RELIABILITY OF FIRST-ORDER SHEAR DEFORMATION MODELS FOR SANDWICH BEAMS

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We are interested in sandwich beams whose skin may be *thick* (as defined by H. G. Allen) and whose core stiffness along the sandwich longitudinal axis may be large enough to influence the deflection (that is, we also account for nonantiplane sandwiches), whereas the core is such that it is allowed to disregard its deformability along its height (the direction of the applied load). For such sandwiches we are particularly interested in investigating the reliability of simple models, such as the first-order shear deformation models, for accurate computation of the deflection in the linear elastic range. We therefore compare different theories on the basis of finite element simulations and focus on the case of a propped cantilever beam supporting a uniform load. In fact, this boundary value problem leads to slightly different conclusions than those previously drawn based on statically determinate cases, such as in three-point bending. The analysis suggests that known models may be largely inaccurate in predicting sandwich behaviour under bending and shear, depending on a peculiarity of the actual sandwich kinematics indirectly describing the interaction between skins and core, in turn due both to material and geometrical properties and to boundary conditions.

1. Introduction

We focus on the reliability of simple models for the accurate computation of the *macroscopic* deflection of sandwich beams, subjected to bending and shear, in the linear elastic range. We use the term macroscopic, as in [Mai et al. 2007], to indicate that part of the deformation that in a homogeneous beam would be described by the Timoshenko theory; this means that we neglect the deformation due to local stress concentrations, which may be extremely important, under certain boundary conditions, in sandwiches having extremely soft cores along the thickness direction [Frostig et al. 1992]. In particular, in the analytical models presented, we will always consider cores able to keep skin distance constant during deformation, besides, of course, being able to transmit shear stresses between the skins. Finite element simulations will show that this is true for quite a large ratio between the elastic moduli of skins and core, provided that the boundary conditions are properly modelled. We will also consider sandwiches whose core stiffness along the longitudinal axis may be relevant, as this may happen in some important applications [Bunn and Mottram 1993; Bardella and Genna 2001]; in other words, the sandwiches here analysed are not necessarily *antiplane*, a term introduced by Allen [1969] to mean a sandwich whose core has negligible normal stress along the longitudinal axis, so that the shear stresses may be taken as uniform.
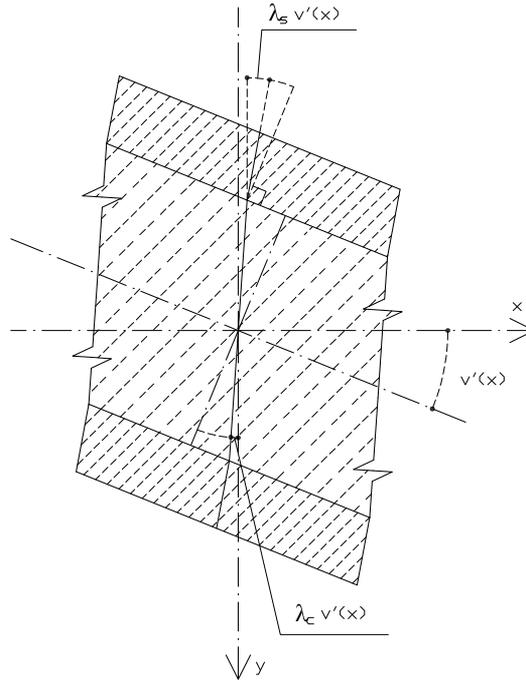
Figure 1. Parameters used for modelling the kinematics of a sandwich beam having identical skins.

We restrict our attention to sandwiches whose skins have identical geometrical and mechanical properties.[1] Also, the load is symmetric with respect to the width of the beam, so that there is no torsion.

It is well known that the relatively high shear compliance of sandwich beams makes an important contribution to their macroscopic deflection [Allen 1969]. Unfortunately, because of the complex kinematics involved, the analytical computation of such a deflection is quite a difficult task, even in the linear elastic range. The relevant kinematics can be approximately represented as in Figure 1. It consists of a piecewise linear warping described by three unknown functions v, λ_c, and λ_s, such that $v(x)$ is the deflection of the sandwich longitudinal axis x and, denoting with $'$ a derivative with respect to x, $\lambda_c(x)v'(x)$ and $\lambda_s(x)v'(x)$ are the total cross section rotations of the core and of the skins, respectively. This description has been used in [Bardella and Genna 2000] as an extension of the kinematics adopted by Allen [1969] (who assumed $\lambda_s = 1$), in order to also account for the effect of not-so-thin skins. Yu [1959] and Krajcinovic [1971] also used the same piecewise linear kinematics for the flexural problem,[2] but they described it in terms of different sets of independent functions of x. Krajcinovic choice allows one to uncouple the Euler–Lagrange equations obtained by minimising the total potential energy (TPE) functional, but the further differentiation required to obtain such a simplification makes things harder concerning the specification of the boundary conditions.

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[1] The extension of the methods described to the case of unequal skins is conceptually simple, albeit analytically entangled.

[2] The kinematics employed in [Krajcinovic 1971] also account for the core compressibility along y and for the bulging; however, these modes of deformation turn out to be related to the axial problem (for which the beam remains straight), uncoupled to the bending problem.

A considerable amount of literature exists on laminate beam models based on kinematics richer than the linear along y involved in the Timoshenko beam model, where transverse normal lines remain straight during deformation, but inclined with respect to the longitudinal axis depending on the amount of shear; this simple model for laminate beams is referred to as first-order shear deformation (FOSD) theory (see, for example, the review of Ghugal and Shimpi [2001]). The zigzag kinematics described in Figure 1 is the base for the so-called discrete-layer theories, often more accurate, but computationally more difficult to deal with than the equivalent single-layer theories; the latter define the displacement field along the longitudinal axis x as a C^∞ continuous polynomial function of y. Examples of single-layer theories are the third-order model of Silverman [1980], which applies to thick-skinned and nonantiplane sandwiches as those here concerned, or the second and third-order shear deformation theories of Khdeir and Reddy [1997], applied to cross-ply laminated beams. More insight on the distinctions among different approaches for the analysis of laminate structures, albeit for plates, can be found in [Yu et al. 2008] and references therein.

Here, in order to test, discuss, and get insight on the reliability of FOSD models (the simplest among the equivalent single-layer models), we compare their predictions with those of models based on the piecewise linear kinematics of Figure 1 and the results of finite element simulations.

In particular, we focus on a propped cantilever sandwich beam supporting a uniform transversal load. This example is of particular interest since, contrary to some statically determinate structures (such as the three-point bending [Bardella and Genna 2000], the four-point bending, and the simply-supported beam subjected to uniform load [Minelli 2007]), its deflection is inaccurately represented by all of the following models:

(i) The model ensuing from the analytical procedure put forward by Bardella and Genna [2000] in order to minimise the TPE, written in terms of the zigzag kinematics of Figure 1, based on the assumption that λ_c and λ_s are constant (that is, independent upon x).

(ii) The model for thick skins and nonantiplane core deriving from Allen's superposition theory [Allen 1969] (still one of the most quoted models for sandwiches).

(iii) The FOSD theory in which the shearing rigidity [Timoshenko and Gere 1990] is evaluated on the basis of the Jourawsky approach [1856] and an energy principle, as accomplished by Bardella and Genna [2000; 2001] (see also [Bert 1973]).

Instead, the finite element solution of the propped cantilever sandwich beam is pretty well represented by the TPE approach in which the variation of λ_c and λ_s along x is accounted for, and this can be recognised by resorting to the Ritz method.

This notwithstanding, the third method listed above, which can be reasonably assumed to be the best possible FOSD model, may be extremely useful in the design because of its simplicity and accuracy (as already been shown for a few cases in [Bardella and Genna 2001; Minelli 2007]; see also [Gordaninejad and Bert 1989]). On the basis of our analysis, we then put forward that this method can be successfully employed when the structure is such that λ_c and λ_s are approximately independent on x, so that, in our opinion, one of the main goals becomes reaching a deep insight on this topic.

