thickness were investigated under different boundary conditions at the edges.

4A. ***Investigation of the natural vibrations of plates using Kirchhoff theory.*** Let's analyze the natural vibrations of a square plate whose thickness varies as $h(x) = [\alpha(6x^2 - 6x + 1) + 1]h_0$ (Figure 1).

The plate material is orthotropic glass-fabric-reinforced plastic with Young's moduli $E_1 = 4.76 \cdot 10^4$ MPa and $E_2 = 2.07 \cdot 10^4$ MPa, shear modulus $G_{12} = 0.531 \cdot 10^4$ MPa, and Poisson's ratios $\nu_1 = 0.149$, $\nu_2 = 0.0647$. The dimensionless frequencies $\bar{\omega} = \omega a^2(\rho h_0/D_0)^{1/2}$ with $D_0 = h_0^3/12 \cdot 10^4$ MPa of the clamped plate as determined by Kirchhoff theory with different numbers of collocation points ($N = 10, 12, 14, 16, 18, 20, 22$) differ a little (Table 1).

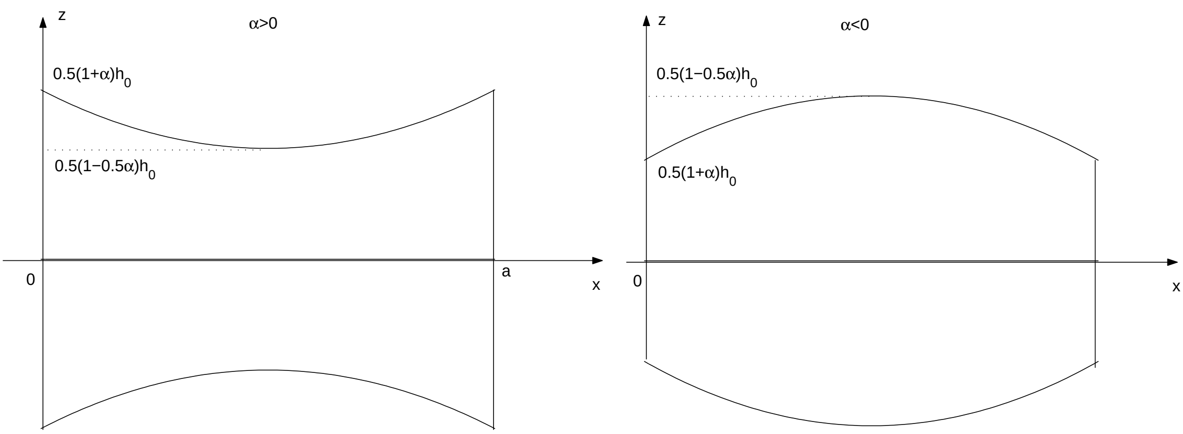


Figure 1. Plate cross-sections for $\alpha > 0$ (left) and $\alpha < 0$ (right).

Tables 2 and 3 collect the dimensionless frequencies ($i = 1, 2, 3$) (ordered by value) of an orthotropic square plate (Kirchhoff theory) for $\alpha \leq 0$ and $\alpha > 0$, respectively, under the boundary conditions (2-5)–(2-7). The number of collocation points is $N = 10$.

α	$\bar{\omega}_i$	N						
		10	12	14	16	18	20	22
0	$\bar{\omega}_1$	61.139	61.132	61.129	61.127	61.127	61.127	61.127
0	$\bar{\omega}_2$	107.188	107.066	107.016	106.994	106.982	106.976	106.972
0	$\bar{\omega}_3$	142.550	142.537	142.532	142.530	142.529	142.528	142.528
0.3	$\bar{\omega}_1$	62.108	62.102	62.099	62.099	62.098	62.098	62.098
0.3	$\bar{\omega}_2$	97.737	97.637	97.598	97.580	97.570	97.566	97.562
0.3	$\bar{\omega}_3$	145.289	145.276	145.271	145.269	145.268	145.268	145.267

Table 1. Values of the dimensionless frequency parameter $\bar{\omega} = \omega a^2 (\rho h_0 / D_0)^{1/2}$, $D_0 = (h_0^3/12) \cdot 10^4$ MPa for a clamped orthotropic square plate with different number of collocation points.

