Journal of
Mechanics of
Materials and Structures

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Volume 3, № 6

June 2008

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 3, No. 6, 2008

EXPLOITATION OF THE DISSIPATION INEQUALITY, IF SOME BALANCES ARE MISSING

WOLFGANG MUSCHIK, VITA TRIANI AND CHRISTINA PAPENFUSS

The balance equations of continuum thermodynamics need constitutive equations in order to solve them under the constraint that the entropy production appearing in the entropy balance equation must be not negative. This dissipation inequality represents the second law of thermodynamics. There are two procedures which exploit the dissipation inequality to obtain constitutive equations which are in agreement with the second law: the Coleman–Noll and the Liu techniques. Here we use the Liu technique in the special case in which not all balance equations are taken into account when exploiting the dissipation inequality. This case is of interest because often not all balances are known, or only the energy balance is considered. It is also proved that in this abridged exploitation of the dissipation inequality, thermodynamic restrictions for the constitutive equations are obtained, so that these satisfy the second law. These restrictions represent a smaller class of materials than that obtained when all balances are taken into account.

1. Introduction

The balance equations of continuum thermodynamics have the shape

$$\dot{\mathbf{X}} + \nabla \cdot \mathbf{Y} = \mathbf{R}. \tag{1}$$

The fields $\mathbf{X}(\mathbf{x}, t)$, $\mathbf{Y}(\mathbf{x}, t)$, and $\mathbf{R}(\mathbf{x}, t)$ can be divided into three classes [Muschik et al. 2001]: They may be wanted (or basic) fields, they may be constitutive equations which are defined on the chosen state space $\mathbf{Z}$ spanned by the fields of the state space variables $\mathbf{z}(\mathbf{x}, t)$,

$$\mathbf{z}(\mathbf{x}, t) \in \mathbf{Z}, \qquad \mathbf{X}(\mathbf{z}(\mathbf{x}, t)), \qquad \mathbf{Y}(\mathbf{z}(\mathbf{x}, t)), \qquad \mathbf{R}(\mathbf{z}(\mathbf{x}, t)), \tag{2}$$

and they may be external given fields $\mathbf{X}(\mathbf{x}, t)$, $\mathbf{Y}(\mathbf{x}, t)$, $\mathbf{R}(\mathbf{x}, t)$.

A special balance is that of the local entropy density $s(\mathbf{x}, t)$

$$\sigma_s = \dot{s} + \nabla \cdot \mathbf{J}_s - r_s \geq 0. \tag{3}$$

This inequality represents the second law of continuum thermodynamics, and is called the dissipation inequality. The fields of the entropy production density $\sigma_s(\mathbf{Z})$, of the entropy flux density $\mathbf{J}_s(\mathbf{Z})$, and of the entropy supply density $r_s(\mathbf{Z})$ are constitutive equations. Consequently, according to Equation (2), these fields depend on the state space variables $\mathbf{z}(\mathbf{x}, t)$, and consequently depend indirectly on space-time.

Keywords: dissipation inequality, reduced set of balances, abridged exploitation of the dissipation inequality, second law exploitation, Coleman–Noll procedure, Liu procedure.

The paper was prepared during Triani's stay at the Institut für Theoretische Physik, TU Berlin, and was partly delivered by Muschik at TRECOP'07, Trends in Continuum Physics, Sept. 16–20, 2007, Lviv/Briukhovichi, Ukraina.

Performing the derivatives in Equation (1), we obtain terms which are linear in the so-called higher derivatives

$$\mathbf{y} := (\dot{\mathbf{z}}, \nabla \mathbf{z}), \tag{4}$$

which are outside the state space, and we obtain other terms from $\mathbf{R}$ in (1) which are independent of the higher derivatives. Consequently, after having performed the derivatives in (1) and (3), we obtain the so-called balances on the state space [Muschik 1990] and the dissipation inequality, which both have the shape of an algebraic system which is linear in the higher derivatives, (4), and are given as

$$\mathbf{A} \cdot \mathbf{y} = \mathbf{C}, \qquad \mathbf{B} \cdot \mathbf{y} \geq D. \tag{5}$$

Here $\mathbf{A}$ and $\mathbf{B}$ are constitutive equations $\mathbf{A}(\mathbf{z}(\mathbf{x}, t))$, $\mathbf{B}(\mathbf{z}(\mathbf{x}, t))$, and $\mathbf{C}$ and D also consist of constitutive equations, or are given external fields $\mathbf{C}(\mathbf{x}, t)$, $D(\mathbf{x}, t)$).