Outline of the paper. The paper is organised as follows. In Section 2 we provide and discuss the formulae relevant for the examples concerned in the paper, and also summarise the theory related to the methods

(i) and (iii) listed above, originally developed in [Bardella and Genna 2000]. In particular, Section 2.1 deals with the approach based on the TPE, in Section 2.2 we give the relevant results related to Allen's superposition method, and Section 2.3 summarises the FODT exploiting Jourawsky's approximate theory of shear and, for the sake of completeness, also considers other possible FOSD methods, including the "thin skin" approximation [Allen 1969]. In Section 3 we compare the results obtained from the application of the above theoretical models with those of finite element simulations. We close, in Section 4, with a discussion and some remarks.

2. The linear elastic deflection of a sandwich beam

We consider a sandwich beam of total length l and width b, consisting of three homogeneous layers. The sandwich has skins with identical thickness, t, and identical mechanical properties, whose relevant Young and shear moduli are denoted by E_s and G_s, respectively. The thickness of the core is indicated with the symbol c, so that the distance between the centres of the two skins is $d = c + t$. The relevant Young and shear moduli of the core are E_c and G_c. Let us recall that, with reference to the terminology used in [Allen 1969], we are interested in a sandwich with thick skins and a nonantiplane core.

2.1. *The total potential energy approach.* The TPE minimum principle requires the choice of an admissible displacement field over the sandwich. This approach leads rapidly to quite involved computations. In [Allen 1969] it is used only in conjunction with the Ritz method, to obtain approximate solutions for sandwiches with antiplane core and skins whose shear deformability turns out to be negligible. As said, we refer our calculations to the kinematics sketched in Figure 1, from which it is straightforward to compute the longitudinal strain ε_x and the engineering shear strain γ_{xy}, then the relevant normal and shear stresses,[3] and write the TPE functional Ψ for a sandwich beam subjected to a transversal (meaning acting along y) force P and a transversal distributed load $q(x)$ as

$$\Psi(v, \lambda_c, \lambda_s) = \frac{G_c bc}{2} \int_0^l \left[(1 - \lambda_c) v' \right]^2 dx + \frac{E_c bc^3}{24} \int_0^l \left[(\lambda_c v')' \right]^2 dx + G_s bt \int_0^l \left[(1 - \lambda_s) v' \right]^2 dx$$

$$+ \frac{E_s bt^3}{12} \int_0^l \left[(\lambda_s v')' \right]^2 dx + \frac{E_s bt}{4} \int_0^l \left\{ \left[(c\lambda_c + t\lambda_s) v' \right]' \right\}^2 dx - P\delta - \int_0^l q(x) v(x) \, dx, \quad (1)$$

in which δ is the displacement of the point where P is applied.[4] The actual solution of the problem corresponds to

$$\min_{(v, \lambda_c, \lambda_s) \in \mathcal{H}} \Psi(v, \lambda_c, \lambda_s),$$

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[3]The computed normal and shear stresses are respectively $\sigma_x^{(i)} = E_i \varepsilon_x^{(i)}$ and $\tau_{xy}^{(i)} = G_i \gamma_{xy}^{(i)}$, with $i = s$ in the skins and $i = c$ in the core. For isotropic materials the *correct* constitutive law would be $\sigma_x^{(i)} = [E_i/(1 - \nu_i^2)] \varepsilon_x^{(i)}$, where ν_i is the Poisson ratio, which links the longitudinal strain with the longitudinal stress for the beam plane stress state characterised by the further kinematic constraint of imposing a vanishing direct strain component along y. Anyway, since the use of that relation would make the beam too stiff, we replace it with the Young modulus E_i, as usual in the engineering calculations for homogeneous beams. This is actually based on the hypothesis of taking a zero direct stress component along y, so $\sigma_y = 0$; such a stress assumption is incompatible with the kinematic approach which the TPE principle is based on, but, as is well known, works for homogeneous beams. The removal of this assumption is a key point in the higher-order theory proposed by Frostig et al. [1992], in which the sandwich core is treated as a plane stress continuum.

[4]If the sandwich were subjected also to axial loading (by, for instance, a distributed load $r(x)$), Ψ would also be dependent on the centre displacement component along x, u_c (see [Krajcinovic 1971] about how to account also for the bulging deformation

where $\mathcal{K}$ is the set enclosing all the compatible fields. Usually the only requirement on λ_c and λ_s is that they be suitably smooth, whereas v must always satisfy some essential boundary conditions, because in any constrained section x_0 at least one of the values $v(x_0)$ and $v'(x_0)$ is preassigned. A case in which one also has to impose essential boundary conditions on λ_c and λ_s is that of a simply-supported beam end with no warping allowed, for which $\lambda_c(x_0) = \lambda_s(x_0)$ (or even $\lambda_c(x_0) = \lambda_s(x_0) = 1$); however, this case, which has been shown to lead to an unfavourable sandwich behaviour [Krajcinovic 1971], will not be considered here.

Within this approach, the bending moment $M(x)$ and the shear force $V(x)$ are given by

$$M = \tfrac{1}{6} E_s b t^3 (\lambda_s v')' + \tfrac{1}{2} E_s b t d \big[(c\lambda_c + t\lambda_s) v' \big]' + \tfrac{1}{12} E_c b c^3 (\lambda_c v')', \tag{2}$$

and

$$V = [G_c b c (1 - \lambda_c) + 2 G_s b t (1 - \lambda_s)] v'. \tag{3}$$

The TPE approach proposed by Allen [1969] is characterised by neglecting the shear compliance of the skins, $G_s \to \infty$. In this case, it is immediately recognisable from the functional (1) that the minimum of $\Psi(v, \lambda_c, \lambda_s)$ is attained when $\lambda_s = 1$ and $\lambda_c \in [-t/c, 1]$, the limits of this range corresponding to both $G_c \to 0$ and $E_c \to 0$ or $G_c \to \infty$, respectively. The fact that λ_c can be either positive or negative implies that this modelling allows the core to be subjected to normal stresses, σ_x, whose moment resultant M_c has opposite sign with respect to the moment M applied to the whole cross section, even though, under this circumstance, M_c is usually far smaller than M. This behaviour, here and henceforth called "normal stress inversion", will be appreciated in discussing the finite element simulation described in Section 3.2.

Moreover, let us note that even though the same elastic moduli are chosen for skins and core, the resulting model remains richer than that described by the Timoshenko theory for homogeneous beams since in this case, as well as in general, $\lambda_c \neq \lambda_s$, and the warping still occurs. On the other hand, by taking the limit $t \to 0$, the model approximates that of Timoshenko in which the shearing rigidity, equal to $G_c b c$, is, as is well known, overestimated.

Only by imposing $\lambda_c(x) = \lambda_s(x)$ it is possible to single out from (2) the *effective* bending stiffness

$$D = E_s \left(\frac{bt^3}{6} + \frac{btd^2}{2} \right) + E_c \frac{bc^3}{12}. \tag{4}$$

Although $\lambda_c(x) = \lambda_s(x)$ seems to be a crude approximation, for it means neglecting the warping, (4) is a completely standard and widely accepted result for a thick skinned and nonantiplane sandwich (see, for example, [Allen 1969; Zenkert 1997], and the methods of Sections 2.2 and 2.3). Hence, in the TPE approach the *hidden* bending stiffness should be lower than D.

