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PERIODIC CONTACT PROBLEMS IN PLANE ELASTICITY

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Various methods for solving the partial contact of surfaces with regularly periodic profiles — which might arise in analyses of asperity level contact, serrated surfaces or even curved structures — have previously been employed for elastic materials. A new approach based upon the summation of evenly spaced Flamant solutions is presented here to analyze periodic contact problems in plane elasticity. The advantage is that solutions are derived in a straightforward manner without requiring extensive experience with advanced mathematical theory, which, as it will be shown, allows for the evaluation of new and more complicated problems. Much like the contact of a single indenter, the formulation produces coupled Cauchy singular integral equations of the second kind upon transforming variables. The integral equations of contact along with both the boundary and equilibrium conditions provide the necessary tools for calculating the surface tractions, often found in closed-form for regularly periodic surfaces. Various loading conditions are considered, such as frictionless contact, sliding contact, complete stick, and partial slip. Solutions for both elastically similar and dissimilar materials of the mating surfaces are evaluated assuming Coulomb friction.

1. Introduction

Understanding the microscopic interaction of real rough surfaces in contact is difficult. Mathematical solutions for the contact of real rough surfaces are complicated and closed form solutions are limited. One simplifying assumption is to approximate asperities or even smooth wavy surfaces as sinusoidal, which allows for greater analytic feasibility, assuming the wavelength is much larger than the amplitude of the wave. Physically, the sine wave represents the first term in a Fourier decomposition of an actual rough surface. Asperity contact and geometrically wavy surfaces are the centerpieces of tribological phenomena such as friction, wear and fracture. Other periodic geometries are of particular tribological and mechanical interest as well. In addition to the normal contact of periodic surfaces, other modes of contact such as sliding, partial slip, and complete stick of mating surfaces become increasingly important when studying different types of failure, such as fretting fatigue, creep failure, and crack nucleation.

The problem of smooth, elastic periodic surfaces in contact has been solved using various methods and in many different contexts; many of the contributions to the area of regularly wavy contact are described below. The first solution is credited to Westergaard [1939], who found a closed form solution for the contact problem of an elastic half-space with a wavy surface by utilizing complex stress functions for a sinusoidal normal pressure. Westergaard derived expressions for the surface tractions and the dependence of the contact area on applied pressure. He also showed that for light loading, that is, for a small area of contact, Hertzian contact of cylinders is recovered. England and Green [1963] also employed complex

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potential functions to solve the problem of an infinite row of punches upon a half-space. Their formulation reduces to an integral of the Abel type to determine the extent of contact.

Using a stress function approach, Dundurs et al. [1973] solved an integral of the Abel type for two wavy surfaces with aligned peaks of the same amplitude. The mixed boundary conditions for the contact and stress free regions lead to a set of dual series equations [Sneddon 1966]. Exploiting the orthogonality of Legendre polynomials and the Mehler–Laplace integral yields closed form solutions for the extent of contact for a given level of loading and distribution of contact pressure. Johnson [1985b] obtained closed form asymptotic results for light loading and nearly complete contact of surfaces having orthogonal two-dimensional waves of equal amplitude and wavelength.

Two books on contact mechanics, [Hills et al. 1993] and [Johnson 1985b], dedicate entire chapters to periodic contact problems. Starting with the Flamant solution for a concentrated force on a half-space, both sources derive integral equations with a Hilbert kernel to determine surface stresses. Analytical solutions for periodic contact are limited in both books; they cite the Westergaard solution and then statistical and numerical methods for rough contact. A recent book by Cai and Lu [2000] extended Muskhelishvili's approach [Muskhelishvili 1992] to periodic plane elasticity. Expressed as Riemann–Hilbert boundary value problems, they are solved using complex variables and Hilbert kernels. Systematically, the authors work out solutions for periodic contact problems, crack problems, anisotropic surfaces, moving loads, and finally doubly periodic problems. Frictionless normal contact and sliding contact with Coulomb friction are worked out for flat, inclined, and circular punches.

Kryshtafovych and Martynyak [2000] considered frictional contact of elastic half-spaces with periodic surface relief for isotropic and anisotropic solid surfaces. The method of interface gaps reduces the problem to a singular integral equation, which is numerically solved. The authors also derived a cotangent kernel for periodic profiles in a general form, which is reduced to a Hilbert kernel. A closed form solution was obtained for an anisotropic half-plane in contact with a rigid wavy body [Krishtafovich et al. 1994]. A series of papers by Kuznetsov on periodic contact with friction [Kuznetsov 1975], with a fluid lubricant [Kuznetsov 1985], for a polymeric material [Kuznetsov and Gorokhovsky 1978], and with depth-varying Poisson's ratio [Kuznetsov 1983] have been published, in which he utilizes the theory of automorphic functions. Panek [1975] solved the normal contact of an elastic, regularly wavy strip pressed between flat surfaces by reposing the problem as an infinite, straight strip indented by regularly wavy surfaces. He determined the local contact stress and heat conduction of the wavy strip by means of dual series equations and a Fredholm integral equation of the second kind.

Nosonovsky and Adams [2000] studied the dry, steady-state frictional sliding of elastic, wavy bodies in two dimensions. A Cauchy singular integral equation of the second kind is derived from integral transforms and Fourier series and then numerically solved using Jacobi polynomials. Based on Staierman's general solutions for periodic contact [Schtaierman 1949], Ciavarella [1998a; 1998b] provided Cattaneo–Mindlin partial slip results for a few periodic geometries with two bodies of the same material. Ciavarella assumes that the periodic profiles chosen demonstrate self-similarity and thus the correction term in the shear traction is of the same form as the normal traction. He also explains why flat punches must either completely stick or slip entirely and cannot undergo partial slip for elastically similar materials. Manners [1998] analyzed more complicated periodic half-space profiles in partial contact with a flat rigid plane. The crux of the analysis is based on a cotangent transform, which allows for a simple method of finding the surface pressure for wavy surfaces with a finite number of harmonics. Manners later extended his analytical method to a numerical one for random rough profiles in order to determine the extent of contact, which is not known a priori [Manners 2003].

Carbone and Mangialardi [2004] recently published a paper on the adhesion and friction between an elastic half-plane and a rigid wavy surface which has application to tire-road contact. By imposing a sliding velocity that is much smaller than the sound velocity in the solids, an eccentricity, e, is created. The authors also require the large-scale viscoelastic behavior to be neglected so that linear elasticity theory can be employed. The eccentricity gives rise to an antisymmetric contribution to the surface stress. The adhesion is modeled by superposing Koiter's solution for an infinite row of collinear cracks with the contact problem, as was first performed by Johnson [1995]. Linear elastic fracture mechanics provides equilibrium values for the eccentricity and contact region of the adhered surfaces. Friction coefficients and conditions for the surfaces to jump in and out of contact are evaluated. Carbone and Decuzzi [2004] solved the solution for an infinitely long elastic beam adhered to a wavy foundation. It was found that the surface energy, amplitude, and thickness of the beam are the main contributors to the deformation of the strip. Adams [2004] considered the adhesive contact of two elastic wavy surfaces assuming Maugis' model of adhesion [Maugis 2000]. Papkovich–Neuber potentials transform the boundary conditions into a triple series, which leads to a singular integral equation that is solved numerically. A loading cycle is developed, which is characterized by discontinuities in the form of a hysteresis.

As shown above, periodic contact problems have received much attention and analysis. At the same time, many rather simple periodic problems in plane elasticity remain unsolved. This fact is partly due to the complicated nature of the mathematical techniques previously used and partly due to a misunderstanding of the capability of the Flamant solution. It will be shown here that for periodic contact, a summation of periodically spaced Flamant solutions must be used. The resulting integral equations result in a cotangent kernel, which can be transformed to a Hilbert kernel by a simple change of variables. This cotangent kernel was first derived by Schtaierman [1949] for a very general case of periodic contact and later by Schmueser and Comninou [1979] for a periodic array of interface cracks. The advantage of this formulation is that solutions can be solved in a closed form in a rather straightforward manner and reduced to the single contact solution in the limit as the period approaches infinity. Moreover, the formulation has a simple physical explanation, as opposed to a gap function [Kryshtafovych and Martynyak 2000], Riemann–Hilbert approach [Cai and Lu 2000] or complex potential approach [England and Green 1963].

The governing equations of elastic contact mechanics and assumptions used are discussed. A periodic array of Flamant solutions on a half-space is used to derive a cotangent kernel for periodic contact problems. Following the general nomenclature and methodology outlined by Hills et al. [1993], an integral equation approach for periodic contact problems is developed. A modification to the kernel to ensure moment equilibrium is introduced and finally the governing equations for periodic plane elasticity are laid out.

Numerous periodic contact problems are solved. A few known solutions are corroborated from the coupled integral equations in order to demonstrate the method. New and more complicated solutions are derived for various periodic profiles, elastically dissimilar materials and types of contact, such as sliding, partial slip and complete stick.

2. Theory of periodic contact in plane elasticity

Assumptions. It will be assumed throughout that the bodies involved are linear elastic, isotropic materials. For isotropic bodies, Poisson's ratio $\nu$and shear modulus μ are used. A small strain theory of elasticity is assumed, where the contact area is significantly smaller than the radii of curvature of the undeformed surfaces. Under these assumptions, the stresses of each surface can be calculated with good accuracy by treating each contacting body as a semiinfinite body bounded by a plane surface, also called a half-space. Such an idealization was first introduced by Hertz [1881].