We now have to exploit the dissipation inequality, $(5)_2$. For this reason, we use the second law in the Coleman–Mizel formulation [1964] (see also [Muschik et al. 2001]):

$$\{\wedge \mathbf{y} \mid \mathbf{A} \cdot \mathbf{y} = \mathbf{C}\} \longrightarrow \mathbf{B} \cdot \mathbf{y} \geq D, \tag{6}$$

This can be proved by an amendment of the second law, “Except in equilibrium subspace, no reversible process directions exist” [Muschik and Ehrentraut 1996], and it shows the material selectivity of the second law; $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and D are not arbitrary, but have to satisfy the second law, (6).

Two celebrated procedures for the exploitation of the second law (in Coleman–Mizel formulation) are the Coleman–Noll procedure [Truesdell and Noll 1965] and the Liu technique [Liu 1972; Muschik and Ehrentraut 1996]. These procedures are different from a mathematical point of view, but are equivalent if all balance equations are taken into account in both procedures [Triani et al. 2008]. Using the Liu procedure, by which the higher derivatives $\mathbf{y}$ are removed from the balances $(5)_1$ and from the dissipation inequality $(5)_2$, we obtain

$$\Lambda \cdot \mathbf{A} = \mathbf{B}, \qquad \Lambda \cdot \mathbf{C} \geq D. \tag{7}$$

The so-called Lagrange parameters Λ are functions on the chosen state space (2).

If the matrix $\mathbf{A}$ has maximal rank, there exists a right-hand reciprocal of $\mathbf{A}$,

$$\mathbf{A} \cdot \bar{\mathbf{A}} = \mathbf{1}, \tag{8}$$

and we obtain from (7) the Lagrange parameters

$$\Lambda = \mathbf{B} \cdot \bar{\mathbf{A}}. \tag{9}$$

Inserting this into $(7)_2$, we obtain the constraints on the constitutive equations in form of an inequality,

$$\mathbf{B} \cdot \bar{\mathbf{A}} \cdot \mathbf{C} \geq D. \tag{10}$$

We now ask the question of what happens when we make an abridged exploitation of the dissipation inequality by not taking all balance equations into account, a procedure which is often performed [Ericksen 1991] (this occurs because people forget some balance equations or do not know all balances). In this paper we prove that an abridged exploitation of the second law restricts the class of materials [Muschik 1990], but this restricted class does satisfy the second law. This means that neglecting balance equations in the exploitation procedure does not result in mistakes with respect to the second law, but the found class of materials is too small with respect to that class derived by taking all balances into account.

2. The general case

We start out with the balances on the state space $(5)_1$ and with the corresponding dissipation inequality $(5)_2$, both in matrix formulation. Because there are more higher derivatives $\mathbf{y}$ than balance equations the matrix $\mathbf{A}$ is singular, which means there exists a set of $\mathbf{y}^0$ spanning the nonvanishing kernel of $\mathbf{A}$,

$$\mathbf{y}^0 \in \ker \mathbf{A} \leftrightarrow \mathbf{A} \cdot \mathbf{y}^0 = \mathbf{0}, \qquad \dim(\ker \mathbf{A}) < n. \tag{11}$$

We now introduce a projector $\mathbf{P}$ as

$$\ker \mathbf{P} \neq \varnothing, \qquad \mathbf{P} \cdot \hat{\mathbf{C}} = \mathbf{0}, \qquad \hat{\mathbf{C}} \neq \mathbf{0}, \tag{12}$$

which reduces the number of balances which are taken into account, $\mathbf{P} \cdot \mathbf{A} \cdot \mathbf{y} = \mathbf{P} \cdot \mathbf{C}$. The projected balances have other solutions $\mathbf{Y}$ than $(5)_1$,

$$\mathbf{P} \cdot \mathbf{A} \cdot \mathbf{Y} = \mathbf{P} \cdot \mathbf{C}. \tag{13}$$

The dissipation inequality $(5)_2$ transforms into another inequality,

$$\mathbf{B}_P \cdot \mathbf{Y} \geq D_P, \tag{14}$$

which belongs to the projected balances (13). First of all, the connection between ($\mathbf{B}$, D) and ($\mathbf{B}_P$, D_P) remains open. The reduced system of balances on the state space (13) and of the dissipation inequality (14) now replaces the original ones, $(5)_1$ and $(5)_2$. The consequences of this replacement are investigated in this paper.