Boundary condition	$\bar{\omega}_i$	α					
		−0.5	−0.4	−0.3	0.2	−0.1	0
A	$\bar{\omega}_1$	58.375	59.012	59.605	60.159	60.674	61,139
	$\bar{\omega}_2$	121.526	119.010	116.240	113.311	110.281	107.188
	$\bar{\omega}_3$	124.339	129.656	134.030	137.585	140.405	142.550
B	$\bar{\omega}_1$	50.227	51.605	52.905	54.129	55.270	56.320
	$\bar{\omega}_2$	102.334	100.723	98.953	97.083	95.147	93.166
	$\bar{\omega}_3$	121.396	126.863	131.384	135.083	138.045	140.330
C	$\bar{\omega}_1$	52.733	52.211	51.624	50.995	50.337	49.659
	$\bar{\omega}_2$	109.151	112.403	110.490	107.334	104.080	100.783
	$\bar{\omega}_3$	116.222	113.496	114.497	116.613	117.777	118.403
D	$\bar{\omega}_1$	43.939	44.009	43.998	43.923	43.790	43.607
	$\bar{\omega}_2$	97.299	95.178	92.905	90.549	88.155	85.755
	$\bar{\omega}_3$	105.958	109.322	111.879	113.750	115.019	115.748
E	$\bar{\omega}_1$	48.701	47.412	46.059	44.676	43.289	41.918
	$\bar{\omega}_2$	95.664	97.177	97.994	98.235	97.985	96.671
	$\bar{\omega}_3$	113.313	110.306	107.049	103.676	100.170	97.306
F	$\bar{\omega}_1$	45.045	49.929	48.698	50.354	51.893	53.306
	$\bar{\omega}_2$	85.738	85.046	84.248	83.378	82.453	81.479
	$\bar{\omega}_3$	119.425	124.989	129.605	133.396	136.448	138.819

Table 2. Effect of thickness variation on the frequency parameter of the Kirchhoff square plate with different boundary conditions ($\alpha \leq 0$).

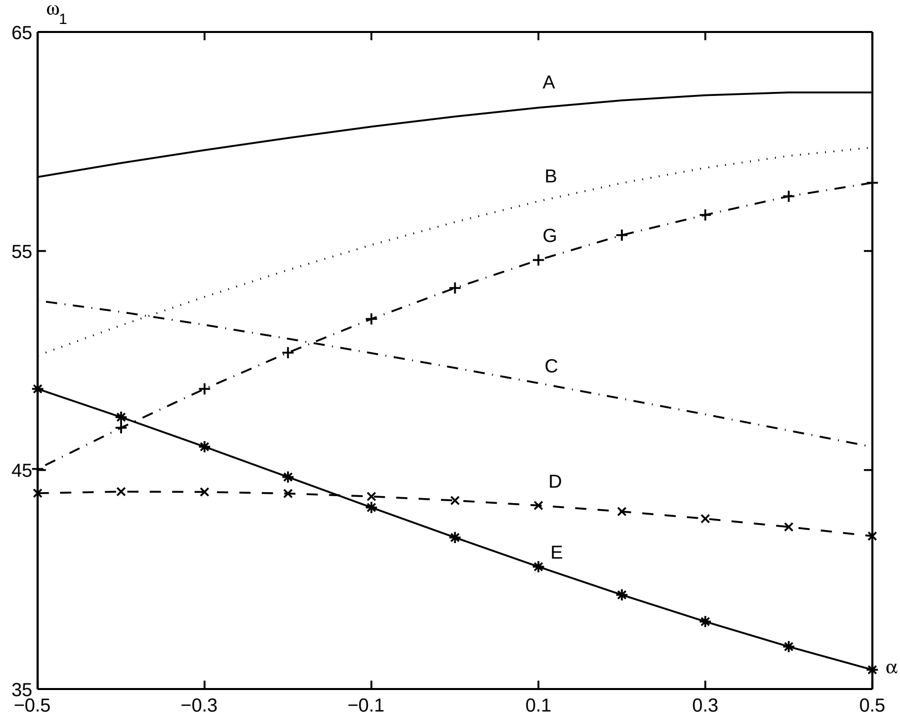
Boundary condition	$\bar{\omega}_i$	α					
		0	0.1	0.2	0.3	0.4	0.5
A	$\bar{\omega}_1$	61.139	61.542	61.871	62.108	62.237	62.238
	$\bar{\omega}_2$	107.188	104.058	100.904	97.737	94.557	91.364
	$\bar{\omega}_3$	142.550	144.062	144.968	145.289	145.035	144.212
B	$\bar{\omega}_1$	56.320	57.266	58.095	58.791	59.339	59.724
	$\bar{\omega}_2$	93.166	91.149	89.104	87.026	84.911	82.749
	$\bar{\omega}_3$	140.330	141.977	143.017	143.469	138.723	131.720
C	$\bar{\omega}_1$	49.659	48.965	48.257	47.535	46.801	46.05
	$\bar{\omega}_2$	100.783	97.477	94.187	90.928	87.713	84.546
	$\bar{\omega}_3$	118.403	118.535	118.208	117.477	116.269	114.688
D	$\bar{\omega}_1$	43.607	43.376	43.098	42.774	42.402	41.982
	$\bar{\omega}_2$	85.755	83.369	81.013	78.694	76.420	74.190
	$\bar{\omega}_3$	115.748	115.981	115.752	115.085	113.998	112.506
E	$\bar{\omega}_1$	41.918	40.583	39.297	38.077	36.933	35.878
	$\bar{\omega}_2$	96.671	93.185	89.743	86.364	83.065	79.858
	$\bar{\omega}_3$	97.306	96.249	94.852	93.149	91.160	88.934
G	$\bar{\omega}_1$	53.306	54.586	55.718	56.642	57.495	58.109
	$\bar{\omega}_2$	81.479	80.460	79.389	78.258	77.055	75.765
	$\bar{\omega}_3$	138.819	140.291	134.815	129.283	123.717	118.135