Contact mechanics. For a distribution of normal and shear tractions — $p(x)$ and $q(x)$ respectively — on a half-space's free surface and depicted in Figure 1, it was shown in [Johnson 1985b] that a pair of coupled integral equations result:

$$\frac{1}{A}\frac{\partial h}{\partial x} = \frac{1}{\pi}\int \frac{p(\xi)}{x-\xi}d\xi - \beta q(x), \tag{1}$$

$$\frac{1}{A}\frac{\partial g}{\partial x} = \frac{1}{\pi}\int \frac{q(\xi)}{x-\xi}d\xi + \beta p(x), \tag{2}$$

where $h(x) = v_1(x) - v_2(x)$ and $g(x) = u_1(x) - u_2(x)$ are the differences in displacements between the upper and lower surfaces, represented by the subscripts 1 and 2 respectively. The integral equations were derived using the Flamant solution. Also, the mismatch in material parameters between the two surfaces couples the two integral equations and results in

$$A = \frac{\kappa_1 + 1}{4\mu_1} + \frac{\kappa_2 + 1}{4\mu_2}, \qquad \beta = \frac{\Gamma(\kappa_1 - 1) - (\kappa_2 - 1)}{\Gamma(\kappa_1 + 1) + (\kappa_2 + 1)}, \tag{3}$$

where A is the compliance between the bodies, β is Dundur's parameter representing the elastic mismatch, and $\Gamma = \mu_2/\mu_1$, as defined in [Hills et al. 1993]. The bulk modulus, κ, is defined as $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plain stress, with subscripts for the respective surfaces. When both bodies are of the same materials, Dundur's parameter is zero. When one body is rigid, it becomes $\beta = (1-2\nu)/2(1-\nu)$.

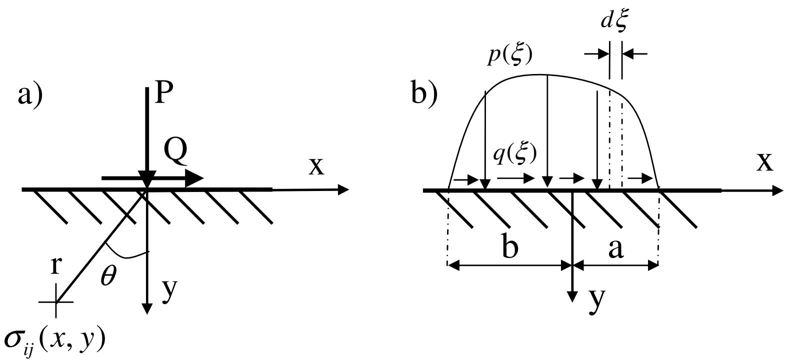

Figure 1. An elastic half-space (a) under a concentrated normal and tangential line force and (b) under an arbitrary distributed normal and tangential pressure.

Lastly, equilibrium of the external forces, P and Q, needs to be ensured and is done so with

$$P = \int p(\xi)\,d\xi, \qquad Q = \int q(\xi)\,d\xi.$$

Equations (1) and (2) are the foundations of plane elastic contact mechanics. Many well-known solutions are obtained using them; see [Hills et al. 1993] and [Johnson 1985b].

Periodic contact mechanics. The solution for periodic regions of distributed normal and shear tractions can be derived in a similar fashion to that of a single region. Taking the Flamant solution as the starting point, consider infinitely many evenly-spaced concentrated normal, compressive forces on a half-space's free surface depicted in Figure 2a. On the free surface, the normal displacements in Cartesian coordinates are

$$v(x) = -P\left(\frac{\kappa+1}{4\pi\mu}\right)\ln|x| - \sum_{n=1}^{\infty} P\left(\frac{\kappa+1}{4\pi\mu}\right)\ln|nL+x| - \sum_{n=1}^{\infty} P\left(\frac{\kappa+1}{4\pi\mu}\right)\ln|nL-x| + \frac{C_2}{2\mu}, \tag{4}$$

where C_2 is an arbitrary rigid body displacement. Taking the derivative of (4) to remove C_2 gives

$$\frac{dv(x)}{dx} = -P\left(\frac{\kappa+1}{4\pi\mu}\right)\frac{1}{x} - \sum_{n=1}^{\infty} P\left(\frac{\kappa+1}{4\pi\mu}\right)\frac{1}{nL+x} - \sum_{n=1}^{\infty} P\left(\frac{\kappa+1}{4\pi\mu}\right)\frac{1}{nL-x}. \tag{5}$$

Using the identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{x+nL} = \frac{\pi}{L}\cot\frac{\pi x}{L},$$

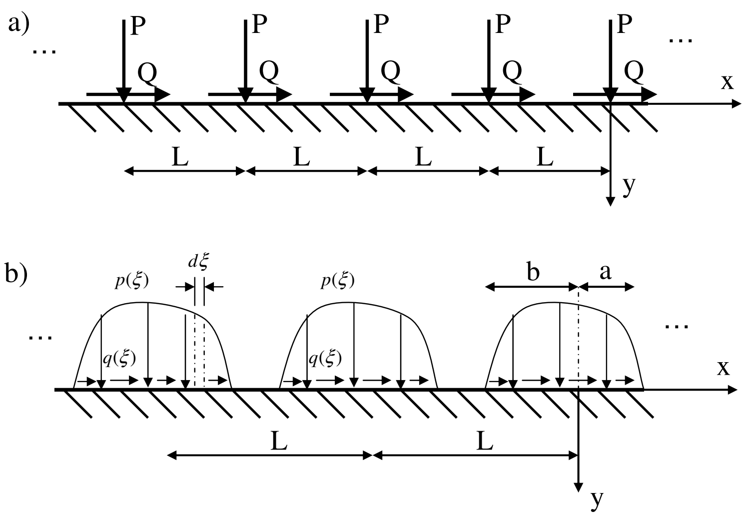

Figure 2. A half-space loaded by (a) an array of evenly spaced concentrated normal and tangential line forces and (b) periodic distributions of normal and shear surface tractions.

allows the displacement derivative in (5) to be rewritten as

$$\frac{dv(x)}{dx} = -P\left(\frac{\kappa+1}{4L\mu}\right)\cot\frac{\pi x}{L}. \tag{6}$$

Equation (6) gives the normal displacement due to evenly spaced concentrated forces on a half-space. This result can now be extended to periodic regions of distributed normal and shear tractions on the half-space's flat free surface, as shown in Figure 2b. The resulting integral equations for contact are

$$\frac{dv(x)}{dx} = \frac{\kappa+1}{4L\mu}\int p(\xi)\cot\Big(\frac{\pi}{L}(x-\xi)\Big)\,d\xi - \frac{\kappa-1}{4\mu}q(x), \tag{7}$$

$$\frac{du(x)}{dx} = \frac{\kappa+1}{4L\mu}\int q(\xi)\cot\Big(\frac{\pi}{L}(x-\xi)\Big)\,d\xi + \frac{\kappa-1}{4\mu}p(x), \tag{8}$$

where the following result was used for the displacements normal to the surface in order to determine the second term in the right hand side of (7),

$$\frac{dv(x)}{dx} = -\frac{\kappa-1}{4\mu}\sum_{n=-\infty}^{\infty}\int q(\xi)\delta(x-\xi+nL)\,d\xi = -\frac{\kappa-1}{4\mu}q(x),$$

and similarly for the tangential displacements in (8). If the relative difference in displacements between two elastic bodies in contact is considered, the integral equations become

$$\frac{1}{A}\frac{\partial h}{\partial x} = \frac{1}{2\pi}\int_b^a p(\xi)\cot\frac{x-\xi}{2}\,d\xi - \beta q(x), \tag{9}$$

$$\frac{1}{A}\frac{\partial g}{\partial x} = \frac{1}{2\pi}\int_b^a q(\xi)\cot\frac{x-\xi}{2}\,d\xi + \beta p(x), \tag{10}$$

where $L = 2\pi$ has been chosen for simplicity. The integral is taken over only a single contact region, $[b, a]$, since the periodicity of the problem guarantees that each period will produce the same result. The equations are easily modified for an arbitrary period, so long as the amplitude of the contacting surface is small compared to the wavelength, L, an assumption critical to the elasticity theory approximation.

Equations (9) and (10) are the foundations of elastic periodic contact mechanics in Cartesian coordinates; accordingly, the equations and their cotangent kernel will be referred to regularly and taken as the starting point of the physical problem to be solved. Before solving specific problems, a few comments should be made. The periodic integral equations resemble those for a single region of contact, (1) and (2), except that the kernel is now a cotangent kernel instead of a Hilbert kernel, whose solutions and transforms have been studied extensively. However, an appropriate change of variables transforms the cotangent kernel into a Hilbert kernel. That is, let

$$u = \tan\xi/2, \qquad v = \tan x/2, \qquad \text{and} \qquad \alpha = \tan a/2. \tag{11}$$

Note that for the rest of the paper, u and v will only be used to represent these transformations and not the normal and tangential displacements defined above. Then (9) becomes

$$\frac{1}{A}\frac{\partial h}{\partial x} = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{p(u)(1+uv)}{(v-u)(1+u^2)}\,du - \beta q(v), \tag{12}$$

where the contact region has been centered on the interval $[-a, a]$. The derivative displacement term is kept in terms of x, but typically needs to be changed to the transformed variables to solve for the surface tractions. The change of variables in (11) will be used and referred to regularly. To simplify the integral in (12) and the corresponding one for the tangential displacements, add and subtract u^2 in the numerator to get

$$\frac{1}{A}\frac{\partial h}{\partial x} = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{p(u)}{v-u}du + \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{up(u)}{1+u^2}du - \beta q(v), \tag{13}$$

$$\frac{1}{A}\frac{\partial g}{\partial x} = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{q(u)}{v-u}du + \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{uq(u)}{1+u^2}du + \beta p(v). \tag{14}$$

The first integral on the right-hand side is a Cauchy singular integral and will also be referred to as the Hilbert kernel. The second integral is a constant which represents physically the moment on the surface due to periodic tractions, $p(u)$, given by

$$M = \frac{1}{L}\int_{-a}^{a} p(x)\tan\frac{\pi x}{L}dx = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{up(u)}{1+u^2}du.$$