Next we ask the question, if $\hat{\mathbf{C}}$ in (12) is a solution of $(5)_1$, then

$$\mathbf{A} \cdot \hat{\mathbf{y}} = \hat{\mathbf{C}}. \tag{15}$$

According to (11), all solutions of this set of balance equations can be written down in the form

$$\hat{\mathbf{y}} = \bar{\mathbf{A}} \cdot \hat{\mathbf{C}} + \mathbf{y}^0, \tag{16}$$

where $\bar{\mathbf{A}}$ is the existing right-hand reciprocal (8) of $\mathbf{A}$ (because $\mathbf{A}$ has less rows than columns and is presupposed to have maximal rank). Introducing (16) into (15) results, by using (11), in

$$\mathbf{A} \cdot \bar{\mathbf{A}} \cdot \hat{\mathbf{C}} = \hat{\mathbf{C}}.$$

According to (8), this shows that (16) is a solution of (15). Because of $(12)_3$, we state that for all solutions of (15)

$$\hat{\mathbf{y}} \notin \ker \mathbf{A}, \tag{17}$$

is valid.

Applying the projector to (15) and using $(12)_2$, we obtain $\mathbf{P} \cdot \mathbf{A} \cdot \hat{\mathbf{y}} = \mathbf{P} \cdot \hat{\mathbf{C}} = \mathbf{0}$, which means that

$$\hat{\mathbf{y}} \in \ker(\mathbf{P} \cdot \mathbf{A}). \tag{18}$$

Because of

$$\mathbf{A}\cdot\mathbf{y}^0=\mathbf{0} \quad \to \quad \mathbf{P}\cdot\mathbf{A}\cdot\mathbf{y}^0=\mathbf{0}, \tag{19}$$

we obtain

$$\mathbf{y}^0\in \ker \mathbf{A} \quad \to \quad \mathbf{y}^0\in\ker(\mathbf{P}\cdot\mathbf{A}).$$

From (18) and (17), it follows that there are $\hat{\mathbf{y}}$ with the property

$$\hat{\mathbf{y}}\notin\ker\mathbf{A} \quad \leftarrow \quad \hat{\mathbf{y}}\in\ker(\mathbf{P}\cdot\mathbf{A}).$$

Consequently, we obtain for the dimension of the kernels

$$\dim(\ker(\mathbf{P}\cdot\mathbf{A}))>\dim(\ker\mathbf{A}). \tag{20}$$

This means that if the balance equations are neglected the dimension of the kernel belonging to the set of the new balance equations becomes greater. This fact also has consequences for the dissipation inequality $(5)_2$.

The balance equations on state space are related to the dissipation inequality by the Coleman–Mizel formulation of the second law [Coleman and Mizel 1964]: "Each solution of the balance equations satisfies the dissipation inequality." This is also true for the projected balances (13) on state space. Consequently, we have the inductions

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{y}^*+\mathbf{y}^0)&=\mathbf{C} &&\to\quad \mathbf{B}\cdot(\mathbf{y}^*+\mathbf{y}^0)\ge D,\\ \mathbf{P}\cdot\mathbf{A}\cdot(\bar{\mathbf{Y}}+\hat{\mathbf{Y}})&=\mathbf{P}\cdot\mathbf{C} &&\to\quad \mathbf{B}_P\cdot(\bar{\mathbf{Y}}+\hat{\mathbf{Y}})\ge D_P.\end{aligned}$$

Because $\mathbf{y}^0$ and $\hat{\mathbf{Y}}$ are arbitrary elements of the kernels of $\mathbf{A}$ and $\mathbf{P}\cdot\mathbf{A}$, we obtain, in order to maintain the dissipation inequalities,

$$\begin{aligned}\mathbf{B}\cdot\mathbf{y}^0=0 &\quad\to\quad \mathbf{B}\perp\ker\mathbf{A},\\ \mathbf{B}_P\cdot\hat{\mathbf{Y}}=0 &\quad\to\quad \mathbf{B}_P\perp\ker(\mathbf{P}\cdot\mathbf{A}).\end{aligned}$$