Setting to zero the first variation of (1) provides a nonlinear differential system of three Euler–Lagrange equations in the three unknown functions $v(x)$, $\lambda_c(x)$, and $\lambda_s(x)$, and the natural boundary conditions. Solving the Euler–Lagrange equations is complicated, so we skip it, whereas it is of some interest to

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mode), as

$$\Psi(v, \lambda_c, \lambda_s, u_c) = \tfrac{1}{2} G_c b c \int_0^l (1 - \lambda_c)^2 (v')^2 dx + \tfrac{1}{24} E_c b c^3 \int_0^l (\lambda_c' v' + \lambda_c v'')^2 dx$$
$$+ G_s b t \int_0^l (1 - \lambda_s)^2 (v')^2 dx + \tfrac{1}{4} E_s b t \int_0^l \big[(c\lambda_c' + t\lambda_s') v' + (c\lambda_c + t\lambda_s) v'' + u_c' \big]^2 dx$$
$$+ E_c b c \int_0^l (u_c')^2 dx + \tfrac{1}{12} E_s b t^3 \int_0^l (\lambda_s' v' + \lambda_s v'')^2 dx - P\delta - \int_0^l q(x) v(x) dx - \int_0^l r(x) u_c(x) dx.$$

consider the natural boundary conditions related to λ_c and λ_s. In fact, in the case of a simply-supported, free to warp, and moment-free beam end (for example, in $x = x_0$), a plausible set of conditions satisfying the natural boundary conditions reads

$$v''(x = x_0) = 0, \qquad \lambda'_c(x = x_0) = 0, \qquad \lambda'_s(x = x_0) = 0. \tag{5}$$

In particular, in the three-point bending, such conditions hold both in $x = 0$ and $x = l$.

The λ-constrained method. As proposed in [Bardella and Genna 2000], the TPE can be simplified if λ_c and λ_s are assumed to be constant (that is, independent upon x). This is a compatible restriction and, also, it is at least consistent with the natural boundary conditions (5) for the case of a simply-supported beam section. This approximation, whose reliability will be discussed later on, allows the derivation of analytical solutions. The TPE functional (1) becomes

$$\Psi_C(v, \lambda_c, \lambda_s) = \frac{G_c bc}{2}(1 - \lambda_c)^2 \int_0^l (v')^2 dx + \frac{E_c bc^3}{24}\lambda_c^2 \int_0^l (v'')^2 dx + G_s bt(1 - \lambda_s)^2 \int_0^l (v')^2 dx$$
$$+ \frac{E_s bt}{4}(c\lambda_c + t\lambda_s)^2 \int_0^l (v'')^2 dx + \frac{E_s bt^3}{12}\lambda_s^2 \int_0^l (v'')^2 dx - \int_0^l q(x)v(x)\, dx - P\delta.$$

By imposing the stationarity of Ψ_C with respect to the kinematic parameters, v, λ_c, and λ_s, one obtains

 (i) an ordinary fourth-order differential equation governing the deformed shape $v(x)$,

(ii) the natural boundary conditions on $v(x)$, and

(iii) two optimum integral equations furnishing the values of both λ_c and λ_s, in terms of $v(x)$, as

$$\lambda_c = \frac{12G_c \int_0^l (v')^2 dx - 6E_s t^2 \lambda_s \int_0^l (v'')^2 dx}{12G_c \int_0^l (v')^2 dx + (6E_s tc + E_c c^2) \int_0^l (v'')^2 dx}, \qquad \lambda_s = \frac{12G_s \int_0^l (v')^2 dx - 3E_s tc\lambda_c \int_0^l (v'')^2 dx}{12G_s \int_0^l (v')^2 dx + 4E_s t^2 \int_0^l (v'')^2 dx}. \tag{6}$$

Here and henceforth we will refer to this strategy for modelling the sandwich deflection as the "λ-constrained method based on the TPE". In the following, for the simple structures of interest here, we will provide analytical solutions for $v(x)$, to be coupled with (6) and given in terms of the parameter

$$\alpha = \sqrt{\frac{12G_c c(1 - \lambda_c)^2 + 24G_s t(1 - \lambda_s)^2}{2E_s t[t^2\lambda_s^2 + 3(c\lambda_c + t\lambda_s)^2] + E_c c^3\lambda_c^2}}, \tag{7}$$

conveniently defined in integrating the Euler–Lagrange equation governing $v(x)$. From the complexity of these solutions, it is evident that this method becomes much too cumbersome for more complicated structures.

Now we try to estimate the *effective* bending stiffness within this method. The assumed independence of λ_c and λ_s on x allows one to define an average curvature

$$\overline{\chi} = \frac{\lambda_c c + \lambda_s t}{d} v'',$$

so that (2) can be rewritten as

$$M = \left(D + \frac{btc}{12}(\lambda_s - \lambda_c)\frac{2E_s t^2 - E_c c^2}{\lambda_c c + \lambda_s t}\right)\overline{\chi}. \tag{8}$$

In regular sandwiches the skins are stiffer than the core, so that $\lambda_s > \lambda_c$ and the sign of the term added to D in the above relation coincides with the sign of

$$\frac{2E_s t^2 - E_c c^2}{\lambda_c c + \lambda_s t}.$$

This is difficult to foresee, although it easily turns out to be negative if the skins are very thin or the core is almost as stiff as the skins.

Three-point bending. In this case $\delta = v(x = l/2)$ is the maximum deflection, $v(x = 0) = v(x = l) = 0$, and $q(x) = 0$. The integration of the Euler–Lagrange equation for v furnishes

$$v(x) = \begin{cases} \dfrac{P\left[\alpha x - \operatorname{sech}(\alpha l/2)\sinh(\alpha x)\right]}{\alpha^3\left[E_s bt^3\lambda_s^2/3 + E_s bt(c\lambda_c + t\lambda_s)^2 + E_c bc^3\lambda_c^2/6\right]}, & x \in [0, l/2], \\ v(l - x), & x \in [l/2, l]. \end{cases} \tag{9}$$

The system constituted by (6)–(7) and (9) can be solved numerically using as initial guesses for α, λ_c, and λ_s the solutions furnished by the Ritz method, by, for instance, approximating $v(x)$ with $\hat{v}(x) = \hat{\delta}\sin(x\pi/l)$. The results of Bardella and Genna [2000; 2001] on three-point bending show a satisfactory accuracy for this method. This therefore gives ground to the hypothesis, also corroborated by the observation (5) about the natural boundary conditions, that in this case λ_c and λ_s are almost constant.

Propped cantilever beam supporting a uniform load **p**. This case is characterised by $q(x) = p$, $P = 0$, $v(x = 0) = 0$, $v'(x = 0) = 0$, and $v(x = l) = 0$. The integration of the Euler–Lagrange equation for $v(x)$, to be coupled with (6)–(7), reads

$$v(x) = \frac{pl^2}{\alpha^2\left[E_s bt^3\lambda_s^2/3 + E_s bt(c\lambda_c + t\lambda_s)^2 + E_c bc^3\lambda_c^2/6\right]}\left\{2\frac{\operatorname{sech}(\alpha l)}{\alpha^2 l^2}[\cosh(\alpha x) - 1] - \left(\frac{x}{l}\right)^2\right.$$
$$\left. + \frac{2[\operatorname{sech}(\alpha l) - 1] + \alpha^2 l^2}{\alpha^2 l^2[\alpha l - \tanh(\alpha l)]}\left[\tanh(\alpha l)[\cosh(\alpha x) - 1] + \alpha x - \sinh(\alpha x)\right]\right\}. \tag{10}$$

When discussing the results for this case, we will also consider the reaction force R within the support, whose estimate is obtained by evaluating (3) at $x = l$.

The Ritz method. The Ritz method allows the direct, albeit approximated, minimisation of the functional (1), without any constraint on the functions λ_c and λ_s. By exploiting this method, Minelli [2007] has verified that assuming the independence of λ_c and λ_s on x seems to be numerically appropriate for some simply-supported beams subjected to symmetric loading (including three and four-point bending).