When $p(u)$ is symmetric, it is seen that $M = 0$ and only the Hilbert kernel integral remains in (13). If $p(u)$ is asymmetric however, $M \neq 0$ and the cotangent kernel can sometimes be modified to eliminate the unprescribed moment on the surface. The shear tractions however cannot produce a moment and must therefore be removed. This issue is discussed by Cai and Lu [2000, pp. 57–60], who comment that an extra term must be included in the kernel

$$\frac{1}{A}\frac{\partial h}{\partial x} = \frac{1}{2\pi}\int_{b}^{a} p(\xi)\left(\cot\frac{x-\xi}{2} - \tan\frac{\xi}{2}\right)d\xi - \beta q(x), \tag{15}$$

$$\frac{1}{A}\frac{\partial g}{\partial x} = \frac{1}{2\pi}\int_{b}^{a} q(\xi)\left(\cot\frac{x-\xi}{2} - \tan\frac{\xi}{2}\right)d\xi + \beta p(x). \tag{16}$$

It is seen that the change of variables from (11) removes the moment term. Thus, (15) and (16) will sometimes be the starting point for solving problems when equilibrium requires removing the unwanted moment. Furthermore, equilibrium is achieved by balancing the total normal and tangential loads

$$P = \int_{-a}^{a} p(x)dx = \int_{-\alpha}^{\alpha}\frac{2p(u)}{1+u^2}du, \tag{17}$$

$$Q = \int_{-a}^{a} q(x)dx = \int_{-\alpha}^{\alpha}\frac{2q(u)}{1+u^2}du. \tag{18}$$

The integral equations (15)–(16) and the conditions (17)–(18) form the basis of solving periodic contact problems. Using the change of variables (11), some well-known solutions will be derived in simpler ways and new solutions will be solved in a closed form in the next sections.

3. Problems in periodic contact: frictionless contact

Normal frictionless contact of periodic profiles. The simplest problems in contact mechanics are indentations without friction in normal contact with the free surface of a half-space. The no-friction assumption decouples the integral equations of periodic contact, thus increasing the availability of closed-form solutions. In this section, various periodic profiles, such as periodic flat punches and sinusoids, are considered. The surface geometry provides the necessary boundary conditions to solve the mixed boundary value problem in order to determine the surface stresses.

Sinusoids: Westergaard's solution. The solution of a wavy surface in contact with a flat surface, as shown in Figure 3, has practical importance in tribology and solid mechanics. The first mathematical solution was credited to Westergaard [1939], who used complex stress functions to obtain the contact stress. Later, Dundurs et al. [1973] employed a stress function approach, which required solving a set of dual series equations. Using the approach delineated in the previous section, the Westergaard solution is rederived here.

The surface profile for a sinusoidal surface with period 2π is

$$h(x) = \delta - \Delta(1 - \cos x), \tag{19}$$

where $h(x)$ is the difference in the normal displacements, δ is the approach and Δ is the amplitude of the undeformed surface profile. Physically, $h(x)$ represents the amount of interpenetration of the bodies and is a geometrical constraint condition. Differentiating Equation (19) to remove the rigid body displacement gives

$$\frac{\partial h(x)}{\partial x} = \Delta \sin x. \tag{20}$$

Transforming (20) using (11) and then substituting into (13) yields

$$\frac{\Delta}{A}\frac{2v}{1+v^2} = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{p(u)}{v-u}\,du + \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{up(u)}{1+u^2}\,du - \beta q(v).$$

It should be noted that the axes are positioned at the center of the contact region to exploit symmetry; this assumption will be employed throughout the paper unless otherwise specified. For frictionless contact or contact of smooth surfaces, $q(x) = 0$, and the integral equations are uncoupled. For two bodies with elastically similar materials, $\beta = 0$, which also removes the coupling term. The symmetry of the problem removes the second integral term on the right-hand side (since $p(u)$ is even) and gives

$$\frac{2\Delta}{A}\frac{v}{1+v^2} = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{p(u)}{v-u}\,du,$$

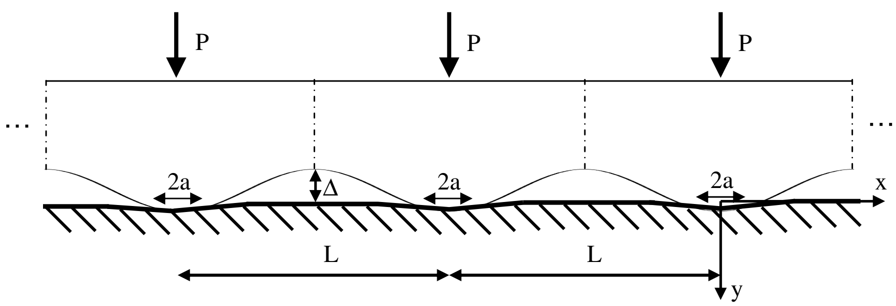

Figure 3. Normal contact of a sinusoidal profile without friction.

which is a Cauchy singular integral equation of the first kind for determining $p(u)$. Section A.1 provides inversion that is nonsingular at both endpoints,

$$p(u) = \frac{2\Delta}{\pi A}\sqrt{\alpha^2 - u^2}\int_{-a}^{\alpha} \frac{v}{1+v^2}\frac{1}{\sqrt{\alpha^2 - v^2}}\frac{1}{v-u}\,dv.$$

Using partial fractions and evaluating the resulting integrals gives

$$p(u) = -\frac{2\Delta}{A}\frac{\sqrt{\alpha^2 - u^2}}{(1+u^2)\sqrt{1+\alpha^2}}.$$

Reverting to original variables produces

$$p(x) = -\frac{\sqrt{2}\Delta}{A}\cos\frac{x}{2}\sqrt{\cos x - \cos a}. \tag{21}$$

If one requires the solution for an arbitrary period L instead of 2π, Equation (21) becomes

$$p(x) = -\frac{2\sqrt{2}\Delta\pi}{AL}\cos\frac{\pi x}{L}\sqrt{\cos\frac{2\pi x}{L} - \cos\frac{2\pi a}{L}}.$$

It is sometimes convenient to express the normal traction in terms of a peak pressure, p_0, in order to find the contact length, a. Let $p_0 = 2\Delta/A$ and invoking equilibrium from (18) gives

$$P = \frac{2p_0}{\sqrt{1+\alpha^2}}\int_{-\alpha}^{\alpha}\frac{\sqrt{\alpha^2 - u^2}}{(1+u^2)^2}\,du.$$

Evaluating the integral results in

$$p_0 = \frac{P(1+\alpha^2)}{\pi\alpha^2} = \frac{2\Delta}{A}. \tag{22}$$

In original variables,

$$p(x) = -\frac{P\sqrt{2}\cos(x/2)\sqrt{\cos x - \cos a}}{2\pi\sin^2(a/2)},$$

which reproduces the Westergaard solution with $\bar{p} = P/2\pi$. It has been previously shown [Dundurs et al. 1973] that, for light loading, the Hertz line contact solution is recovered.

Periodic inclined punches. The problem of evenly spaced indenters on a half-space, shown in Figure 4, was originally studied by England and Green [1963] using complex potential functions. A general result for symmetric and/or antisymmetric loading was developed and the particular case of inclined flat punches was solved. The problem was later solved by Cai and Lu [2000]. Their solution is rederived here using the method of the previous section. As the slope of the punches approaches zero, the result for periodic flat punches is recovered.

For an inclined punch, the derivative of the surface profile is given by

$$\frac{\partial h(x)}{\partial x} = \varepsilon, \tag{23}$$

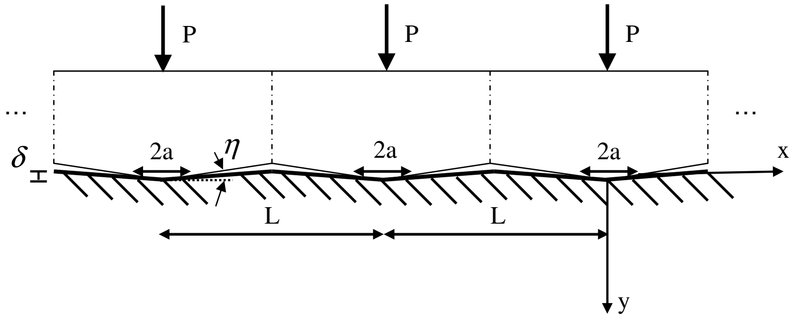

Figure 4. Frictionless normal contact of evenly spaced blunt wedges.

where ε is the slope of the punch. Substituting Equation (23) into (13) for frictionless contact ($q(u) = 0$) where now $p(u)$ is not symmetric gives

$$\frac{\varepsilon}{A} = \frac{1}{\pi}\int_{-\alpha}^{\alpha} \frac{p(u)}{(v-u)}\,du + \frac{1}{\pi}\int_{-\alpha}^{\alpha} \frac{up(u)}{(1+u^2)}\,du.$$

[Söhngen 1954] suggested solving this type of integral equation by decomposing $p(u)$ into a homogeneous part, $p_h(u)$, and an inhomogeneous part, $p_i(u)$, which results in two integral equations

$$\frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{p_h(u)}{v-u}\,du = 0, \tag{24}$$

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha} \frac{p_i(u)}{v-u}\,du + \frac{1}{\pi}\int_{-\alpha}^{\alpha} \frac{up_i(u)}{1+u^2}\,du = \frac{\varepsilon}{A}. \tag{25}$$

The first integral is recognized as the periodic flat punch solution, or the homogenous case of the integral equation. The second integral represents the contribution due to the slope of the indenter. Inverting (24) according to the appropriate formula in Section A.1 (with singularities at both endpoints) gives

$$p_h(u) = \frac{C}{\sqrt{1-(u/\alpha)^2}}. \tag{26}$$

The equilibrium (18) gives

$$C = -\frac{P\sqrt{1+\alpha^2}}{2\alpha\pi}. \tag{27}$$

Putting (27) into (26) gives the solution for the complete contact of periodic flat punches on an elastic half-space without friction

$$p_h(x) = -\frac{P\sqrt{2}\cos(x/2)}{2\pi\sqrt{\cos x - \cos a}}, \qquad |x| < a. \tag{28}$$