Table 3. Effect of thickness variation on the frequency parameter of square Kirchhoff plates with different boundary conditions ($\alpha > 0$).

Figure 2 on the next two pages shows the dimensionless vibration frequencies $\bar{\omega}_i$ of a square orthotropic plate with different boundary conditions as a function of the parameter α. The maximum and minimum frequencies $\bar{\omega}_2$ and $\bar{\omega}_3$ correspond to a reorganization of the vibration modes. The frequency of the clamped plate is maximal among all frequencies computed for different boundary conditions and values of α. The first frequency for the boundary conditions of type D changes weakly in comparison with other boundary conditions. The modes of the natural vibrations of a plate with boundary conditions of type G are presented in Figure 3.

4B. ***Studying the natural vibrations of plates based on Mindlin's theory.*** To evaluate the accuracy of the technique proposed, we analyze the dimensionless vibration frequencies $\bar{\omega} = (\omega a^2/\pi^2)\sqrt{\rho h_0/D}$ of an isotropic square constant-thickness plate, collected in Table 4, for the edge length a, $\nu = 0.3$, $h_0/a = 0.1$. Here the results calculated by Mindlin's and the three-dimensional theories [Liew and Teo 1999] using the various approaches of determining the frequencies as applied to the simply supported edges $y = 0$, $y = a$ are presented. In the case of Mindlin's theory, we used the proposed approach with 16 collocation points. The solution is searched for by application of the trigonometric functions

$$w = \hat{w}(x)\sin\frac{m\pi y}{a}, \qquad \psi_x = \hat{\psi}_x(x)\sin\frac{m\pi y}{a}, \qquad \psi_y = \hat{\psi}_y(x)\cos\frac{m\pi y}{a}, \tag{4-1}$$



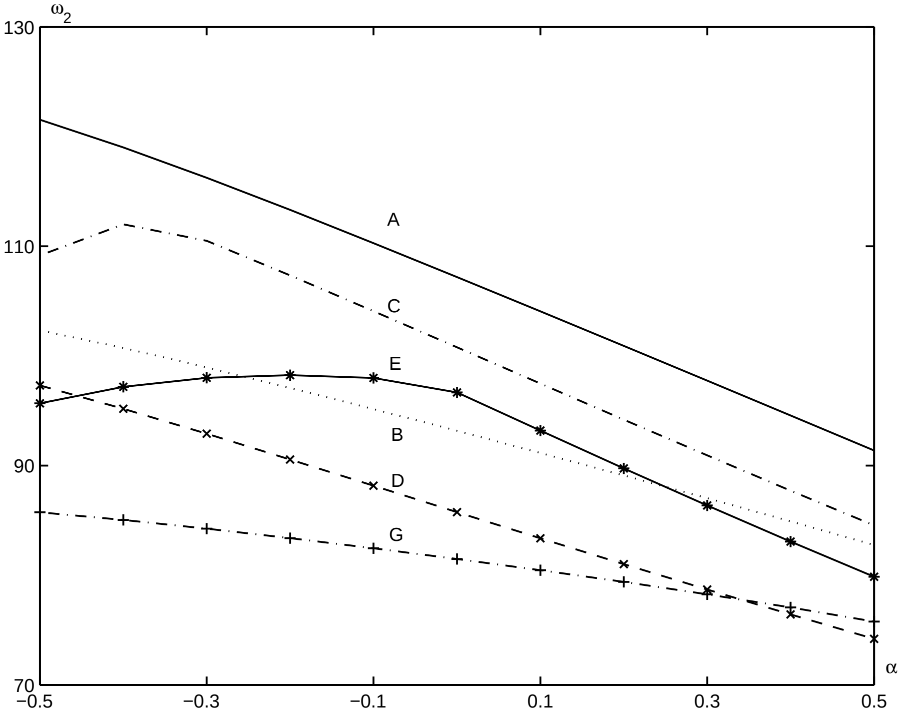


Figure 2. Dimensionless vibration frequencies $\bar{\omega}_1$ (top) and $\bar{\omega}_2$ (bottom) for the Kirchhoff square orthotropic plate with different boundary conditions as a function of α. The corresponding graphs for $\bar{\omega}_3$ are shown on the next page.