Propped cantilever beam subjected to a uniform load. In this paper, we show that the "λ-constrained method based on the TPE" is quite inaccurate for this case, where we can obtain accurate results only by applying the Ritz method to the TPE, by choosing a discretisation dependent on 15 weights, to be determined by setting to zero the first variation of the TPE functional, (1). In particular, λ_s and λ_c are free from essential boundary conditions and have been approximated by means of a third-order polynomial (4 weights) and a seventh-order polynomial (8 weights), respectively, whereas the approximate deflection

$\hat{v}(x)$ has been chosen to depend on 3 weights ($w_1 \equiv \hat{v}(l/2)$, w_2, and w_3) in such a way that the natural boundary condition imposing zero overall moment at the support is a priori fulfilled[5], as

$$\hat{v}(x) = 4w_1\left[3\left(\frac{x}{l}\right)^2 - 5\left(\frac{x}{l}\right)^3 + 2\left(\frac{x}{l}\right)^4\right] + w_2\left[-\frac{5}{8}\left(\frac{x}{l}\right)^2 + \frac{19}{8}\left(\frac{x}{l}\right)^3 - \frac{11}{4}\left(\frac{x}{l}\right)^4 + \left(\frac{x}{l}\right)^5\right]$$
$$+ w_3\left[-\frac{27}{16}\left(\frac{x}{l}\right)^2 + \frac{93}{16}\left(\frac{x}{l}\right)^3 - \frac{41}{8}\left(\frac{x}{l}\right)^4 + \left(\frac{x}{l}\right)^6\right]. \quad (11)$$

2.2. *The Allen superposition method.* We give the formulae for the Allen superposition method [Allen 1969], applied to the cases we are concerned with here. The method was formulated for antiplane sandwiches and then extended to the nonantiplane case by adopting the further approximation that the displacement field along the sandwich core be linear.

Three-point bending. The maximum deflection [Allen 1969] is given by

$$\delta = \frac{Pl^3}{48D} + \frac{Pl}{4\hat{G}\hat{A}}\left(1 - \frac{E_s bt^3}{6D}\right)^2\left(1 - \frac{2}{\alpha_A l}\tanh\frac{\alpha_A l}{2}\right),$$

in which D is the bending stiffness as given by (4),

$$\hat{A} = \frac{bd^2}{c}, \quad \hat{G} = \frac{G_c}{1 + \frac{c^2}{6nt(t+c)}}, \quad n = \frac{E_s}{E_c}, \quad \alpha_A = \sqrt{\frac{6\hat{G}\hat{A}}{E_s bt^3\left(1 - \frac{1}{1+3(d/t)^2}\right)}}.$$

Note that for the antiplane case $\hat{G} = G_c$.

Propped cantilever beam supporting a uniform load. For this case we have derived the relevant results as follows. Allen's method gives the deflection as the superposition of two transversal displacements as $v(x) = v_1(x) + v_2(x)$, where, for the case concerned here,

$$v_1(x) = -\frac{6}{E_s bt(t^2+3d^2)}\left[\frac{C_1}{\alpha_A^3}\sinh(\alpha_A x) + \frac{C_2}{\alpha_A^3}\cosh(\alpha_A x) + \frac{p}{6}\left(lx^3 - \frac{x^4}{4}\right) - R\frac{x^3}{6}\right] + C_4\frac{x^2}{2} + C_5 x + C_6,$$

$$v_2(x) = \frac{6}{\alpha_A^2 E_s bt^3}\left[\frac{C_1}{\alpha_A}\sinh(\alpha_A x) + \frac{C_2}{\alpha_A}\cosh(\alpha_A x) + \frac{p}{2}(2lx - x^2) - Rx\right] + C_3,$$

in which R is the reaction force within the support (at $x = l$) that, together with the six integration constants C_i ($i = 1, \ldots, 6$) must be obtained by imposing seven independent boundary conditions. In Allen's method some boundary conditions are arbitrary. Based on the finite element model employed in order to test the various theories presented, we have chosen the boundary conditions

$$v_1(x = 0) = 0, \qquad v_1'(x = 0) = 0, \qquad v_1(x = l) = 0,$$
$$v_2(x = 0) = 0, \qquad v_2'(x = 0) = 0, \qquad v_2(x = l) = 0,$$
$$\left(1 + \frac{3d^2}{t^2}\right)v_1''(x = l) + v_2''(x = l) = 0,$$

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[5]The λ-constrained method is useful also because it could be exploited to obtain good shape functions, such as (10) for the propped cantilever beam. However, finding the optimum discretisation for the functions involved in the TPE is beyond the scope of this paper.

the last one imposing that the overall moment must vanish at $x = l$. In solving this system a limit of this model immediately emerges: the estimate of the reaction R turns out to be equal to the classical value of the homogeneous Bernoulli beam theory, $R = 3pl/8$, independently upon the sandwich geometry and material constants. We obtain for the midspan displacement

$$v\left(\frac{l}{2}\right) = \frac{pl^4}{32E_s bt(t^2 + 3d^2)}\left\{1 + \frac{18}{(\alpha_A l)^2}\left(\frac{d}{t}\right)^2\left[6 - \left(1 + \frac{10}{\alpha_A l}\sinh\frac{\alpha_A l}{2}\right)\mathrm{sech}^2\frac{\alpha_A l}{4}\right]\right\},$$

where, by the way, we would find it artificial to single out a "pure bending term" involving the bending stiffness D given by (4).

2.3. First-order shear deformation models.
Beside their analytical complication, both the Allen and the total potential energy methods have other drawbacks, most notably that of failing to yield accurate (or, even, *correct*) results for special cases, such as that of thin skins, or that of a homogeneous beam. For this reason FOSD models are developed in order to try and obtain the simplest possible models able to predict the sandwich deflection. Within this framework, the deflection of a sandwich beam is computed, under general loading and constraint conditions, by exploiting the well known formulae for the homogeneous Timoshenko beam. Each different FOSD model is characterised by a different choice of the shearing rigidity S [Timoshenko and Gere 1990], whereas there seems to be complete agreement among researchers about the use of the bending stiffness D provided by (4).

For statically determinate beams, the maximum deflection reads

$$\delta = \frac{Ql^3}{\omega_1 D} + \frac{Ql}{\omega_2 S},$$

where l is the beam length, Q is a proper combination of the transversal loads applied to the beam (Q is the magnitude of their resultant if they are all equally oriented), and ω_1 and ω_2 are numerical constants which depend on both loading and constraint conditions.[6]

Instead, in the case of a propped cantilever beam supporting a uniform transversal load p, the formulae for the midspan displacement and the reaction force within the support, respectively, read

$$v\left(\frac{l}{2}\right) = \frac{pl^4}{24D + 8l^2 S}\left(\frac{21}{16} + \frac{l^2 S}{24D} + \frac{3D}{l^2 S}\right), \qquad R = \frac{12D + 3l^2 S}{24D + 8l^2 S}pl.$$

In general, a complicated problem arises from the need to evaluate the sandwich shearing rigidity S.[7] In fact, as already pointed out, the presence of a relatively soft core and of thick skins makes the kinematics of a sandwich beam much different from that of a standard homogeneous beam. Hence, one of the limits of this approach is that it cannot predict local stress concentrations, that, in sandwich beams in which the

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[6]For instance, for a simply-supported beam subjected to four-point bending,

$$\omega_1 = \frac{32}{\left(1 - l_p/l\right)\left[1 - \frac{1}{3}\left(1 - l_p/l\right)^2\right]}, \qquad \omega_2 = \frac{4}{1 - l_p/l}, \tag{12}$$

where l_p is the distance between the two symmetrically applied concentrated loads, each of magnitude $P/2$, and $Q = P$. Of course the particular case of three-point bending is obtained by setting $l_p = 0$ in (12), so that $\omega_1 = 48$ and $\omega_2 = 4$.

[7]Even for homogeneous beams there is a body of literature, starting with Timoshenko [1921], about how to evaluate the shearing rigidity; see [Renton 1991; Hutchinson 2001] and references therein.

stiffness of the core is much lower than that of the skins, are amplified due to the inapplicability of Saint-Venant's principle. For instance, this approach neither can account for the stiffening due to overhangs (which is instead qualitatively describable with Allen's superposition theory [Allen 1969]), nor has any chance to capture the "normal stress inversion" in the core (as defined in Section 2.1).

A FOSD model based on an energy principle and Jourawsky's theory. Bardella and Genna [2000] have put forward a FOSD model based on the classical approximate shear force treatment by Jourawsky [1856] and on an energy principle by which it is possible to obtain the estimate S_E of the shearing rigidity S by equating the external and internal works of deformation,

$$\frac{V^2}{2S_E} = \frac{1}{2}\int_{A_c}\frac{\tau_c^2}{G_c}\,dA + \frac{1}{2}\int_{A_s}\frac{\tau_s^2}{G_s}\,dA, \tag{13}$$

where V is the shear force applied to the sandwich cross section, τ_c and τ_s are the total shear stresses in the core and skins, respectively, and A_c and A_s are the surfaces occupied by the core and the skins (of areas cb and $2tb$, respectively).