Refer to (25) for the inhomogeneous solution and let $C_1 = \varepsilon/A - C_2$. Recognizing that the second integral only produces a constant

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha} \frac{up_i(u)}{1+u^2}\,du = C_2,$$

then the integral equation is rewritten as

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{p_i(u)}{v-u}du = C_1.$$

Inverting for $p_i(u)$ (see Section A.1) produces

$$p_i(u) = \frac{C_1}{\pi\sqrt{\alpha^2-u^2}}\int_{-\alpha}^{\alpha}\frac{\sqrt{\alpha^2-s^2}}{u-s}ds = -\frac{C_1 u}{\pi\sqrt{\alpha^2-u^2}}.$$

To solve for C_1, insert $p_i(u)$ into (25), obtaining

$$C_1 = \frac{\pi\varepsilon}{A}\sqrt{\alpha^2+1}.$$

Returning to original variables gives

$$p(x) = p_i + p_h = -\frac{\varepsilon\sqrt{2}\sin(x/2)}{A\sqrt{\cos x - \cos a}} - \frac{P\sqrt{2}\cos(x/2)}{2\pi\sqrt{\cos x - \cos a}}, \tag{29}$$

which is the result in [England and Green 1963; Cai and Lu 2000]. It is seen that for $\varepsilon = 0$, (29) is the flat punch solution in (28). For incomplete penetration, the pressure must be positive, in which case it is required that for period L and contact length $2a$,

$$P \geq -\frac{L\varepsilon}{A}\tan\frac{\pi a}{L}.$$

Now that the method has been verified against some well-known solutions, new solutions in periodic contact are solved in the following sections.

Indentation by blunt periodic wedges. In Figure 4, the problem of a periodic array of wedges is shown. To ensure that the small strain assumption is valid, the wedge must be blunt, which means that the angle η is small. The normal displacement derivative is

$$\frac{\partial h(x)}{\partial x} = -\eta\,\mathrm{sgn}(\tan x/2).$$

The function changes sign as necessary for periodic wedges with period 2π. Anticipating a symmetric stress distribution, $p(u)$, the governing integral equation is

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{p(u)}{v-u}du = \frac{\eta}{A}\,\mathrm{sgn}(v). \tag{30}$$

This integral can be found in [Hills et al. 1993] with solution

$$p(u) = -\frac{2\eta}{\pi A}\cosh^{-1}\frac{\alpha}{|u|},$$

or in original variables with arbitrary period, L, is

$$p(x) = -\frac{2\eta}{A\pi}\cosh^{-1}\frac{\tan(\pi a/L)}{\tan(\pi|x|/L)}.$$

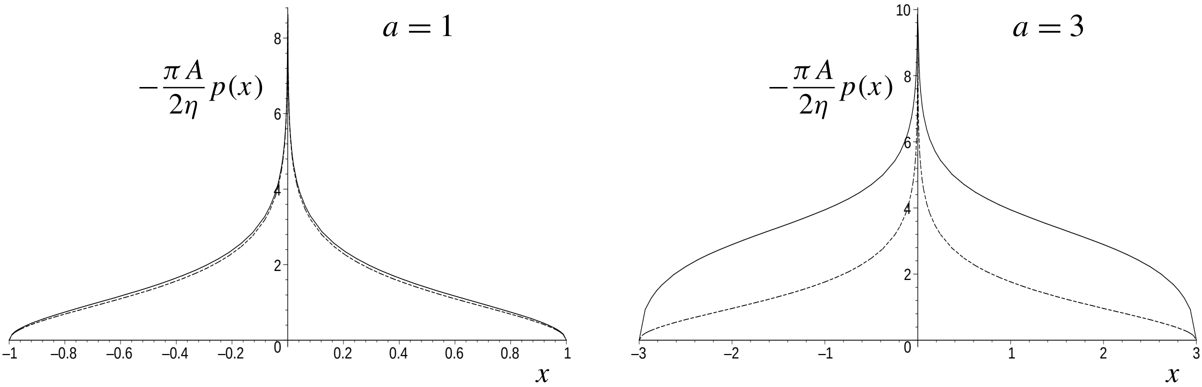

Figure 5. Normal surface tractions for single (dotted lines) and periodic (solid lines) blunt wedge indenters for wavelength $L = 2\pi$, for $a = 1$ and $a = 3$.

In the limit as $L \to \infty$, the solution for a single blunt wedge (see [Truman et al. 1995] for instance) is recovered. The normal surface tractions are plotted in Figure 5 to compare the single wedge to the periodic array of wedges. When the contact region is small compared to the wavelength, the two results are almost indistinguishable. As the contact length approaches the wavelength, the plots become noticeably different. The normalization factor is chosen for convenience, but could easily be adjusted in terms of the total load, P, by evaluating the equilibrium of surface tractions. The main feature of the blunt wedge indenter is the singularity at the tip due to the discontinuity in the displacement derivative.

4. Problems in periodic contact: sliding contact

Sliding contact of elastically dissimilar materials. It was assumed that either the bodies in contact were elastically similar or that the contact was frictionless. As a result, the integral equations were uncoupled and solved using standard techniques. In this section, the sliding contact of elastically dissimilar materials with friction is considered. The empirical Coulomb friction law is used here, which states:

(i) the frictional force is proportional to the normal force multiplied by the coefficient of friction;

(ii) the frictional force opposes the direction of relative motion;

(iii) neither the apparent contact area nor the velocity during gross sliding affects the magnitude of the frictional force;

(iv) the motion is assumed to be quasistatic.

While these observations typically apply to rigid bodies, it is reasonable to extend the model to the analysis of frictional elastic contacts in two dimensions. More complicated studies suggest a nonlinear relationship between normal loading and frictional force [Urbakh et al. 2004]; however only Coulomb friction will be considered here.

By assuming that the shear tractions are proportional to the normal force, limited everywhere by friction and independent of the speed, except for sign, one obtains

$$q(x) = -f p(x), \tag{31}$$

where f is the coefficient of friction. Substituting Equation (31) into (16) yields

$$\frac{1}{A}\frac{\partial h}{\partial x} = \frac{1}{2\pi}\int_b^a p(\xi)\left(\cot\frac{x-\xi}{2} - \tan\frac{\xi}{2}\right)d\xi + \beta f p(x), \tag{32}$$

which is a Cauchy singular integral equation of the second kind, where the solution technique is outlined in Section A.2. Another consequence of introducing friction is that the contact patch, $2a$, is no longer expected to be centered. Instead, an eccentricity, e, is introduced into the displacement derivative, which corresponds to the rotation due to the sliding of elastically dissimilar materials. Equation (32) is used as the starting point for much of the problems solved below.

Sliding contact of periodic flat punches. For periodic flat punches sliding on an elastic half-space, the derivative of the surface profile is proportional to a constant, which represents a rotation due to the sliding of elastically dissimilar materials. However, the problem has more practical relevance when the periodic punches are rigid, which means that the punch is not free to rotate and the resulting boundary condition becomes

$$\frac{\partial h(x)}{\partial x} = 0. \tag{33}$$

Since there is no rotation, (32) must be used. Substituting (33) leads to

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{p(u)}{(v-u)}du + \beta p(v) = 0.$$

For a rigid punch in contact with an elastic half-space in plane strain contact, the Dundurs' parameter, β, is

$$\beta = \frac{1-2\nu}{2(1-\nu)},$$

although the definition in (3) can be still be used without much loss of generality. Returning to the Cauchy singular integral equation of the second kind, Section A.2 gives an inversion for $p(u)$ that is singular at both endpoints, namely $p(u) = C(\alpha-u)^{m-1}(\alpha+u)^{-m}$, where $\tan m\pi = 1/\beta f$ and $0 < m < 1$. To find C, equilibrium of the total load, (18), gives

$$P = 2C\int_{-\alpha}^{\alpha}\frac{(\alpha-u)^{m-1}(\alpha+u)^{-m}}{1+u^2}du.$$

The integral is solved using complex variables and is given in Appendix B. Solving for C,

$$C = -\frac{P\sqrt{1+\alpha^2}\sin m\pi}{2\pi\,\sin\big[m\pi - \phi(2m-1)\big]},$$

where $\tan\phi = 1/\alpha$. Returning to original variables,

$$p(x) = -\frac{P\sin m\pi\big(\tan(a/2)-\tan(x/2)\big)^{m-1}\big(\tan(a/2)+\tan(x/2)\big)^{-m}}{2\pi\cos(a/2)\sin\big[m\pi - \phi(2m-1)\big]},$$

where $\tan\phi = \cot(a/2)$. When $m = 1/2$, the result for frictionless contact in Equation (28) is recovered. The equation is singular at both end points, $x = \pm a$, as expected for complete contact.

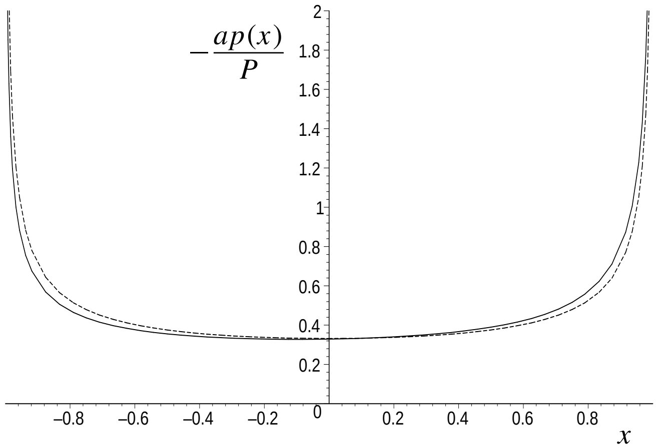

Figure 6. The surface pressure for a evenly spaced rigid flat punches with $f = 0.5$, $\nu = 0.3$ and $L = 2\pi$, for frictionless (dotted curve) and sliding (solid curve) contact for $a = 1$, sliding from left to right.