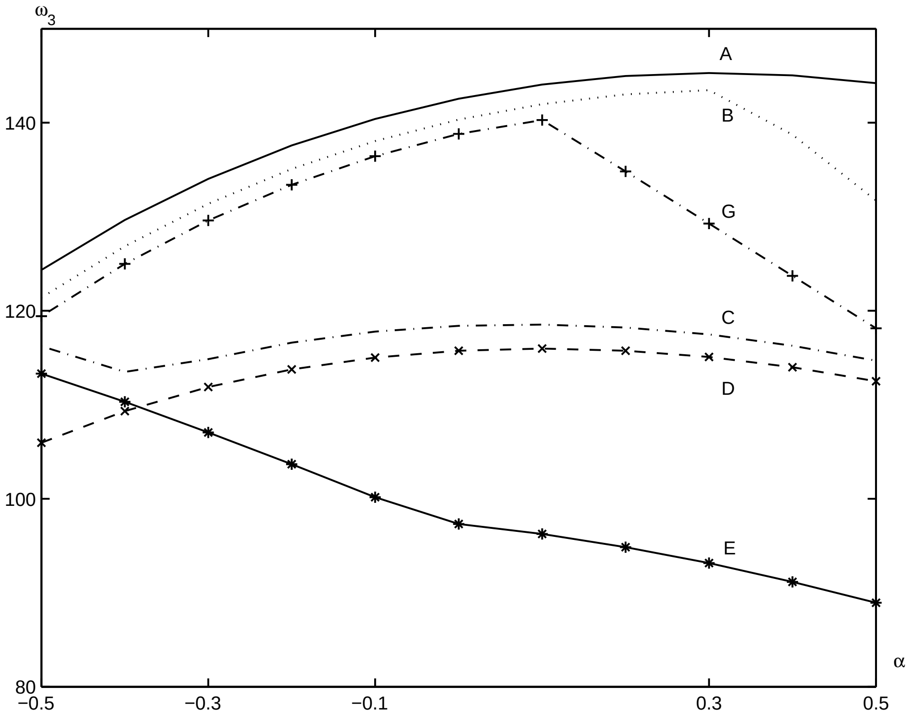


Figure 2, continued.

followed by the discrete-orthogonalization method. Application of the proposed technique yields small differences between the second and third frequencies corresponding to one half-wave in one coordinate direction and two half-waves in the other. Mindlin's theory gives lower frequencies in comparison with those obtained by the spatial theory.

Based on Mindlin's theory and the proposed technique, we studied the spectrum of natural vibrations of a square ($0 \le x \le a$, $0 \le y \le a$) orthotropic plate (the material is a glass-fabric-reinforced composite with Young's moduli $E_1 = 4.76 \cdot 10^4$ MPa and $E_2 = 2.07 \cdot 10^4$ MPa, shear modulus $G_{12} = 0.531 \cdot 10^4$ MPa, $G_{13} = 0.501 \cdot 10^4$ MPa, $G_{23} = 0.434 \cdot 10^4$ MPa and Poisson's ratios $\nu_1 = 0.149$, $\nu_2 = 0.0647$) for a

$\bar{\omega}$	Three-dimensional theory [Bhat et al. 1990]	Mindlin's theory	
		Spline-collocation method	By expansion (4-1)
$\bar{\omega}_1$	1.9342	1.9320	1.9320
$\bar{\omega}_2$	4.6222	4.6073	4.6073
$\bar{\omega}_3$	4.6222	4.6074	4.6073
$\bar{\omega}_4$	7.1030	7.0818	7.0717
$\bar{\omega}_5$	8.6617	8.6153	8.6153

Table 4. Comparison of frequency parameters calculated for the isotropic square plate of constant thickness with simply supported edges using different theories and methods.