This results in

$$\begin{aligned}\dim(\operatorname{span}\mathbf{B})+\dim(\ker\mathbf{A})&=n,\\ \dim(\operatorname{span}\mathbf{B}_P)+\dim(\ker\,(\mathbf{P}\cdot\mathbf{A}))&=n.\end{aligned}$$

Subtracting both the equations from each other,

$$\dim(\operatorname{span}\mathbf{B})-\dim(\operatorname{span}\mathbf{B}_P)+\dim(\ker\mathbf{A})-\dim(\ker(\mathbf{P}\cdot\mathbf{A}))=0,$$

and according to (20) this results in

$$\dim(\operatorname{span}\mathbf{B})>\dim(\operatorname{span}\mathbf{B}_P). \tag{21}$$

The inequality (21) can be interpreted as follows: If balance equations are not taken into account when exploiting the dissipation inequality, the found class of materials [Muschik et al. 2001] becomes smaller and the dissipation inequality remains valid. Neglecting balance equations in the process of exploiting the dissipation inequality does not result in violations of the second law, but the acquired class of materials is too small. Correct exploitation, considering all balance equations, results in a greater class of materials than when we neglect some of the balances.

3. An example

3.1. ***Unabridged system of balances.*** We consider a state space [Muschik et al. 2001; Muschik 2004] spanned by the internal energy density ε and by two additional vector fields, $\mathbf{z}$ and $\mathbf{w}$,

$$\mathbf{Z} = \{\varepsilon, \mathbf{z}, \mathbf{w}\}. \tag{22}$$

The local balances (3), of internal energy ε and entropy s, read

$$\dot{\varepsilon} + \nabla \cdot \mathbf{q} - r = 0, \qquad \dot{s} + \nabla \cdot \mathbf{J}_s - r_s \geq 0, \tag{23}$$

where $\mathbf{q}$ is the heat flux density and r the heat supply density. The second law of thermodynamics forces the entropy production (3) to be nonnegative.

The set of governing equations for $\mathbf{z}$ and $\mathbf{w}$ is assumed to have balance form, and represent constraints for the state space variables

$$\dot{\mathbf{z}} + \nabla \cdot \Psi = \pi, \qquad \dot{\mathbf{w}} + \nabla \cdot \Delta = \delta. \tag{24}$$

Here Ψ is the flux of $\mathbf{z}$, Δ is the flux of $\mathbf{w}$, and π and δ are their productions and supplies, respectively. The relaxation equations of $\mathbf{z}$ and/or $\mathbf{w}$ are included by setting $\Psi \equiv \mathbf{0}$ and/or $\Delta \equiv \mathbf{0}$.

Let us now introduce two additional constitutive functions M and $\mathbf{W}$, which are not in the state space, and for which we also have balances

$$\dot{M}(\varepsilon, \mathbf{z}, \mathbf{w}) + \nabla \cdot \Upsilon(\varepsilon, \mathbf{z}, \mathbf{w}) = \Sigma(\varepsilon, \mathbf{z}, \mathbf{w}), \tag{25}$$

$$\dot{\mathbf{W}}(\varepsilon, \mathbf{z}, \mathbf{w}) + \nabla \cdot \Xi(\varepsilon, \mathbf{z}, \mathbf{w}) = \Omega(\varepsilon, \mathbf{z}, \mathbf{w}). \tag{26}$$

The balance laws (23)–(26) and the entropy inequality (3) can be written in matrix formulation [Muschik et al. 2001], with the shape of $(5)_1$ and $(5)_2$, as

$$\mathbf{A} \cdot \mathbf{y} = \mathbf{C}, \qquad \mathbf{B} \cdot \mathbf{y} \geq D, \tag{27}$$

where $\mathbf{y}$ represents the higher derivatives of the chosen state space (22), $\mathbf{y} = \{\dot{\varepsilon}, \dot{\mathbf{z}}, \dot{\mathbf{w}}, \nabla\varepsilon, \nabla\mathbf{z}, \nabla\mathbf{w}\}$. The matrices $\mathbf{A}$ and $\mathbf{C}$ follow from the five balance equations (23)–(26),