The normal tractions are no longer symmetric, as shown in Figure 6. However, the frictionless result is not very different from the sliding one for the given parameters. If however the punch were no longer rigid but elastic, the material mismatch would lead to greater discrepancy between the results, as would an increase in the coefficient of friction. It is also seen that the corners of the flat punch lead to singularities in the pressure distribution at the edge of the contact.

Sliding contact of sinusoids. The sliding of elastically dissimilar sinusoidal surfaces was studied by Nosonovsky and Adams [2000], who numerically solved the dry steady-state frictional sliding incorporating wave speeds, and later by Carbone and Mangialardi [2004], who used linear fracture mechanics to determine the coefficient of friction due to the adhesive forces.

For a given coefficient of friction, the sliding of elastically dissimilar sinusoids is solved here in a closed-form for the first time. This new result is compared to the frictionless case, except for an eccentricity, e, due to the dissimilar elastic constants. The difference of the displacement derivatives for two wavy surfaces is

$$\frac{\partial h(x)}{\partial x} = \Delta \sin(x - e), \tag{34}$$

where e represents the phase shift due to sliding and Δ is a function of the gap and amplitude(s) to ensure that the displacements are continuous at the edge of the contact region. Substituting (34) into (32) with the usual change of variables (and with an eye toward the integrals in Appendix B) yields

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha} \frac{p(u)}{v-u}\,du + \beta p(v) = \frac{\Delta}{A} w(v), \tag{35}$$

where $e_1 = \cos e$, $e_2 = \sin e$ and $w(v) = \left(e_1 \dfrac{2v}{1+v^2} - e_2 \dfrac{1-v^2}{1+v^2}\right)$; this determines the eccentricity, e, from the consistency condition

$$\int_{-\alpha}^{\alpha} w(v)(\alpha - v)^{-m}(\alpha + v)^{m-1}\,dv = 0.$$

This integral may seem different from those in Appendix B with respect to their powers; however, setting $m = 1 - n$ transforms the powers to the desired form. The eccentricity is found to be

$$e = \tan^{-1}\left(\frac{\cos\lambda}{\sin\lambda - \alpha^2\sqrt{1+\alpha^2}m(1-m)}\right),$$

where $\lambda = (m\pi - \phi(2m-1))$ and $\tan\phi = 1/\alpha$. For $m = 1/2$ the eccentricity vanishes, as expected from the absence of sliding.

The integral equation in Equation (35) is now inverted for $p(u)$ according to the appropriate formula in Section A.2 as

$$p(u) = \frac{\Delta}{A}\left[\beta f w(u) - \frac{1}{\pi}\int_{-\alpha}^{\alpha} w(v)\frac{(\alpha-v)^{-m}(\alpha+v)^{m-1}}{v-u}\,dv\right](\alpha-v)^m(\alpha+v)^{1-m}.$$

Using partial fractions and evaluating the Cauchy integrals which cancel with the βf terms above results in

$$p(u) = -\frac{2\Delta}{A\pi(1+u^2)}(\alpha-v)^m(\alpha+v)^{1-m}\int_{-\alpha}^{\alpha}\big(e_1(1-uv)+e_2(v+u)\big)\frac{(\alpha-v)^{-m}(\alpha+v)^{m-1}}{1+v^2}\,dv.$$

Calculating the integrals according to Appendix B, the surface stress is

$$p(u) = -\frac{2\Delta}{A(1+u^2)}\frac{1}{\sqrt{1+\alpha^2}}\frac{\big[\sin(\lambda+e) - u\cos(\lambda+e)\big]}{\sin\pi m}(\alpha-v)^m(\alpha+v)^{1-m}.$$

In dimensional variables, the final result for the normal surface tractions is

$$p(x) = -\frac{2\Delta}{A}\frac{\cos(a/2)\cos(x/2)}{\sin\pi m}\sin\left[\lambda+e-\frac{x}{2}\right]\left(\tan\frac{a}{2}-\tan\frac{x}{2}\right)^m\left(\tan\frac{a}{2}+\tan\frac{x}{2}\right)^{1-m}. \tag{36}$$

The surface stress can also be expressed in terms of the total load, P, by substituting (31) into the equilibrium equation in (17) and exploiting the solutions given in Appendix B. For $m = 1/2$, (36) reduces to the frictionless contact of sinusoids given in (21) as expected.

Equation (36) is the first closed form solution for the sliding of an elastic wavy surface. It may prove useful in analyzing real rough surface whose average amplitude and wavelength have been evaluated in order to obtain pressure approximations or even to back-out the coefficient of friction of the mating surfaces.

The Coulomb friction again causes the sliding result to be asymmetric, but it is only slightly different from the frictionless result; see Figure 7. The friction reduces the pressure at the front edge of the indenter but increases it at the rear. The plots become more disparate by considering two elastic bodies or increasing the coefficient of friction. The smooth nature of the wavy profiles does not produce singularities in the surface stress and vanishes at the edge of the contact.

5. Problems in periodic contact: partial slip contact of similar materials

Partial slip contact: similar materials. In the previous section, sliding contact problems of various periodic geometries — in which the entire surface of the indenter moves in the same direction — were solved in a closed-form. The elastically dissimilar materials produced asymmetry in the solutions.

A different type of contact is considered in this section, namely partial slip contact of elastically similar materials. Typically in partial slip problems, the normal load is held fixed and the tangential load

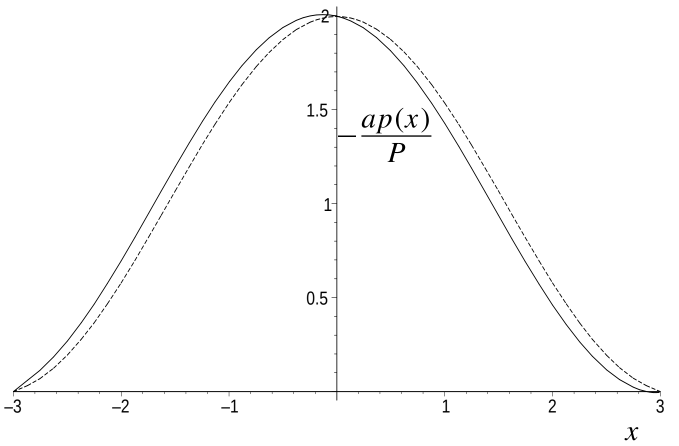

Figure 7. The surface pressure for a rigid wavy surface with $f = 0.5$, $\nu = 0.3$ and $L = 2\pi$ for frictionless (dotted curve) and sliding (solid curve) contact for $a = 3$.

is increased from zero causing two types of contact regions. One region consists of a contact area that is a stick region, and the other region of the contact area slips such that, shear tractions remain bounded. This requires the region to undergo "microslip." Partial slip analyses are particularly useful for analyzing fretting fatigue problems [Hills and Nowell 1994].

A comprehensive examination of partial slip theory and problems for similar materials was undertaken by Ciavarella [1998a; 1998b], who based his analysis on Cattaneo and Mindlin's solution. Their method was to write the shear tractions, $q(x)$, in the contact region as a linear combination of the normal traction and a correction term, that is,

$$q(x) = -fp(x) + q^*(x), \tag{37}$$

where f is the coefficient of friction, $p(x)$ is the normal traction, and $q^*(x)$ is a correction to the shear traction to ensure a finite coefficient of friction at the edge of contact. It should be noted that $q^*(x)$ is zero in the slip zones, meaning the shear traction is simply given by Coulomb's friction law.

Based on this approach, Ciavarella derived closed form solutions for many indenter geometries, including a few involving periodic indenters. More importantly, he summarized his findings on the theory in a few rules which are quoted exactly below:

(1) If the indenter profile is symmetric and *self-similar*, the corrective solution is of the same shape as the normal pressure in the contact area for any load.

(2) No partial slip solution can be predicted where the stick zone lies entirely within a flat region of the punch; in other words, flat regions are either entirely in full stick or are in full slip conditions.

(3) If in normal indentation there is no change of relative rotation, then the points that come into contact last are the first to slip.

(4) If the indenter profile has discontinuities, these affect the tangential load, stick area dimension relation in the same was as they affect the normal load, contact area dimension.

When solving periodic problems, Ciavarella does not differentiate between a single indenter and periodic indenters; he presupposes that the shear correction term has the same form as the normal pressure

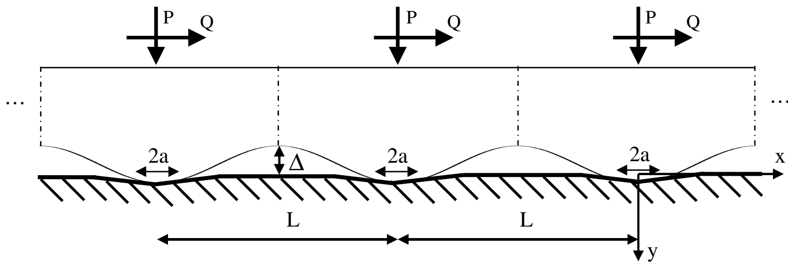

Figure 8. Partial slip contact of a sinusoidal profile for elastically similar materials.

solution. In (1) above, surfaces are assumed self-similar. Periodic solutions are not necessarily self-similar, because they have a wavelength which introduces a length scale.

In this section Ciavarella's solution for periodic contact of similar materials will be derived using the previous technique. The standard change of variables changes the periodic problem to a single contact patch.