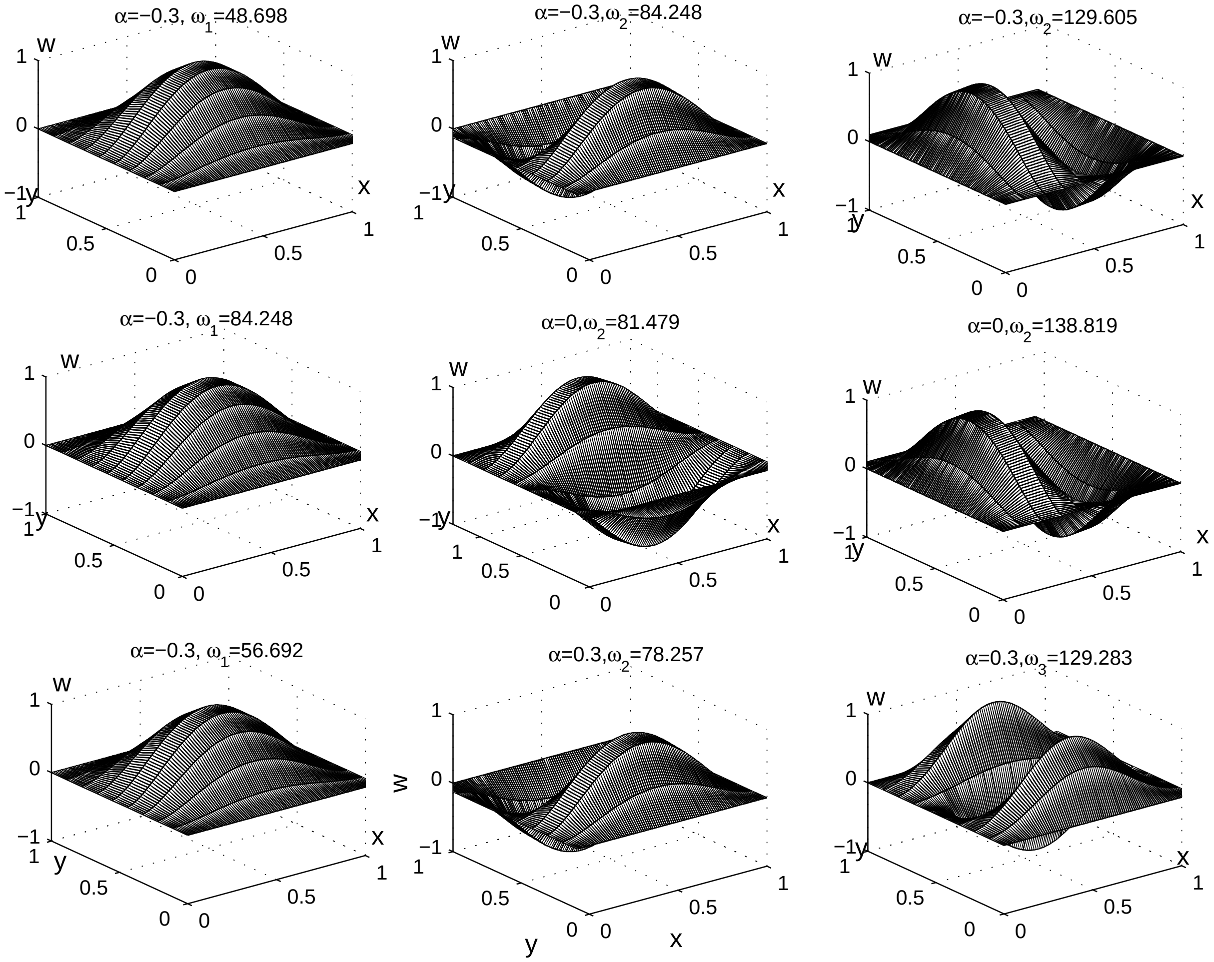


Figure 3. Natural vibration modes of the Kirchhoff orthotropic plate with boundary conditions of type G.

thickness given by the law

$$h(x) = \left[\alpha\left(6\frac{x^2}{a^2} - 6\frac{x}{a} + 1\right) + 1\right]h_0. \tag{4-2}$$

In this case the weight of the plate is independent of α.

To evaluate the accuracy of the proposed technique as applied to plates of varying thickness, we will consider the plate with the boundary conditions (2-7). In this case the solution of the problem may be presented in the form of Equations (4-1). Table 5 collects the values of dimensionless vibration frequencies $\bar{\omega} = \omega a_0^2(\rho h_0/D_0)^{1/2}$ ($D_0 = h_0^3 \cdot 10^4$ MPa, $h_0 = a_0 = 1$ m) for an orthotropic plate with $h_0/a = 0.1$, $\alpha = 0.4$ with and without the use of splines for different numbers of collocation points. It should be noted that the convergence of the method is faster for modes with one half-wave in the OY-direction ($\bar{\omega}_1$, $\bar{\omega}_3$). To reach the desired accuracy, if the number of half-waves increases ($\bar{\omega}_2$, $\bar{\omega}_5$ are two half-waves, $\bar{\omega}_4$ is three half-waves), it is necessary to increase the number of collocation points.