By initially assuming that plane sandwich sections remain plane during the deformation, one can develop the classical beam analysis for shear stresses to evaluate them over the sandwich cross section [Jourawsky 1856]. This allows the shear stresses to be estimated by some averages of them.[8] By considering the shear stress components along the shear force direction y and its normal z, Jourawsky's theory estimates $\overline{\tau}_{xz} = 0$ everywhere, whereas

$$\overline{\tau}_{xy}^{(s)} = \frac{VE_s}{2D}(td + c^2/4 - y^2),$$

$$\overline{\tau}_{xy}^{(c)} = \frac{V}{2D}[E_s td + E_c(c^2/4 - y^2)],$$

so that we take $\tau_s \approx \overline{\tau}_{xy}^{(s)}$ and $\tau_c \approx \overline{\tau}_{xy}^{(c)}$ in (13) and, after some lengthy algebra, we can approximate S_E with the following shear rigidity S_J, that can be written by singling out the form factor of a rectangular homogeneous beam as

$$S_J = \frac{5}{6}\left(\frac{1}{bcG_c(1+\alpha_c)} + \frac{4}{2btG_s(1+\alpha_s)}\right)^{-1}, \tag{14}$$

in which the interaction coefficients α_s and α_c, whose values provide a measure of the deviation from the Reuss bound of the shear stiffness, turn out to be, with $n = E_s/E_c$,

$$\alpha_c = \frac{\dfrac{c^3 t}{3n} + c^2 t^2\left(1 + \dfrac{7}{3n}\right) + 2ct^3\left(7 + \dfrac{4}{3n}\right) + 35t^4 + \dfrac{32t^5}{c} + \dfrac{32t^6}{3c^2}}{\dfrac{c^4}{6n^2} + \dfrac{5c^2 td}{3n} + 5t^2 d^2}, \tag{15}$$

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[8]Accounting for the *exact* linear elastic solution for the shear stresses (for example, provided in [Pagano 1970]) would lead to a much too cumbersome algebraic problem.

$$\alpha_s = \frac{\dfrac{c^6}{n^2 t^4} + \dfrac{12c^5}{nt^3} + \dfrac{12c^4}{t^2}\left(3 + \dfrac{2}{n}\right) + \dfrac{16c^3}{t}\left(9 + \dfrac{1}{n}\right) + 230c^2 + 167ct + 48t^2}{10c^2 + 25ct + 16t^2}. \tag{16}$$

The formulae (14)–(16) particularise nicely to the case of homogeneous beams, obtainable in various ways (for example, $t \to 0$, or $c \to 0$, or $n \to 1$ and $G_c/G_s \to 1$). This result turns out to be an explicit particularisation of the procedure proposed by Bert [1973] for the simplified treatment of the shear deformation in beams with heterogeneous cross section.

We finally note that the shearing rigidity (14)–(16) evaluated by the Jourawsky theory is related to a cubic displacement field that can be somehow richer than the piecewise linear warping the TPE approach is based on (see Figure 1). Therefore, contrary to what was pointed out about the bending stiffness (see Section 2.2), the shearing rigidity hidden behind the TPE minimisation can be higher than (14)–(16). Therefore, between these two methods, we find it difficult to a priori establish which is the most accurate.

The FOSD model based on thin skins and antiplane core. For the sake of completeness, in the discussion we will also consider a simpler model, in which the shearing rigidity is estimated under the assumptions of thin skins and antiplane core [Allen 1969], $S_{TA} = G_c bd$.

Discussion of other FOSD models. In order to avoid the lengthy computation required to obtain (14)–(16), which can be seen to be equivalent to the computation of the so-called form factor, some authors, for some other heterogeneous beams [Mai et al. 2007], prefer a less accurate estimate of the shearing rigidity. This is obtained by scaling the Voigt bound of the shear stiffness by the shear coefficient, defined as the ratio between the maximum shear strain and the shear strain average over the section, still evaluated through Jourawsky's theory [Timoshenko and Gere 1990]. This procedure may lead to highly inaccurate estimates of S (and in fact this is what happens in our benchmark) if the shear moduli of skins and core are very different from each other. In fact, under this circumstance, the Voigt bound is much too large. For this reason, we do not report the expression for the estimate of S related to this scheme, that could instead be appropriate for composite beams whose homogeneous parts all have their centre lying on the neutral axis.

Instead, we found out that multiplying the Reuss bound of the shear stiffness by the ratio between the maximum shear stress and its average turns into a better estimate of S, that reads

$$S_R = \frac{G_c}{1 + 2G_c/G_s} \frac{b^2(c + 2t)^2}{2D}\left(E_s td + E_c \frac{c^2}{4}\right).$$

Even though the use of S_R will be shown provide better results than those obtainable by employing S_J for the particular example of Section 3.2, in general, S_R is expected to be a worse estimate than S_J, and this is evident from the inadequate particularisation of S_R to the homogeneous case.

3. Comparison with finite element simulations

In order to verify the accuracy of the methods reported in Section 2, we have run some finite element simulations by means of the code ABAQUS [2006], on an arbitrary geometry of sandwich beam for several values of the ratios $n = E_s/E_c$ and t/c.

The mesh (recognisable in the deformed shape of Figure 2, related to a propped cantilever beam supporting a uniform load) consists of eight-noded plane stress continuum elements with reduced integration. The accuracy of the results has been checked by properly refining the mesh. How concentrated loads and constraints are modelled is extremely important when n becomes large, because of the lack of Saint-Venant's principle for these beams. Depending on the way concentrated loads are actually applied, there may be, for instance, the need to account for the core deformation along y [Frostig et al. 1992]. Since we are here interested in the *macroscopic* beam deflection (that neglecting the effects of the load diffusion, as explained in Section 1), in our analyses, concentrated external loads are modelled by applying four equal concentrated forces in the following four points of the section concerned: the top point of the upper skin ($y = -c/2 - t$), both the upper and the lower interface points between skin and core ($y = \pm c/2$), and the bottom of the lower skin ($y = c/2 + t$). Any support is modelled by constraining the displacement along y of all the nodes of the section concerned. Analogously, the encastré condition is enforced by pinning all the section nodes; the deflection is taken as the displacement of the section centre ($y = 0$).
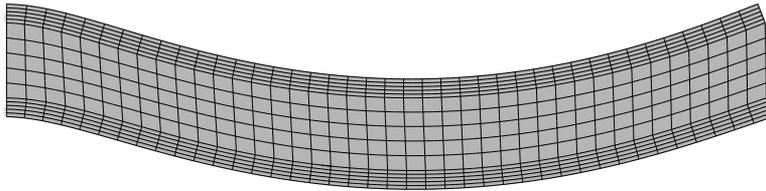
Figure 2. Finite element simulation of the deflection of a propped cantilever beam supporting a uniform load, deformed shape. The relevant geometrical and material parameters are $l = 200\,\text{mm}$, $t = 5\,\text{mm}$, $c = 20\,\text{mm}$, $E_s = 16000\,\text{MPa}$, $G_s = 6154\,\text{MPa}$, $E_c = 100\,\text{MPa}$, and $G_c = 50\,\text{MPa}$.

3.1. ***Three-point bending and other statically determinate cases.*** The results found in [Bardella and Genna 2000; 2001] are shown in Figure 3 for four relevant approaches among those considered in the previous section. The results are given in terms of the percentage error on the midspan deflection, computed using the finite element solution as a reference solution, and plotted against the ratio t/c for four different values of n. In the nonantiplane cases the relevant shear modulus of each layer is obtained by dividing the Young modulus by 2.6, whereas the antiplane case is approximated by $n = 1600000$, $G_s = E_s/2.6$, and $G_c = E_s/6.4$.