Sinusoids in partial slip. Since elastically similar bodies are considered here, it means that the governing integral equations are uncoupled ($\beta = 0$). As opposed to the sliding contact problems in the previous section, symmetric surface tractions are once again expected, as shown in Figure 8.

For a sinusoidal indenter, the stick zone is anticipated at the center of the contact, since the loading, geometry and material properties are symmetric about the x and y axes. The slip zones occur at the edges of contact region and are the points which are last to come into contact. The shear tractions can be expressed by (37) in the stick zone and (31) in the slip zone and the governing integral equation for the shear tractions is

$$\frac{1}{2\pi}\int_{-a}^{a} q(\xi)\cot\frac{x-\xi}{2}\,d\xi = \frac{1}{A}\frac{\partial g(x)}{\partial x}, \qquad |x| < b, \tag{38}$$

which is (10) for similar materials. Also, since the relative tangential displacements due to the normal load, P, are constant, it follows that $\partial g(x)/\partial x = 0$ in the stick zone. Thus, substituting (37) into (38) gives

$$\frac{1}{2\pi}\int_{-b}^{b} q^*(\xi)\cot\frac{x-\xi}{2}\,d\xi = \frac{f}{2\pi}\int_{-a}^{a} p(\xi)\cot\frac{x-\xi}{2}\,d\xi, \qquad |x| < b, \tag{39}$$

where $p(x)$ is the Westergaard solution found in Equation (21). In order to solve for the shear correction term in (39), the change of variables in (28) is used as well as $\gamma = \tan(b/2)$ to get

$$\frac{1}{\pi}\int_{-\gamma}^{\gamma}\frac{q^*(u)}{v-u}\,du + \frac{1}{\pi}\int_{-\gamma}^{\gamma}\frac{uq'(u)}{1+u^2}\,du = -\frac{2f\Delta}{A\pi}\int_{-\alpha}^{\alpha}\frac{(1+uv)\sqrt{\alpha^2-u^2}}{(1+u^2)^2(v-u)\sqrt{1+\alpha^2}}, \qquad |v| < \gamma. \tag{40}$$

Noting the symmetry of $q^*(x)$ eliminates the second term on the left-hand side of (40). The right-hand side is simplified by first adding and subtracting v^2 in the numerator and yields

$$\frac{1}{\pi}\int_{-\gamma}^{\gamma}\frac{q^*(u)}{v-u}\,du = -\frac{2f\Delta}{A\pi\sqrt{1+\alpha^2}}\left((1+v^2)\int_{-\alpha}^{\alpha}\frac{\sqrt{\alpha^2-u^2}}{(1+u^2)^2(v-u)}\,du + v\int_{-\alpha}^{\alpha}\frac{\sqrt{\alpha^2-u^2}}{(1+u^2)^2}\,du\right), \quad |v| < \gamma. \tag{41}$$

The second term on the right-hand side of (41) is easily evaluated. The remaining term on the right-hand side is simplified by using partial fractions, which gives

$$\frac{1}{\pi}\int_{-\gamma}^{\gamma}\frac{q^*(u)}{v-u}\,du = -\frac{2f\Delta}{A}\frac{v}{1+v^2}, \qquad |v|<\gamma.$$

Inverting for $q^*(x)$ (Section A.1) and again using partial fractions results in

$$q^*(u) = -\frac{2f\Delta\sqrt{\gamma^2-u^2}}{A(1+u^2)\sqrt{1+\gamma^2}}, \qquad |u|<\gamma.$$

Reverting to original variables gives the final result for the shear correction term

$$q^*(x) = -\frac{\sqrt{2}f\Delta}{A}\cos\frac{x}{2}\sqrt{\cos x-\cos b}. \tag{42}$$

It is seen that indeed the shear correction term (42) has the same form as the normal tractions, (21), which is also revealed upon inspection of (39). But it has been shown that it indeed has the same form for regularly periodic indenters. Lastly, satisfying equilibrium requires that

$$Q^* = \frac{2q_0^*}{\sqrt{1+\gamma^2}}\int_{-\gamma}^{\gamma}\frac{\sqrt{\gamma^2-u^2}}{(1+u^2)^2}\,du.$$

Evaluating the integral results in

$$q_0^* = \frac{Q(1+\gamma^2)}{\pi\gamma^2} = -\frac{\sqrt{2}f\Delta}{A}. \tag{43}$$

Combining (22) with (43) gives the result in a convenient form to determine the stick zone

$$\frac{Q}{fP} = 1-\frac{\sin^2(\pi b/L)}{\sin^2(\pi a/L)}, \tag{44}$$

which is the result in [Ciavarella 1998b] for sinusoidal indenters. The result is expressed in terms of period, L, for practicality. Here, Q and P are understood to be the total normal and shear loads.

Other periodic profiles under partial slip. Other new results can be obtained for various periodic indenter profiles. For instance, Ciavarella [1998b] provides the result for a single wedge, power-law or polynomial punch

$$\frac{Q}{fP} = 1-\left(\frac{b}{a}\right)^k, \tag{45}$$

where $k = 1, 2, 4, 6$ is a power representing a wedge, parabolic (Hertzian) and high-order polynomial indenters respectively. His method is easily extended to periodic indenters as well. Starting with Equation (9) with $\beta = 0$,

$$\frac{1}{A}\frac{\partial h}{\partial x} = \frac{1}{2\pi}\int_{-a}^{a} p(\xi)\cot\frac{x-\xi}{2}\,d\xi. \tag{46}$$

Changing variables and inverting for $p(u)$ gives

$$p(u) = \frac{1}{\pi A}\sqrt{\alpha^2-u^2}\int_{-\alpha}^{\alpha}\frac{h'(x)}{\sqrt{\alpha^2-v^2}(v-u)}\,dv, \qquad |u|<\alpha, \tag{47}$$

where $h'(t) = \partial h/\partial x$ is the derivative of the indenter profile in terms of the original variable x. Substituting (47) into the equilibrium equation, (30), gives the total load, P as

$$P = -\frac{1}{A\pi}\int_{-\alpha}^{\alpha}\frac{h'(x)}{\sqrt{\alpha^2-v^2}}dv\int_{-\alpha}^{\alpha}\frac{\sqrt{\alpha^2-u^2}}{(1+u^2)(v-u)}du.$$

Interchanging the order of integration and then using partial fractions results in

$$P = -\frac{\sqrt{1+\alpha^2}}{A}\int_{-\alpha}^{\alpha}\frac{vh'(x)}{\sqrt{\alpha^2-v^2}(1+v^2)}dv. \tag{48}$$

Equation (48) directly relates the surface profile to the total load. A similar calculation is done for the shear correction load, Q^*, by first substituting (46) into (39) to get

$$\frac{1}{2\pi}\int_{-b}^{b}\frac{q^*(\xi)}{f}\cot\frac{x-\xi}{2}d\xi = \frac{1}{A}\frac{\partial h(x)}{\partial x}. \tag{49}$$

Inverting (49) for $q^*(u)$ with changed variables, invoking equilibrium and finally solving for Q^* gives

$$Q^*/f = \frac{\sqrt{1+\gamma^2}}{A}\int_{-\gamma}^{\gamma}\frac{vh'(x)}{\sqrt{\gamma^2-v^2}(1+v^2)}dv. \tag{50}$$

Integrating (37) over the contact region and substituting (50) yields

$$Q = fP + \frac{f\sqrt{1+\gamma^2}}{A}\int_{-\gamma}^{\gamma}\frac{vh'(x)}{\sqrt{\gamma^2-v^2}(1+v^2)}dv. \tag{51}$$

Combining (51) with (48) produces the final result for the ratio of the shear load to the normal load for periodic indenters

$$\frac{Q}{fP} = 1 - \frac{\sqrt{1+\gamma^2}\displaystyle\int_{-\gamma}^{\gamma}\frac{vh'(x)}{\sqrt{\gamma^2-v^2}\,(1+v^2)}dv}{\sqrt{1+\alpha^2}\displaystyle\int_{-\alpha}^{\alpha}\frac{vh'(x)}{\sqrt{\alpha^2-v^2}\,(1+v^2)}dv}. \tag{52}$$

Depending on the nature of $h'(x)$, other closed-form solutions for periodic indenters can be derived from Equation (52). For instance, the partial slip of periodic blunt wedges is solved by first noting

$$h'(x) = \eta\,\mathrm{sgn}(x). \tag{53}$$

Substituting (53) into (52) and integrating yields

$$\frac{Q}{fP} = 1 - \frac{\tanh^{-1}\left(\gamma/\sqrt{1+\gamma^2}\right)}{\tanh^{-1}\left(\alpha/\sqrt{1+\alpha^2}\right)} = 1 - \frac{\tanh^{-1}\sin(\pi b/L)}{\tanh^{-1}\sin(\pi a/L)}, \tag{54}$$

where L is the period of the indenters. For a small contact region, $a/L \ll 1$, (54) reduces to (45) for $k = 1$ as anticipated. For a general power-law profile, the displacement derivative is

$$h'(x) = -\eta k|x|^{k-1}\,\mathrm{sgn}(x). \tag{55}$$

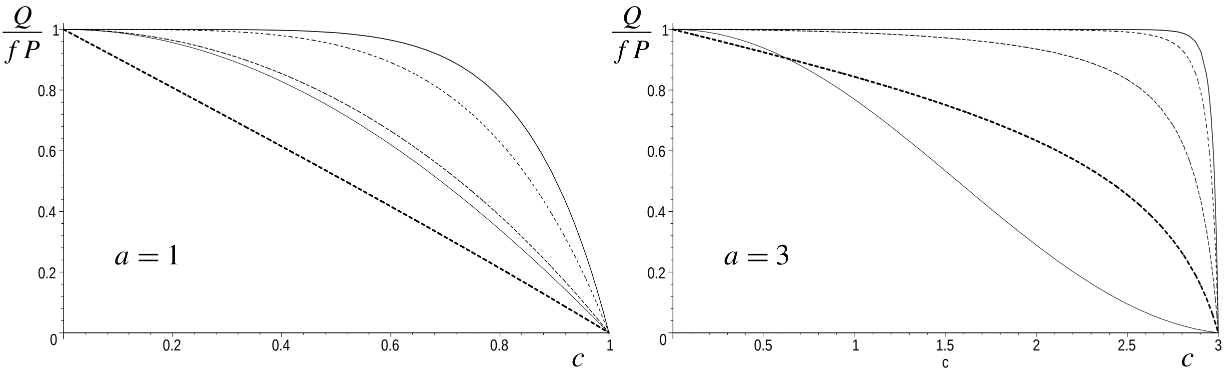

Figure 9. Ratio of shear force to normal load as a function of slip zone, c, for $a = 1, 3$ and various power-law profiles. Curves from top to bottom: $k = 6$ profile, quadratic profile, parabolic profile, wedge profile, simple profile.