$\bar{\omega}$	Without splines	Using splines					
		N=9	N=11	N=13	N=15	N=17	N=19
$\bar{\omega}_1$	0.1469	0.1469	0.1469	0.1469	0.1469	0.1469	0.1469
$\bar{\omega}_2$	0.2039	0.2065	0.2052	0.2046	0.2043	0.2041	0.2040
$\bar{\omega}_3$	0.2959	0.2959	0.2959	0.2959	0.2959	0.2959	0.2959
$\bar{\omega}_4$	0.3218	0.3356	0.3286	0.3255	0.3240	0.3232	0.3227
$\bar{\omega}_5$	0.3405	0.3419	0.3411	0.3408	0.4307	0.3406	0.3405

Table 5. Comparison of frequency parameters calculated for the isotropic square plate of constant thickness with and without splines for various numbers of collocation points.

Table 6 presents the three first vibration frequencies of a square orthotropic plate ($h/a = 0.1$) for the above-mentioned stiffness coefficients and the law of thickness distribution (4-2) for different values of the parameter α and boundary conditions (2-6)–(2-8) at the ends. The results were obtained from Mindlin's theory. The frequency of plates with all edges being clamped is largest for the boundary conditions under consideration and different values of α. The computations we carried out make it possible to analyze the effect of variance in thickness (with the weight of the plate being constant) and the type of boundary conditions on the distribution of dynamic characteristics of the orthotropic plate within the framework of the applied theory.

Table 7 collects vibration frequencies $\bar{\omega} = a^2\omega\sqrt{\rho h_0/D_0}$, $D_0 = h_0^3 \cdot 10^4$ MPa for a square plate, calculated by Kirchhoff's theory and Mindlin's theory. The plate made from an orthotropic material with the parameters mentioned above and thickness varying according to (4-2).

Boundary condition	$\bar{\omega}_i$	α				
		-0.2	-0.1	0	0.1	0.2
A	$\bar{\omega}_1$	0.1304	0.1393	0.1482	0.1572	0.1665
	$\bar{\omega}_2$	0.2422	0.2445	0.2469	0.2496	0.2525
	$\bar{\omega}_3$	0.2798	0.2855	0.2907	0.2956	0.3002
B	$\bar{\omega}_1$	0.1112	0.1192	0.1275	0.1360	0.1449
	$\bar{\omega}_2$	0.2327	0.2344	0.2364	0.2386	0.2412
	$\bar{\omega}_3$	0.2561	0.2607	0.2651	0.2693	0.2753
C	$\bar{\omega}_1$	0.1200	0.1245	0.1289	0.1333	0.1379
	$\bar{\omega}_2$	0.2016	0.2020	0.2024	0.2029	0.2034
	$\bar{\omega}_3$	0.2718	0.2770	0.2817	0.2859	0.2896
D	$\bar{\omega}_1$	0.0847	0,0841	0.0836	0.0829	0.0823
	$\bar{\omega}_2$	0.1826	0.1807	0.1787	0.1768	0.1750
	$\bar{\omega}_3$	0.2266	0.2273	0.2271	0.2264	0.2249

Table 6. Effect of thickness variation on the frequency parameter of square Mindlin plates with different boundary conditions.

Kirchhoff's theory	Mindlin's theory						
	$a=10h_0$	$a=15h_0$	$a=20h_0$	$a=25h_0$	$a=30h_0$	$a=35h_0$	$a=40h_0$
$\alpha=0$							
17.943	14.819	16.308	16.976	17.388	17.577	17.754	17.920
$\alpha=0.2$							
18.834	15.430	17.550	17.840	18.250	18.630	18.811	18.865

Table 7. Comparison of frequency parameter $\bar{\omega}=a^2\omega\sqrt{\rho h_0/D_0}$, ($D_0=h_0^3\cdot 10^4$ MPa) calculated for square clamped orthotropic plate by Kirchhoff's and Mindlin's theories.