It is apparent that the Jourawsky-based FOSD method of page 1196 is the only one exhibiting acceptable accuracy over the whole range of variable considered; also, beside the inaccurate thin skin approximation, it is the only one whose equations particularise without relevant problems to both $t/c \to 0$ and $n \to \infty$.

Beside the fact that the complicated λ-constrained method of minimizing the TPE functional becomes too stiff when the skins are very thin,[9] it is not as accurate as expected. Since Minelli [2007] has verified,

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[9]This does not hold if the core is antiplane, the case in which the TPE approach becomes difficult to deal with from the numerical viewpoint if $t/c \ll 1$.
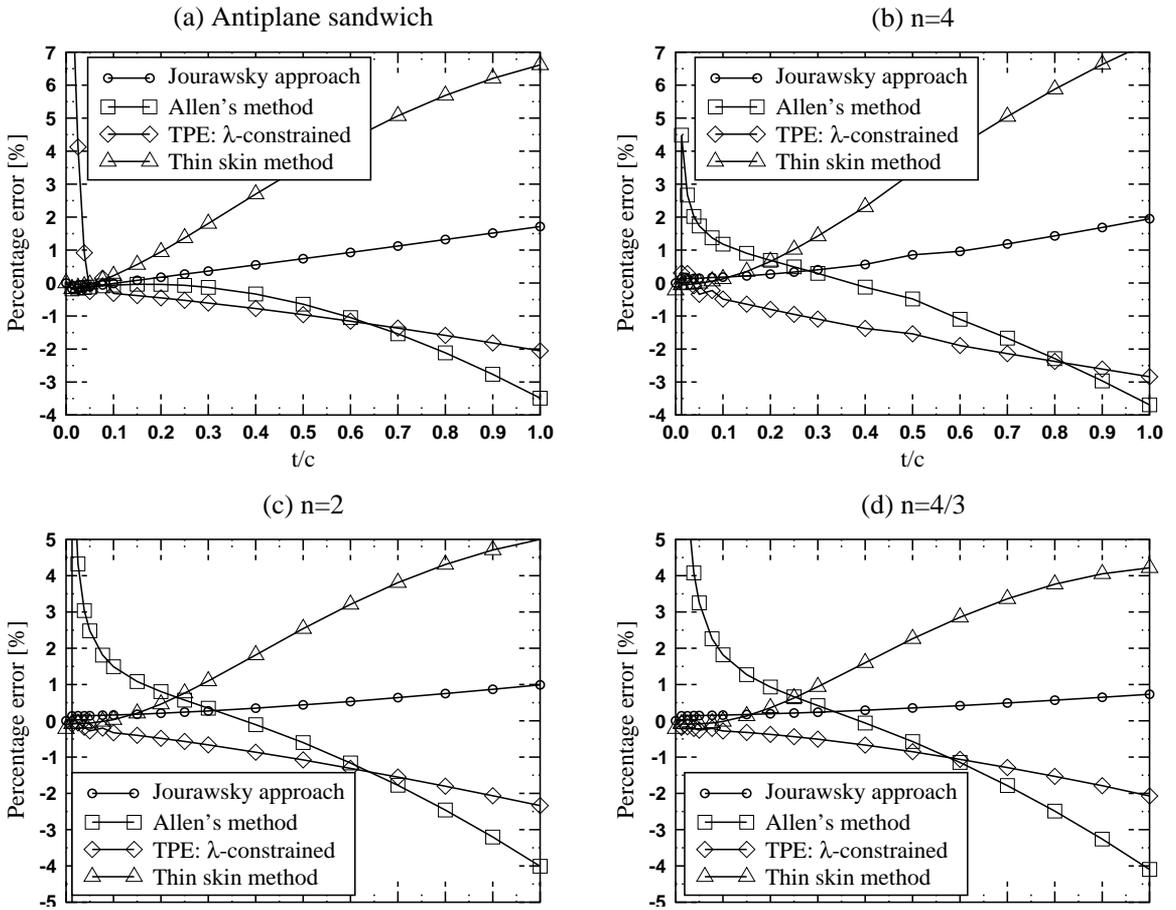
Figure 3. Comparison among different methods for evaluating the maximum deflection of a three-point bending sandwich beam with respect to finite element results, for the antiplane sandwich (upper left) and for $n = 4$, $n = 2$, and $n = \frac{4}{3}$.

by exploiting the Ritz method, that assuming λ_c and λ_s constant is numerically quite appropriate for three-point bending, we argue that the reason for the not-so-good accuracy lies in the inconsistency between the use of a trilinear kinematic model and the strain field ensuing from the shear stress distribution. Hence, the source of error is mostly due to the TPE approach itself, not to the approximated way the minimum is attained with the λ-constrained scheme.

In [Bardella and Genna 2000; 2001] similar results are given for a even larger range of t/c, in particular $t/c \in (0, 3]$. Also, for such unrealistic geometries, the FOSD method based on Jourawsky's theory turns out to be the most accurate, overall.

Similar conclusions as those above have been drawn by Minelli [2007] for other cases, such as those of simply-supported beams subjected to uniform load and four-point bending.

Gordaninejad and Bert [1989], based on [Bert 1973], developed and implemented into finite elements a FOSD model also suitable for sandwiches of unequal skins and layers with bimodular materials, whose

particularisation to the case of a symmetric cross section turns out to be coincident with the Jourawsky-based FOSD method of page 1196. For three different types of cross section, Gordananinejad and Bert successfully tested the method for the cases of cantilever and simply-supported beams subjected to uniform loads, even though they just compared the results with the classical ones, consisting of a FOSD theory in which the shearing rigidity is given by the Voigt upper bound of the shear stiffness.

3.2. *Propped cantilever beam supporting a uniform load.* We consider a sandwich beam characterised by the following relevant geometrical and material parameters: $l = 200$ mm, $t = 5$ mm, $c = 20$ mm, $E_s = 16000$ MPa, $G_s = E_s/2.6$, $E_c = 100$ MPa, and $G_c = E_c/2$. Many other combinations of parameters have been tested without finding any other relevant results beside those emerging from the quite realistic situation underlined by the given data. The finite element deformed shape is given in Figure 2, where, by the way, one can observe the trilinear warping at the simply-supported right end ($x = l$). Moreover, we note that in this case the effect of the core deformation along its thickness is negligible, even though $n = 160$; of course, this is also due to the way in which the constraints have been modelled.

For the models considered, Table 1 reports the percentage errors computed with respect to the finite element solution, referring to the midspan displacement $v(l/2)$, the reaction force R at the right end support ($x = l$), and the moment within the encastré ($x = 0$), computed as $M_0 = pl^2/2 - Rl$. Moreover, we consider the following global percentage errors on the discrepancy between the finite element and theoretical deformed shapes,

$$\mathcal{E}_1 = \frac{\sum_{i=1}^{N} |v_i^{(FE)} - v_i^{(TH)}|}{\sum_{i=1}^{N} |v_i^{(FE)}|} \times 100, \qquad \mathcal{E}_2 = \frac{\sum_{i=1}^{N} (v_i^{(FE)} - v_i^{(TH)})^2}{\sum_{i=1}^{N} (v_i^{(FE)})^2} \times 100, \qquad (17)$$

where $v_i^{(FE)}$ is the displacement evaluated by means of the finite element simulation in the i node on the beam longitudinal axis x, $v_i^{(TH)}$ is the corresponding displacement predicted by the theory considered, and N is the number of nodes lying on x in the mesh employed ($N = 41$, in the case concerned here).