For $k = 2$, which physically corresponds to evenly spaced Hertzian indenters, the ratio of shear to normal loads is

$$\frac{Q}{fP} = 1 - \frac{\sqrt{\gamma^2+1}-1}{\sqrt{\alpha^2+1}-1}. \tag{56}$$

Returning to original variables with period, L, changes (56) into

$$\frac{Q}{fP} = 1 - \frac{\cos(\pi a/L)}{\cos(\pi b/L)} \frac{1-\cos(\pi b/L)}{1-\cos(\pi a/L)}.$$

When the contact region is small, $a/L \ll 1$, the result for $k = 2$ in (54) is recovered. In a similar fashion, higher order power-law and polynomial indenter profiles can be solved. For $k = 4$ in (55), the ratio of the transmitted forces is found to be

$$\frac{Q}{fP} = 1 - \frac{\cos(\pi a/L)}{\cos(\pi b/L)} \frac{\left(\tan^2(\pi b/L) + 2\cos(\pi b/L) - 2\right)}{\left(\tan^2(\pi a/L) + 2\cos(\pi a/L) - 2\right)}.$$

For $k = 6$, the solution is

$$\frac{Q}{fP} = 1 - \frac{\cos(\pi a/L)}{\cos(\pi b/L)} \frac{\left(3\tan^4(\pi b/L) - 4\tan^2(\pi b/L) - 8\cos(\pi b/L) + 8\right)}{\left(3\tan^4(\pi a/L) - 4\tan^2(\pi a/L) - 8\cos(\pi a/L) + 8\right)}.$$

Further higher-order power-law indenters can be calculated in a similar manner. The results for the partial slip of elastically similar profiles are plotted below in Figure 9. As the power of the profile increases, the slip zone size varies weakly under small loads and strongly under larger loads, which is the same observation made in [Ciavarella 1998b]. The higher order polynomial indenters are close to a flat punch profile, which must either completely stick or slip and is discussed next.

Another solution of interest is the partial slip of evenly spaced flat punches on a half-space of similar material. Ciavarella notes that elastically similar flat punches must either completely slip or completely stick. The shear tractions are always proportional to the normal tractions, which are given by

$$q(x) = -\frac{Q\sqrt{2}\cos(\pi x/L)}{L\sqrt{\cos(2\pi x/L)-\cos(2\pi a/L)}}, \qquad |x| < a. \tag{57}$$

It should not be surprising then that Equation (57) resembles (28), which is the frictionless normal contact solution for flat punches. When $|Q| = fP$, full slip occurs throughout the contact region. The solution is singular at the edge of the contact area, as expected for flat punch indenters.

6. Problems in periodic contact: partial contact of dissimilar materials

Partial slip contact: dissimilar materials. The partial slip contact of elastically dissimilar materials is considered here, which means that the governing integral equations are coupled. Before solving the partial slip problem, which is mathematically complicated, the no slip or complete stick problem is solved for periodic profiles. Physically, complete stick occurs when the friction force is large enough to prevent any slip. Even though the integral equations are still coupled for complete stick, some closed-form solutions are calculated in this section.

One simplifying assumption to uncouple the integral equations is to assume the effect of the shear tractions on the normal displacements is negligible, the Goodman approximation [Goodman 1962]. This simplification removes the coupling in (15). The advantage is that the normal contact solutions are known and as a first approximation can then be substituted into the shear traction (16). Solutions for flat punches and sinusoidal profiles are worked out for this type of analysis.

Lastly, partial slip solutions for elastically dissimilar periodic flat punches and sinusoids are solved based upon the procedure of Spence [1973], who used a self-similarity argument when evaluating the displacement derivatives. In light of the previous section, it is now known that even though regularly periodic profiles are not self-similar, the appropriate transformation of variables changes the problem to a single contact patch which *is* self-similar; then the solution techniques for a single indenter can be exploited. Although the materials are dissimilar, the removal of the coupling term ensures that the normal tractions are symmetric for symmetric indenters, which in turn causes the shear tractions to also be symmetric about the centerline of the contact region. The stick region is anticipated in the center of the contact region with slip regions at the edges.

No slip contact of periodic flat punches: coupled. In this section, the problem of elastically dissimilar flat punches in contact with an elastic half-space is solved for the completely coupled case. It may be more practical to assume that the flat punches are rigid, but the solution here is solved more generally in terms of Dundur's parameter. For flat punches, the displacement derivates are

$$\frac{\partial h(x)}{\partial x} = \frac{\partial g(x)}{\partial x} = 0. \tag{58}$$

This reduces the integral equations of contact, (15)–(16), in changed variables to

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{p(u)}{(v-u)}\,du - \beta q(v) = 0, \qquad \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{q(u)}{(v-u)}\,du + \beta p(v) = 0.$$

The solution technique for the above coupled integral equations is described in [Johnson 1985b], where the solution is

$$F(v) = -\frac{\lambda}{\sqrt{1-\lambda^2}}\frac{C}{\pi\sqrt{\alpha^2-v^2}}\left(\frac{\alpha-v}{\alpha+v}\right)^{i\eta}, \tag{59}$$

where $F(u) = p(u) + iq(u)$, $i = \sqrt{-1}$, $\lambda = -i/\beta$, and

$$\eta = \frac{1}{2\pi}\ln\frac{1+\beta}{1-\beta}.$$

Note that η is a constant based on the material parameter β here and not the slope for a blunt wedge used above. The constant, C, in (59) is determined from the equilibrium equation

$$\int_{-\alpha}^{\alpha}\frac{2F(v)}{1+v^2}\,dv = P + iQ.$$

The final result for the surface tractions in terms of the original variable is

$$p(x) + iq(x) = \frac{P+iQ}{\sqrt{1-\beta^2}}\frac{\sqrt{2}\cos(x/2)}{2\pi\sqrt{\cos x - \cos a}}\left(\frac{\tan(a/2)+\tan(x/2)}{\tan(a/2)-\tan(x/2)}\right)^{i\eta}. \tag{60}$$

It is seen that the singularities at $x = \pm a$ are complex for the completely coupled case. As the wavelength of the periodic punches becomes infinite, the result for a single flat indenter given in [Johnson 1985b] is recovered. When $P = 0$, (57) for the shear tractions due to a tangential force, Q, is recovered.

No slip contact of periodic flat punches: approximation. Because of the limited availability of closed-form solutions for the completely coupled integral equations, it is often convenient to introduce the Goodman approximation, which neglects the contributions of the shear stress on the normal displacements. The advantage is that the frictionless normal contact solutions developed in previous section can be used to derive the shear tractions without much loss of accuracy. Recalling the solution for frictionless normal contact of a flat punch

$$p(x) = -\frac{P\sqrt{2}\cos(x/2)}{2\pi\sqrt{\cos x - \cos a}},$$

and using Equation (58) causes the integral equation for the shear traction, (14), to be

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{q(u)}{(v-u)}\,du - \frac{\beta P\sqrt{1+\alpha^2}}{2\pi\sqrt{\alpha^2-v^2}} = 0. \tag{61}$$

Inverting (61) for $q(u)$ according to the formula in Section A.1 with singularities at both endpoints gives

$$q(u) = \frac{\beta P\sqrt{1+\alpha^2}}{2\pi^2\sqrt{\alpha^2-u^2}}\int_{-\alpha}^{\alpha}\frac{dv}{v-u} + \frac{Q\sqrt{2}\sqrt{1+\alpha^2}}{2\pi\sqrt{\alpha^2-u^2}}, \tag{62}$$

where the second term corresponds to the shear traction solution for a flat punch, (57). Evaluating the integral in (62) yields

$$q(u) = \frac{\beta P\sqrt{1+\alpha^2}}{2\pi^2\sqrt{\alpha^2-u^2}}\ln\left(\frac{\alpha+u}{\alpha-u}\right) + \frac{Q\sqrt{2}\sqrt{1+\alpha^2}}{2\pi\sqrt{\alpha^2-u^2}}.$$

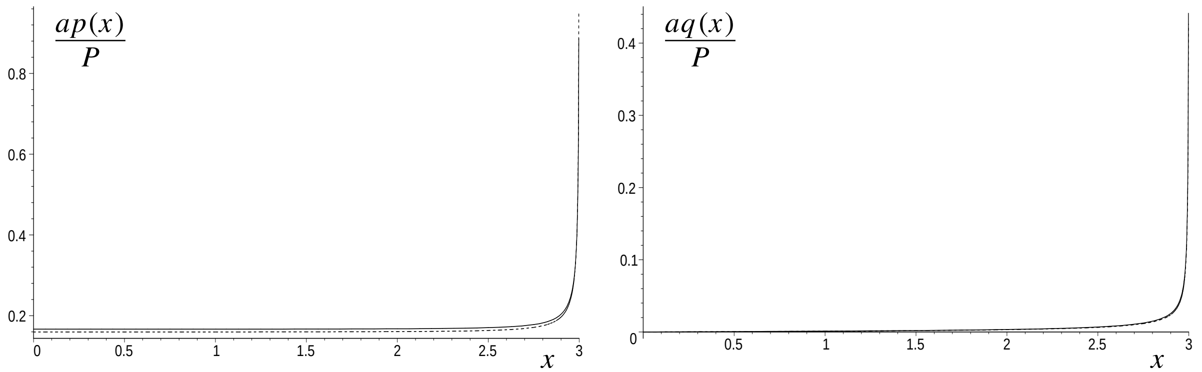

Figure 10. Normal (left) and tangential (right) tractions for the slipless contact of periodic rigid flat punches, for $\nu = 0.3$, $f = 0.215$, $L = 2\pi$, and $a = 3$. On the left, the solid curves is for the coupled normal force and the dashed curve for the frictionless normal force. On the right panel, the curves for the coupled tangential force and the Goodman tangential force are confounded.