Table 8 collects the first frequencies $\bar{\omega}=a_0^2\omega\sqrt{\rho h_0/D_0}$, ($D_0=h_0^3\cdot 10^4$ MPa) for the orthotropic rectangular plate with the stiffness parameters identical to those already mentioned, and edges $a=a_0\cdot\beta$, $b=a_0/\beta$, $h_0/a_0=0.05$ for different boundary conditions (2-6)–(2-8) at the plate ends. (For this selection of geometrical parameters of the weight of the plate is independent of edge length.)

	β	α						
		-0.3	-0.2	-0.1	0	0.1	0.2	0.3
A	1	0.03987	0.04075	0.04167	0.04261	0.04359	0.04461	0.04565
	2	0.09002	0.08888	0.08719	0.08513	0.08281	0.08026	0.07754
	1/2	0.10105	0.10637	0.11163	0.11686	0.12204	0.12719	0.13234
	5/4	0.04426	0.04396	0.04366	0.04338	0.04311	0.04286	0.04253
	4/5	0.04783	0.04999	0.05219	0.05439	0.05659	0.05880	0.06102
B	1	0.03530	0.03436	0.03541	0.03547	0.03553	0.03559	0.03566
	2	0.08769	0.08765	0.08664	0.08486	0.08266	0.08018	0.07749
	1/2	0.08189	0.08398	0.08588	0.08758	0.08909	0.09040	0.09150
	5/4	0.04223	0.04164	0.04105	0.04045	0.04987	0.03930	0.03876
	4/5	0.03932	0.04003	0.04071	0.04133	0.04190	0.04241	0.04288
C	1	0.03262	0.03407	0.03552	0.03695	0.03842	0.03987	0,04133
	2	0.04571	0.04483	0.04385	0.04279	0.04169	0.04053	0.03933
	1/2	0.10077	0.10061	0.11141	0.11660	0.12185	0.12701	0.13217
	5/4	0.02787	0.02852	0.02908	0.02966	0.03026	0.03087	0.3150
	4/5	0.04525	0.04766	0.05005	0.05244	0.05480	0.05717	0.05954
D	1	0.02238	0.02220	0.02200	0.02179	0.02158	0.02140	0.02125
	2	0.04443	0.04367	0.04279	0.04185	0.04083	0.03975	0.03861
	1/2	0.06379	0.06296	0.06188	0.06057	0.05906	0.05738	0.05555
	5/4	0.02151	0.02162	0.02166	0.02200	0.02165	0.02158	0.02148
	4/5	0.02840	0.02811	0.02774	0.02732	0.02687	0.02639	0.02491

Table 8. Effect of thickness variation on the frequency parameter $\bar{\omega}=a^2\omega\sqrt{\rho h_0/D_0}$ ($D_0=h_0^3\cdot 10^4$ MPa, $h_0/a_0=0.05$, $a=a_0\cdot\beta$, $b=a_0/\beta$) for the orthotropic rectangular plate with different boundary conditions.