The sole method providing results accurate enough is the direct Ritz method, with the discretisation described in Section 2.1, whose weights for the deflection (11) turn out to be $w_1 \approx 2.9585(p/b)$ mm^3/N,

Percentage errors	Error on $v(l/2)$	Error on R	Error on M_0	$\mathcal{E}_1$, (17)$_1$	$\mathcal{E}_2$, (17)$_1$
TPE: Ritz (15 weights)	−3.23	−1.21	6.72	3.62	0.137
TPE: Ritz (λ_s constant)	−3.35	4.39	−24.4	3.73	0.142
TPE: λ-constrained	−6.40	7.16	−39.9	7.46	0.518
Scaled Reuss bound	12.4	3.84	−21.4	18.6	3.58
Jourawsky approach	21.6	4.50	−25.1	28.5	7.64
Allen's method	26.9	−11.5	64.2	38.6	17.1
Thin skin method	51.2	6.28	−35.0	60.0	33.0

Table 1. Percentage errors, with respect to the results of a finite element simulation, of various models in predicting the midspan deflection $v(l/2)$, the support reaction force R, the encastré moment M_0, and the global deformed shape of a propped cantilever beam subjected to a uniform load. Geometric and material parameters: $l = 200$ mm, $t = 5$ mm, $c = 20$ mm, $E_s = 16000$ MPa, $G_s = 6154$ MPa, $E_c = 100$ MPa, and $G_c = 50$ MPa.
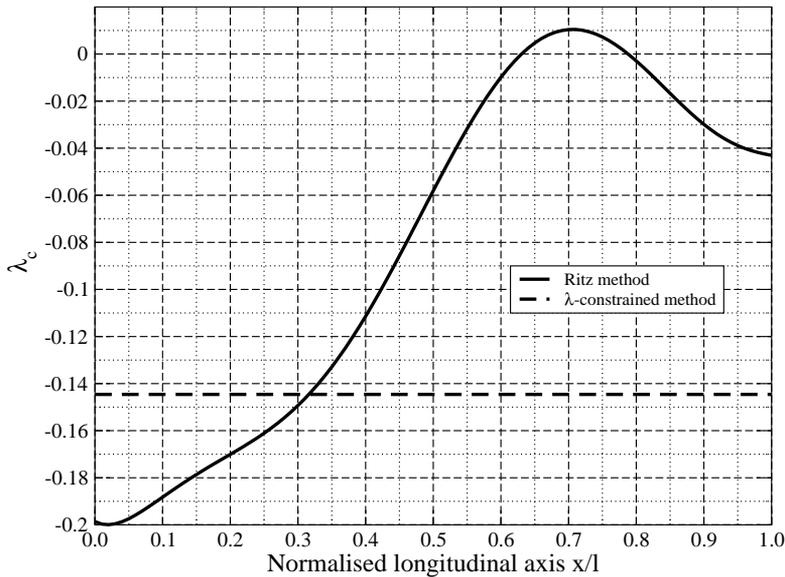
Figure 4. Ritz method for the variation of λ_c along the longitudinal axis of a propped cantilever beam supporting a uniform load. Geometric and material parameters: $l = 200\,\text{mm}$, $t = 5\,\text{mm}$, $c = 20\,\text{mm}$, $E_s = 16000\,\text{MPa}$, $G_s = 6154\,\text{MPa}$, $E_c = 100\,\text{MPa}$, and $G_c = 50\,\text{MPa}$.

$w_2 \approx -229.9(p/b)\,\text{mm}^3/\text{N}$, and $w_3 \approx 64.57(p/b)\,\text{mm}^3/\text{N}$. Such a solution highlights that for this case λ_c varies between a minimum value a little bit greater than $-t/c \equiv -0.2$ (which, as observed in Section 2.1, corresponds to an inefficient core if $\lambda_s \equiv 1$; see also [Allen 1969]) attained at the encastré ($x = 0$) and a maximum close to 0 in $x \approx 0.7l$ (see Figure 4). Even though λ_s varies within a quite limited range, as shown in Figure 5, its variation, too, has to be accounted for if one wants to be accurate in the stress computation. In fact, assuming λ_s constant (without changing the discretisation adopted for v and λ_c, so that the minimisation now depends on 12 weights) leads to an error on the reaction force R equal to 4.39%, producing a largely incorrect value of the moment M_0 at the encastré, which turns out to be underestimated by 24.4%. We have also verified that this discrepancy remains almost unaltered by improving the discretisations for v and λ_c while keeping λ_s constant (its value turns out to be $\lambda_s \approx 0.9915$). As it can be observed from Figures 4 and 5, the natural boundary conditions (5), in $x = l$, are approximately well met by the chosen discretisation (recall that we have a priori imposed the natural boundary conditions on $v(x)$ only).

Also the λ-constrained method to minimise the TPE provides a perhaps acceptable approximation of the deformed shape, but it is quite inaccurate in the reaction force computation. We note that an error of 7.16% on R leads to an underestimation of M_0 of about 40%, and, of course, this strongly and negatively affects the normal stress evaluation. This method gives in this case the stiffest response; this is confirmed by computing the estimate (8), by which the effective bending stiffness is larger than D by about 13%.

The actual inefficiency of the core at the encastré is confirmed in Figure 6, where, given a load $p = 1\,\text{N/mm}$ and a width $b = 50\,\text{mm}$, we report the contour of the normal stress σ_x obtained by means
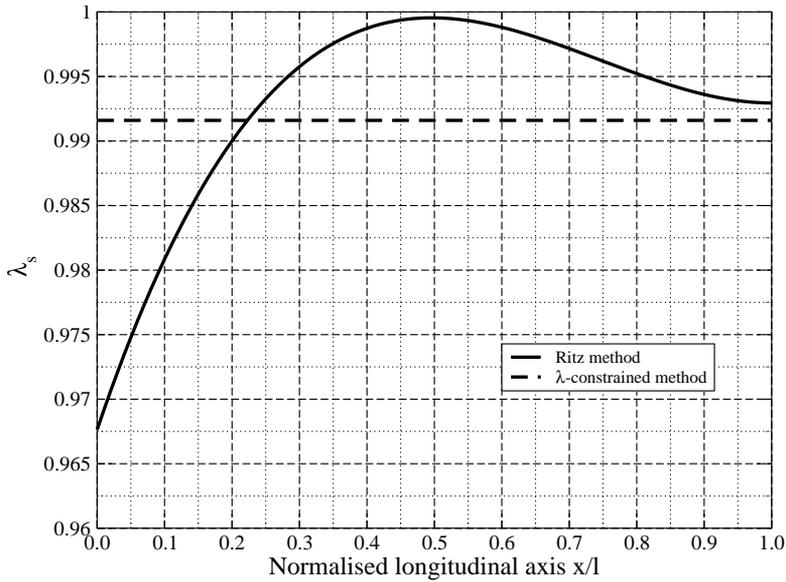
Figure 5. Ritz method for the variation of λ_s along the longitudinal axis of a propped cantilever beam supporting a uniform load. Geometric and material parameters: $l = 200\,\text{mm}$, $t = 5\,\text{mm}$, $c = 20\,\text{mm}$, $E_s = 16000\,\text{MPa}$, $G_s = 6154\,\text{MPa}$, $E_c = 100\,\text{MPa}$, and $G_c = 50\,\text{MPa}$.
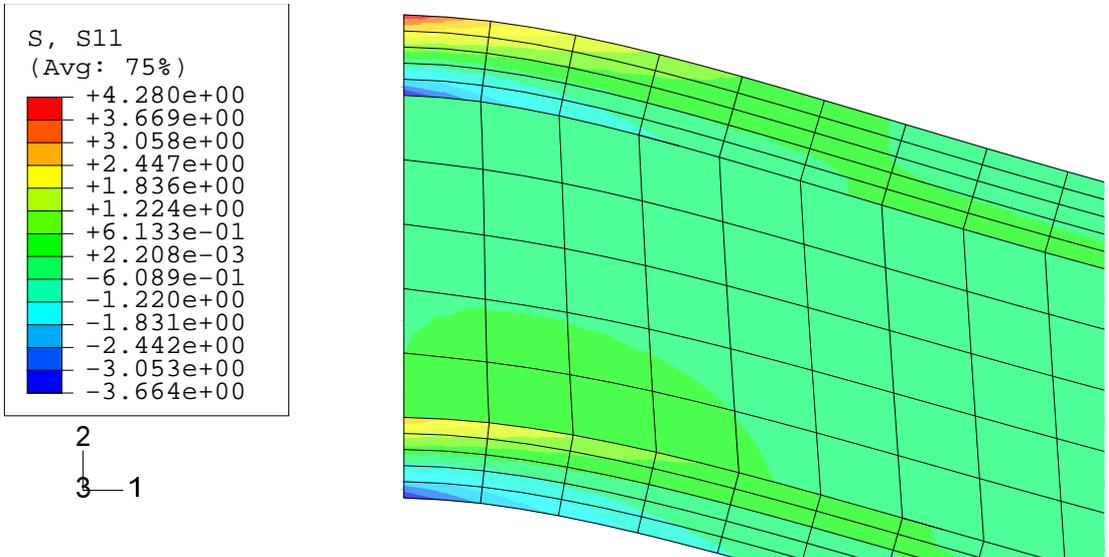
Figure 6. Finite element simulation of the deflection of a propped cantilever beam supporting a uniform load: normal stresses along the longitudinal axis at the encastré. The relevant data are $p = 1\,\text{N/mm}$, $l = 200\,\text{mm}$, $t = 5\,\text{mm}$, $c = 20\,\text{mm}$, $b = 50\,\text{mm}$, $E_s = 16000\,\text{MPa}$, $G_s = 6154\,\text{MPa}$, $E_c = 100\,\text{MPa}$, and $G_c = 50\,\text{MPa}$.