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & \frac{\partial \mathbf{q}}{\partial \varepsilon} & \frac{\partial \mathbf{q}}{\partial \mathbf{z}} & \frac{\partial \mathbf{q}}{\partial \mathbf{w}} \\ 0 & 1 & 0 & \frac{\partial \Psi}{\partial \varepsilon} & \frac{\partial \Psi}{\partial \mathbf{z}} & \frac{\partial \Psi}{\partial \mathbf{w}} \\ 0 & 0 & 1 & \frac{\partial \Delta}{\partial \varepsilon} & \frac{\partial \Delta}{\partial \mathbf{z}} & \frac{\partial \Delta}{\partial \mathbf{w}} \\ \frac{\partial M}{\partial \varepsilon} & \frac{\partial M}{\partial \mathbf{z}} & \frac{\partial M}{\partial \mathbf{w}} & \frac{\partial \Upsilon}{\partial \varepsilon} & \frac{\partial \Upsilon}{\partial \mathbf{z}} & \frac{\partial \Upsilon}{\partial \mathbf{w}} \\ \frac{\partial \mathbf{W}}{\partial \varepsilon} & \frac{\partial \mathbf{W}}{\partial \mathbf{z}} & \frac{\partial \mathbf{W}}{\partial \mathbf{w}} & \frac{\partial \Xi}{\partial \varepsilon} & \frac{\partial \Xi}{\partial \mathbf{z}} & \frac{\partial \Xi}{\partial \mathbf{w}} \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} r \\ \pi \\ \delta \\ \Sigma \\ \Omega \end{bmatrix}. \tag{28}$$

From the entropy balance $(23)_2$, it follows that

$$\mathbf{B} = \begin{bmatrix} \frac{\partial s}{\partial \varepsilon} & \frac{\partial s}{\partial \mathbf{z}} & \frac{\partial s}{\partial \mathbf{w}} & \frac{\partial \mathbf{J}_s}{\partial \varepsilon} & \frac{\partial \mathbf{J}_s}{\partial \mathbf{z}} & \frac{\partial \mathbf{J}_s}{\partial \mathbf{w}} \end{bmatrix}, \qquad D = r_s. \tag{29}$$

The balances and the dissipation inequality (27) are now exploited by the Liu procedure [Liu 1972; Muschik and Ehrentraut 1996], by which the higher derivatives are removed. We introduce the so-called Lagrange parameters,

$$\Lambda := \{\, \lambda^{\varepsilon} \;\; \lambda^{\mathbf{z}} \;\; \lambda^{\mathbf{w}} \;\; \lambda^{M} \;\; \lambda^{\mathbf{W}} \,\}, \tag{30}$$

and, taking (28)–(29) into account, the Liu equations, $(7)_1$, become

$$\begin{aligned}
\lambda^{\varepsilon}+\lambda^{M}\frac{\partial M}{\partial\varepsilon}+\lambda^{\mathbf{W}}\cdot\frac{\partial\mathbf{W}}{\partial\varepsilon}&=\frac{\partial s}{\partial\varepsilon},\\
\lambda^{\mathbf{z}}+\lambda^{M}\frac{\partial M}{\partial\mathbf{z}}+\lambda^{\mathbf{W}}\cdot\frac{\partial\mathbf{W}}{\partial\mathbf{z}}&=\frac{\partial s}{\partial\mathbf{z}},\\
\lambda^{\mathbf{w}}+\lambda^{M}\frac{\partial M}{\partial\mathbf{w}}+\lambda^{\mathbf{W}}\cdot\frac{\partial\mathbf{W}}{\partial\mathbf{w}}&=\frac{\partial s}{\partial\mathbf{w}},\\
\lambda^{\varepsilon}\frac{\partial\mathbf{q}}{\partial\varepsilon}+\lambda^{\mathbf{z}}\cdot\frac{\partial\Psi}{\partial\varepsilon}+\lambda^{\mathbf{w}}\cdot\frac{\partial\Delta}{\partial\varepsilon}+\lambda^{M}\frac{\partial\Upsilon}{\partial\varepsilon}+\lambda^{\mathbf{W}}\cdot\frac{\partial\Xi}{\partial\varepsilon}&=\frac{\partial\mathbf{J}_s}{\partial\varepsilon},\\
\lambda^{\varepsilon}\frac{\partial\mathbf{q}}{\partial\mathbf{z}}+\lambda^{\mathbf{z}}\cdot\frac{\partial\Psi}{\partial\mathbf{z}}+\lambda^{\mathbf{w}}\cdot\frac{\partial\Delta}{\partial\mathbf{z}}+\lambda^{M}\frac{\partial\Upsilon}{\partial\mathbf{z}}+\lambda^{\mathbf{W}}\cdot\frac{\partial\Xi}{\partial\mathbf{z}}&=\frac{\partial\mathbf{J}_s}{\partial\mathbf{z}},\\
\lambda^{\varepsilon}\frac{\partial\mathbf{q}}{\partial\mathbf{w}}+\lambda^{\mathbf{z}}\cdot\frac{\partial\Psi}{\partial\mathbf{w}}+\lambda^{\mathbf{w}}\cdot\frac{\partial\Delta}{\partial\mathbf{w}}+\lambda^{M}\frac{\partial\Upsilon}{\partial\mathbf{w}}+\lambda^{\mathbf{W}}\cdot\frac{\partial\Xi}{\partial\mathbf{w}}&=\frac{\partial\mathbf{J}_s}{\partial\mathbf{w}},
\end{aligned}\tag{31}$$