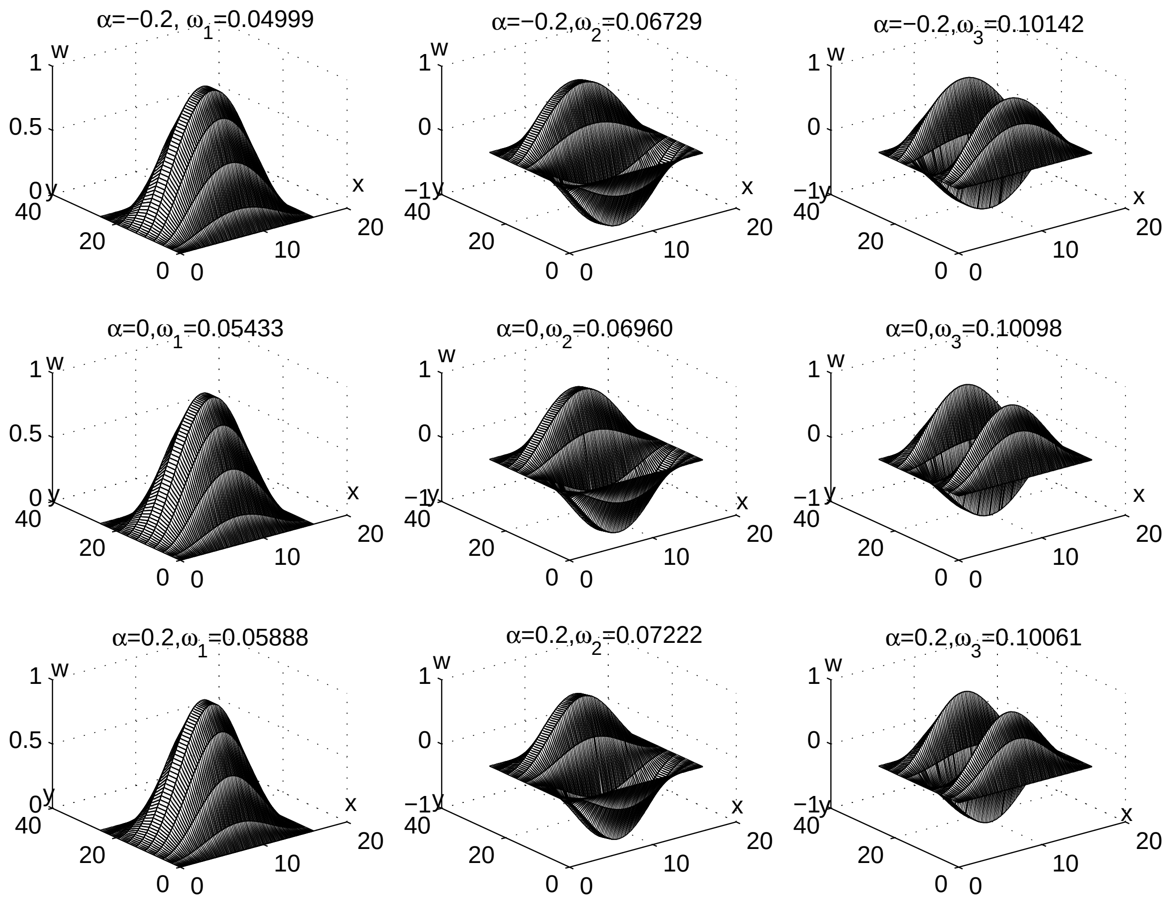


Figure 4. Natural vibration modes of a Mindlin orthotropic rectangular plate with $\beta = 0.8$, $h_0/a_0 = 0.05$ and boundary conditions of type A.

Figure 4 shows forms of deflection for vibrations of a rectangular ($\beta = 0.8$) orthotropic plate with clamped edges and various values of the parameter α. The corresponding graphs for a square plane with $\alpha = 0.2$ are shown in Figure 5. Note that for the third mode the number of half-waves changes along two coordinate directions in the case of square ($m = 2$, $n = 1$) and rectangular ($m = 1$, $n = 3$) plates with insignificant variation of the plate geometry.

5. Conclusions

The paper proposes a spline-collocation approach to study the natural vibrations of orthotropic rectangular variable-thickness plates within the framework of classical (of Kirchhoff's type) and refined (of Timoshenko-Mindlin's type) theory of plates. The approach includes two stages. At the first stage an initial eigenvalue problem for the systems of partial differential equations is reduced to an eigenvalue problem for the system of high-order ordinary differential equations by representing the desired solution in the form of truncated series of spline-collocations and choosing collocation points for the domain under consideration. Application of the spline-approximation makes it possible to satisfy boundary conditions

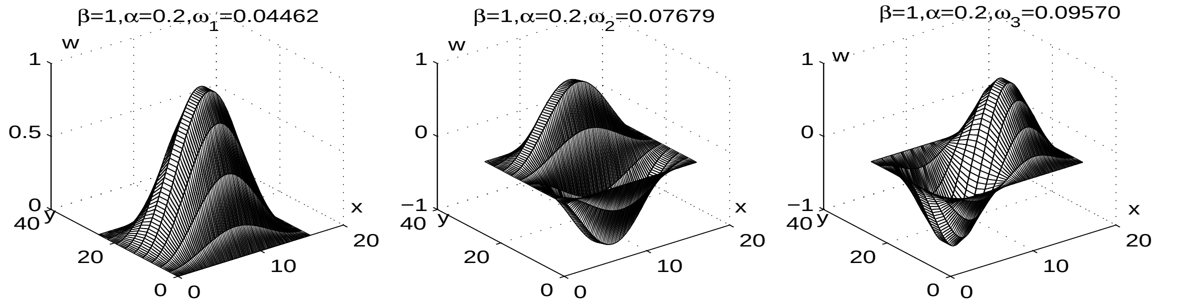


Figure 5. Natural vibration modes of a Mindlin orthotropic square plate with $\beta = 0.8$, $h_0/a_0 = 0.05$ and boundary conditions of type A.

at the plate edges exactly. The one-dimensional eigenvalue problems obtained are solved by the stable numerical discrete-orthogonalization method in combination with the step-by-step search method that provides a highly accurate solution. The reliability of the results obtained is estimated. Some new problems for natural vibrations of rectangular plates with varying thickness under different boundary conditions at plate edges are solved. The dynamic response of the plate is studied within the framework of different plate theories depending on the law of thickness variation.

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