of the finite element analysis. Each skin mostly bends locally around its own midaxis parallel to z, both skins being subjected to substantial compression and tension in their lower and upper parts, respectively.

The foregoing observations imply that in this case the only procedure, among those presented, able to describe the stress distribution is the TPE minimisation accounting for the variations of both λ_c and λ_s. However, the chosen 15 parameter discretisation could be improved to obtain more accurate values for the stresses. In fact, for example, the normal stress in the skins is given by

$$\sigma_x = E_s\big[(\lambda_c' v' + \lambda_c'' v)c/2 + (\lambda_s' v' + \lambda_s'' v)(y - c/2)\big], \qquad y \in [-c/2 - t, -c/2] \cup [c/2, c/2 + t],$$

and, at the encastré, the employed discretisation allows one to predict, for both skins,

$$-2.23\,\text{MPa} \le \sigma_x \le +3.20\,\text{MPa},$$

to be compared with the contour of Figure 6; to this purpose, note that the maximum value reported in the legend is an approximated extrapolation, whereas the relevant maximum stress computed in a Gauss point is $\sigma_x = +3.54\,\text{MPa}$. We also note that computing the bending moment M_0 by evaluating Equation (2) in $x = 0$ leads to a value different by about 10% from $pl^2/2 - Rl$. Even though such a discrepancy diminishes as the discretisation is improved, in order to describe phenomena such as the slightly different bending behaviour of the two skins, due to the fact that in the finite element model the load is actually applied on the upper sandwich surface, one should also account for the core deformability along y (see, for example, [Krajcinovic 1971; Frostig et al. 1992]). On the other hand, one could obtain results closer to those predicted by the beam theory adopted if the distributed load were uniformly applied as a body force in the finite element simulation; in this case, for the normal stress computed at the Gauss points at the encastré section, we obtain $-2.73 \le \sigma_x \le +3.28\,\text{MPa}$, equally distributed on both skins.

Finally, we note that the FOSD order model based on Jourawsky's theory is much more accurate than Allen's superposition model, mostly concerning the support reaction force. In this regard, the accuracy of the Jourawsky approach is far better even than that obtained by means of the TPE λ-constrained method. The results of the scaled Reuss bound (the FOSD model described just before Section 3) are believed to be so good by chance, and are reported just for the sake of curiosity. The thin skin approximation fails badly, also because here the skins are quite thick.

4. Discussion and concluding remarks

We have considered various methods for computing the deflection $v(x)$ of sandwich beams, of identical skins, subjected to bending moment and shear, mostly aiming at finding insight on the reliability of FOSD models, that is, those making use of the standard formulae for displacements of the homogeneous Timoshenko beam.

A simple and often accurate FOSD method, put forward in [Bardella and Genna 2000] (see also [Bert 1973]), consists of estimating the shearing rigidity by means of an energy principle and the Jourawsky approximate treatment [Jourawsky 1856] of the shear problem for beams. The only difference between this Jourawsky approach and other FOSD theories, such as the well known method based on the approximation of thin skins and antiplane core [Allen 1969], lies in the evaluation of the shearing rigidity S. Instead, all these methods estimate the bending stiffness D by assuming that plane sandwich sections

remain plane during deformation, corresponding to a linear displacement field, whereas the estimation of S by means of Jourawsky's theory turns out to be based on a cubic displacement field.

The FOSD method based on the Jourawsky approach has been shown to furnish accurate results for the cases of three and four-point bending and for simply-supported beams subjected to uniform load [Gordaninejad and Bert 1989; Bardella and Genna 2000; 2001; Minelli 2007], all statically determinate, symmetrically loaded structures in which the bending moment does not change sign along the beam length. Concerning the interlaminar shear stress computation, Heller [1969] determined the conditions under which there is a satisfactory agreement between the results of Yu's model [1959] (based, as our TPE approach, on the kinematics of Figure 1) and those obtained by means of Jourawsky's formulae, still in sandwich beams whose shear force is statically determinate. In particular, Heller exploited the cases of a cantilever beam with a concentrated load at the free end and of a beam with both ends fixed, subjected to a uniform load. This allowed Heller to skip the crucial problem of choosing a FOSD model for the deflection evaluation. For such particular cases, Heller concluded that the Jourawsky theory is accurate enough if $n < 100$ or if $t/c \ll 1$ (see also [Allen 1969]).

Clearly, one of the advantages of any FOSD method consists of allowing the use of the classical formulae for the deflection of homogeneous beams to compute the deflection of sandwich beams, whereas both the Allen superposition method and the TPE approach may require, for every different set of boundary conditions, a new ad hoc integration that may be quite complicated.

The TPE approach is rigorously based on a suitable kinematics describing the warping. Here, we have considered the trilinear warping represented in Figure 1, known to provide an accurate description of the sandwich behaviour. Because of the chosen kinematics, hidden behind the minimisation of the TPE functional there has to be a bending stiffness lower than, and, probably, a shearing rigidity larger than those employed in the FOSD model based on the Jourawsky approach. We think that this is the main reason why it is a priori uncertain which of these methods provides more accurate results.

Moreover, the analytical minimisation of the TPE functional based on the trilinear kinematics of Figure 1 is unfeasible, even for the simplest sandwich structures. Bardella and Genna [2000] have found a closed-form solution based on the hypothesis that both the ratio between the rotations of the core and of the longitudinal axis, λ_c, and the ratio between the rotations of the skins and of the longitudinal axis, λ_s, are constant (or independent on the position along the longitudinal axis x). This hypothesis turns out to be appropriate for the cases mentioned above, in which the Jourawsky approach also exhibits good accuracy, often also in terms of stresses.

We have found that in the case of a propped cantilever beam supporting a uniform load, for a particular but not totally unrealistic choice of geometrical and material parameters, the direct numerical minimisation of the TPE functional (by a proper approximation of the unknown functions — v, λ_c, λ_s — involving 15 weights to be computed by imposing stationarity) is the only way, among those presented, to provide accurate results in terms of both displacement and bending moment, having also shown that the assumption of independence of both λ_c and λ_s upon x is inappropriate in this case. In particular, even though λ_s turns out to vary within a limited range ($\lambda_s \in (0.968, 1)$ in the benchmark considered), we have observed that assuming λ_s to be constant (in the literature it is even often set $\lambda_s = 1$) in the TPE minimisation can be acceptable for the deflection computation, but may lead to quite inaccurate values for the stresses.

Finally, based on all above, we put forward that the convenient FOSD method based on Jourawsky's theory should always be able to describe the deflection v when λ_c and λ_s turn out to be approximately constant over the sandwich length, which seems to happen mostly when the bending moment does not change sign along the sandwich length. Getting a better insight on this could be extremely useful for engineering practice.

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