and the residual inequality, $(7)_2$, becomes

$$\lambda^{\varepsilon}r+\lambda^{\mathbf{z}}\cdot\pi+\lambda^{\mathbf{w}}\cdot\delta+\lambda^{M}\Sigma+\lambda^{\mathbf{W}}\cdot\Omega\geq r_s.\tag{32}$$

The Liu equations (31) and the residual dissipation inequality (32) represent the constraints on the constitutive equations M, $\mathbf{W}$, $\mathbf{q}$, Ψ, Δ, Υ, and Ξ caused by the second law.

According to (8) and $(7)_1$ the Lagrange parameters are given by (9), and the constraints on the constitutive equations have the form of an inequality, as in (10).

3.2. ***Reduced system of balances.*** Now we introduce the matrix projector $\mathbf{P}$, cutting the balance (26) as

$$\mathbf{P}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}.\tag{33}$$

The original system of balances, (27), is now replaced by the system (13)

$$\mathbf{P}\cdot\mathbf{A}\cdot\mathbf{Y}=\mathbf{P}\cdot\mathbf{C},\quad \mathbf{Y}\neq\mathbf{y},\qquad \dim(\operatorname{span}\mathbf{Y})>\dim(\operatorname{span}\mathbf{y}),\qquad \dim(\operatorname{span}(\mathbf{P}\cdot\mathbf{C}))<\dim(\operatorname{span}\mathbf{C}).$$

With this replacement we are not taking into account the balance of the constitutive function $\mathbf{W}$ according to (33). The Liu equations and the residual dissipation inequality, obtained by using the Liu technique [Muschik et al. 2001], have the same algebraic form as (7) in the case of complete exploitation, considering all balances,

$$\Lambda_P\cdot\mathbf{P}\cdot\mathbf{A}=\mathbf{B}_P,\qquad \Lambda_P\cdot\mathbf{P}\cdot\mathbf{C}\geq D_P.\tag{34}$$

According to (30) we have

$$\Lambda_P=\{\,\lambda_P^{\varepsilon}\ \ \lambda_P^{\mathbf{z}}\ \ \lambda_P^{\mathbf{w}}\ \ \lambda_P^{M}\ \ \lambda_P^{\mathbf{W}}\,\},\qquad \Lambda_P\cdot\mathbf{P}=\{\,\lambda_P^{\varepsilon}\ \ \lambda_P^{\mathbf{z}}\ \ \lambda_P^{\mathbf{w}}\ \ \lambda_P^{M}\ \ 0\,\}=:\Lambda_P^0.\tag{35}$$

The projected balances of (34),

$$\Lambda_P^0\cdot\mathbf{A}=\mathbf{B}_P,\qquad \Lambda_P^0\cdot\mathbf{C}\geq D_P,\tag{36}$$

differ from (7) by setting $\lambda^{\mathbf{W}}$ formally to zero. Consequently, (36) becomes, if (31) and (32) are taken into account,

$$\begin{aligned}
\lambda^{\varepsilon}_P + \lambda^M_P \frac{\partial M}{\partial \varepsilon} &= \frac{\partial s}{\partial \varepsilon}\Big|_P,\\
\lambda^{\mathbf{z}}_P + \lambda^M_P \frac{\partial M}{\partial \mathbf{z}} &= \frac{\partial s}{\partial \mathbf{z}}\Big|_P,\\
\lambda^{\mathbf{w}}_P + \lambda^M_P \frac{\partial M}{\partial \mathbf{w}} &= \frac{\partial s}{\partial \mathbf{w}}\Big|_P,\\
\lambda^{\varepsilon}_P \frac{\partial \mathbf{q}}{\partial \varepsilon} + \lambda^{\mathbf{z}}_P \cdot \frac{\partial \Psi}{\partial \varepsilon} + \lambda^{\mathbf{w}}_P \cdot \frac{\partial \Delta}{\partial \varepsilon} + \lambda^M_P \frac{\partial \Upsilon}{\partial \varepsilon} &= \frac{\partial \mathbf{J}_s}{\partial \varepsilon}\Big|_P,\\
\lambda^{\varepsilon}_P \frac{\partial \mathbf{q}}{\partial \mathbf{z}} + \lambda^{\mathbf{z}}_P \cdot \frac{\partial \Psi}{\partial \mathbf{z}} + \lambda^{\mathbf{w}}_P \cdot \frac{\partial \Delta}{\partial \mathbf{z}} + \lambda^M_P \frac{\partial \Upsilon}{\partial \mathbf{z}} &= \frac{\partial \mathbf{J}_s}{\partial \mathbf{z}}\Big|_P,\\
\lambda^{\varepsilon}_P \frac{\partial \mathbf{q}}{\partial \mathbf{w}} + \lambda^{\mathbf{z}}_P \cdot \frac{\partial \Psi}{\partial \mathbf{w}} + \lambda^{\mathbf{w}}_P \cdot \frac{\partial \Delta}{\partial \mathbf{w}} + \lambda^M_P \frac{\partial \Upsilon}{\partial \mathbf{w}} &= \frac{\partial \mathbf{J}_s}{\partial \mathbf{w}}\Big|_P,
\end{aligned} \tag{37}$$

and

$$\lambda^{\varepsilon}_P r + \lambda^{\mathbf{z}}_P \cdot \pi + \lambda^{\mathbf{w}}_P \cdot \delta + \lambda^M_P \Sigma \geq {r_s}^P. \tag{38}$$

The equations (37) and the dissipation inequality (38) are the thermodynamic constraints due to the second law, if the balance (26) of the constitutive function $\mathbf{W}$ is not taken into account. These relations are analogous to (31) and to (32). We will compare them in the next section.

3.3. ***Comparison.*** A comparison of the original Liu equations, (31), with the projected ones, (37), shows how different spaces are generated by the Lagrange parameters. Because, according to (30) and $(35)_2$, $5 = \dim(\text{span } \Lambda) > \dim(\text{span } \Lambda^0_P) = 4$ is valid, we obtain, according to the maximal rank of $\mathbf{A}$ and to $(7)_1$ and $(36)_1$, $\dim(\text{span } \mathbf{B}) > \dim(\text{span } \mathbf{B}_P)$, the inequality (21), which was expected according to the considerations of the general case.

The class of materials becomes smaller by the reduction of the balances. This yields a comparison of the projected balances, (37), with the original ones, (31). All solutions of the projected balance equations are also solutions of the original balance equations in the case of $\lambda^{\mathbf{W}} = 0$.

3.3.1. *The supplies.* Comparing the dissipation inequalities (32) and (38), we see that the energy supply r is insensitive to canceling (26). The same is presupposed for the entropy supply

$$r_s^P \doteq r_s. \tag{39}$$

Because energy supply and entropy supply are connected by the temperature Θ,

$$r_s = \frac{r}{\Theta}, \tag{40}$$

and we obtain from (32) and (38) with (39) and (40)

$$\left[\lambda^{\varepsilon} - \frac{1}{\Theta}\right] r + \lambda^{\mathbf{z}} \cdot \pi + \lambda^{\mathbf{w}} \cdot \delta + \lambda^M \Sigma + \lambda^{\mathbf{W}} \cdot \Omega \geq 0, \qquad \left[\lambda^{\varepsilon}_P - \frac{1}{\Theta}\right] r + \lambda^{\mathbf{z}}_P \cdot \pi + \lambda^{\mathbf{w}}_P \cdot \delta + \lambda^M_P \Sigma \geq 0.$$

Because the energy supply is independent of all the other quantities which appear in these dissipation inequalities and because the sign of the energy supply can be positive or negative, we obtain

$$\lambda^{\varepsilon} = \frac{1}{\Theta} = \lambda^{\varepsilon}_{P}.$$

In the next section it is proved that the entropy production is sensitive to the reduction of the balance equations.

3.3.2. *The entropy production.* We now calculate relations between ($\mathbf{B}$, D) and ($\mathbf{B}_P$, D_P). From (7) and (34) we obtain

$$\mathbf{B} = \Lambda \cdot \mathbf{A}, \quad \mathbf{B}_P = \Lambda_P \cdot \mathbf{P} \cdot \mathbf{A}, \qquad \Lambda \cdot \mathbf{C} \geq D, \quad \Lambda_P \cdot \mathbf{P} \cdot \mathbf{C} \geq D_P,$$

which results, by use of $(35)_2$, in

$$\mathbf{B} - \mathbf{B}_P = (\Lambda - \Lambda^0_P) \cdot \mathbf{A}. \tag{41}$$

Because $\mathbf{A}$ is of maximal rank, and $\Lambda \neq \Lambda^0_P$ follows from (30) and $(35)_2$, we obtain from (41) that $\mathbf{B} \neq \mathbf{B}_P$.

Starting out with $(30)_2$ and $(36)_2$, and taking $(29)_2$ and (39) into account,

$$\Lambda \cdot \mathbf{C} \geq D = r_s, \qquad \Lambda^0_P \cdot \mathbf{C} \geq D_P = r^P_s = r_s,$$

and we obtain for the entropy production

$$\sigma = \Lambda \cdot \mathbf{C} - r_s, \quad \sigma_P = \Lambda^0_P \cdot \mathbf{C} - r_s \qquad \longrightarrow \qquad \sigma \neq \sigma_P. \tag{42}$$

Taking (9) and the corresponding relation, $\Lambda^0_P = \mathbf{B}_P \cdot \bar{\mathbf{A}}$, into account, (42) results in $\sigma = \mathbf{B} \cdot \bar{\mathbf{A}} \cdot \mathbf{C} - r_s \geq 0$ and $\sigma_P = \mathbf{B}_P \cdot \bar{\mathbf{A}} \cdot \mathbf{C} - r_s \geq 0$. We have proved that the entropy production changes when the balance equations are reduced. But it is not clear that the entropy production becomes smaller in the reduced case compared with the original one. Comparing (32) with (38), we do not know that $\lambda^{\mathbf{W}} \cdot \Omega$ is positive. Beyond that, we do not know the values of the remaining terms in which Λ is replaced by Λ^0_P.

4. Conclusion

It is well known that the second law represents a constraint on the constitutive equations of a system under consideration [Muschik et al. 2001], which means the second law, represented by the dissipation inequality, is material-selective. When exploiting the dissipation inequality by the Liu technique, usually one has to take into account all balance equations of the system [Triani et al. 2008]. There are several reasons why not all balances would be included in the exploitation of the dissipation inequality: not all balance equations are known, only the energy balance is taken into account [Ericksen 1991], or some balances are forgotten. In these cases, an interesting question arises: What happens if not all balances are taken into account when exploiting the dissipation inequality? The answer is that we obtain a smaller class of materials than in the nonreduced case! This smaller class of materials does obey the second law. Therefore, no mistakes appear with respect to the second law, if we forget some balances in its exploitation: we are punished with a smaller class of materials, which has a different nonnegative entropy production than in the nonreduced case. This result is important with respect to the fact that there are former papers in which an abridged exploitation of the dissipation inequality was performed without any comment or notice.

The results mentioned above are obtained by choosing an abstract large state space (no after-effects) [Muschik et al. 2001] on which the constitutive equations are defined, starting out with the Coleman–Mizel formulation of the second law [Coleman and Mizel 1964; Muschik and Ehrentraut 1996], using the Liu procedure for exploiting the dissipation inequality [Liu 1972; Muschik and Ehrentraut 1996], and by presupposing that the entropy supply is insensitive to reducing the number of balances.

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Received 7 Feb 2008. Revised 26 Mar 2008. Accepted 2 Apr 2008.

WOLFGANG MUSCHIK: muschik@physik.tu-berlin.de
Institut für Theoretische Physik, EW 7-1, Technische Universität Berlin, Hardenbergstraße 36, 10623 Berlin, Germany

VITA TRIANI: vitrian@virgilio.it
Dipartimento di Matematica ed Informatica, Universitá degli Studi della Basilicata, Campus Macchia Romana, Potenza, 85100, Italy

CHRISTINA PAPENFUSS: c.papenfuss@gmx.de
Institut für Mechanik, Technische Universität Berlin, Strasse des 17. Juni, 10623 Berlin, Germany