Returning to original variables, the shear traction with the Goodman approximation is

$$q(x) = -\frac{\beta P\sqrt{2}\cos(x/2)}{2\pi^2\sqrt{\cos x - \cos a}} \ln\frac{\tan(a/2)+\tan(x/2)}{\tan(a/2)-\tan(x/2)} + \frac{Q\sqrt{2}\cos(x/2)}{2\pi\sqrt{\cos x - \cos a}}. \tag{63}$$

As the period goes to infinity, the result for a single flat indenter is recovered. The coupled normal and shear traction solutions in the previous section, (60), are compared below with the frictionless, (28), and Goodman approximation, (63), respectively.

Figure 10 demonstrates that the Goodman approximation is not very different than that for the coupled case, even as the contact length approaches the period. In fact, the shear tractions are essentially indistinguishable. One interesting thing to note about the solution is the symmetry about the origin. Because of the symmetric normal tractions and symmetric indenter, the shear tractions are also symmetric about the center line of the contact region, although the materials are dissimilar. This is a consequence of neglecting the coupling term in the normal displacement governing integral equation, which was not done in the sliding contact solutions.

No slip contact of sinusoids: Goodman approximation. The no slip problem for a sinusoidal surface in contact with an elastic half-space is developed in a similar manner to the flat punch solution. However, the tangential displacement derivative is

$$\frac{\partial g(x)}{\partial x} = C\Delta|\sin x|,$$

where C is an unknown constant to be determined. The absolute value function comes from the self-similarity argument made by Spence [1973]. The premise is that the stick zone is proportional to the slip zone; see (44). Thus, as the shear load is steadily increased from zero, the particles entering the stick zone undergo a prestrain that is proportional to $|\sin x|$, which is the absolute distance from the center of the contact region. For a single indenter, the prestrain is proportional to x, which is recovered for small

x (or large L) in the periodic problem. It is also noted that the tangential displacement derivative must be an even function in order to ensure the symmetry of the solution.

Recalling from (21) the normal tractions for a sinusoidal profile, the integral (18) upon changing variables becomes

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{q(u)}{(v-u)}\,du-\frac{\beta p_0\sqrt{\alpha^2-v^2}}{\sqrt{1+\alpha^2}(1+v^2)}=\frac{2\Delta C}{A}\,\frac{|v|}{1+v^2},\qquad |v|<\alpha. \tag{64}$$

Inverting (64) for $q(u)$ with no singularities at the endpoints gives

$$q(u)=\frac{\sqrt{\alpha^2-u^2}}{\pi}\int_{-\alpha}^{\alpha}\frac{F(v)}{\sqrt{\alpha^2-v^2}(v-u)}\,dv, \tag{65}$$

with consistency condition

$$\int_{-\alpha}^{\alpha}\frac{F(v)}{\sqrt{\alpha^2-v^2}}\,dv=0,$$

where

$$F(v)=\frac{\beta p_0\sqrt{\alpha^2-v^2}}{\sqrt{1+\alpha^2}(1+v^2)}+\frac{2\Delta C}{A}\,\frac{|v|}{1+v^2}.$$

The consistency condition is evaluated, and it is found that

$$C=\frac{-p_0\beta A}{2\Delta}\,\frac{\tan^{-1}(\alpha)}{\tanh^{-1}(\alpha/\sqrt{1+\alpha^2})}.$$

The integral equation, (65), is solved using partial fractions. After some work, the result is

$$q(u)=\frac{p_0\beta}{\pi(1+u^2)}\left[\frac{\sqrt{\alpha^2-u^2}}{\sqrt{1+\alpha^2}}\ln\left|\frac{\alpha+u}{\alpha-u}\right|+\frac{u\tan^{-1}(\alpha)}{\tanh^{-1}(\alpha/\sqrt{1+\alpha^2})}\ln\left(\frac{\alpha+\sqrt{\alpha^2-u^2}}{\alpha-\sqrt{\alpha^2-u^2}}\right)\right], \tag{66}$$

where $\alpha=\tan(a/2)$ and $u=\tan(x/2)$ are in terms of the original variables. The solution for a single Hertzian indenter is provided by Johnson [1985b] and corresponds to when the period between the sinusoids approaches infinity and the contact region is small in the above solution.

Partial slip contact of periodic flat punches: Goodman approximation. The next step is to evaluate the partial slip solution with the Goodman approximation. This type of partial slip analysis with the Goodman approximation was first done for a single flat indenter [Spence 1973] and then Hertz contact [Spence 1975; Hills and Sackfield 1985]. The main difference from the complete stick analyses above are that the shear tractions are now split into stick and slip regions with

$$\int_{-\alpha}^{\alpha}\frac{q(u)}{(v-u)}\,du=2f\int_{\gamma}^{\alpha}\frac{up(u)}{v^2-u^2}\,du+2\int_{0}^{\gamma}\frac{uq(u)}{v^2-u^2}\,du, \tag{67}$$

where $\gamma=\tan(b/2)$ and $q(u)=fp(u)\,\mathrm{sgn}(\beta u)$ in the slip region but is unknown within the stick region. The symmetry about the centerline has also been exploited. For periodic flat punches, Equation (67) is substituted into (66) to get

$$\frac{2}{\pi}\int_{0}^{\gamma}\frac{uq(u)}{v^2-u^2}\,du=G(v),\qquad |v|<\gamma, \tag{68}$$

where

$$G(v) = \frac{2fp_0\sqrt{1+\alpha^2}}{\pi}\int_\gamma^\alpha \frac{u\,du}{(v^2-u^2)\sqrt{\alpha^2-v^2}} + \frac{\beta p_0\sqrt{1+\alpha^2}}{\sqrt{\alpha^2-v^2}}, \tag{69}$$

which has the flat punch solution, (28), substituted for $p(u)$ with $p_0 = P/2\pi$. Inverting (68) for $q(u)$ gives

$$q(u) = \frac{2u}{\pi\sqrt{\gamma^2-u^2}}\int_0^\gamma \frac{\sqrt{\gamma^2-v^2}G(v)}{u^2-v^2}\,dv. \tag{70}$$

Upon substituting (69) into (70) and interchanging the order of integration in the first term, $q(u)$ becomes

$$q(u) = \frac{2u\sqrt{1+\alpha^2}}{\pi\sqrt{\gamma^2-u^2}}\left[fp_0\int_\gamma^\alpha \frac{\sqrt{v^2-\gamma^2}}{(v^2-u^2)\sqrt{\alpha^2-v^2}}\,dv - \beta p_0\int_0^\gamma \frac{\sqrt{\gamma^2-v^2}}{(u^2-v^2)\sqrt{\alpha^2-v^2}}\,dv\right].$$

It is necessary that $q(u)$ be bounded at $u=\gamma$, which requires that

$$fK'\left(\frac{\gamma}{\alpha}\right) = \beta K\left(\frac{\gamma}{\alpha}\right), \tag{71}$$

which relates the friction coefficient, f, to the elastic mismatch, β, through the complete elliptic integrals of the first kind, which are defined in Appendix B. The $'$ notation on $K'(\gamma/\alpha)$ refers to the complementary argument, which means replacing γ/α with $\sqrt{1-(\gamma/\alpha)^2}$. Once the singularity at $u=\gamma$ is removed by assuming (71), the integral (68) can be inverted according to the formula in Section A.1 for nonsingular endpoints. After some lengthy calculations, the shear tractions in the stick region are found to be

$$q(x) = \frac{fP\sqrt{2}\cos(x/2)}{2\pi\sqrt{\cos x-\cos a}}\,\frac{F\left(\sin^{-1}\dfrac{\tan(x/2)}{\tan(b/2)}, \dfrac{\tan(b/2)}{\tan(a/2)}\right)}{K\left(\dfrac{\tan(b/2)}{\tan(a/2)}\right)}\,\mathrm{sgn}\left(\beta\tan\frac{x}{2}\right), \qquad |x| < \tan\frac{b}{2}, \tag{72}$$

where $F(\phi,k)$ is an incomplete elliptic integral of first kind, which is defined in Appendix B.

The shear tractions for the partial slip of evenly spaced rigid flat punches are plotted in Figure 11. For aesthetics, the coefficient of friction is chosen to be $f=0.215$, so that $b\approx 0.5$, which is half of the entire contact length. The shear tractions are symmetric about the center of the contact region and transition from the partial slip result, Equation (72), in the stick region to Coulomb friction, (28), in the slip region at $x=b$, producing a discontinuity in the shear tractions. It is interesting to note the continuity of the plot despite the abrupt change from stick to slip at $x=b$; that is because the partial slip solution contains the normal traction solution, but is corrected by the elliptic functions, the coefficient of friction and the sign function.

As another interesting side note, it turns out that (71) is valid for all regularly periodic profiles that display self-similarity in a single contact patch, that is, in a single period.

Partial slip contact of sinusoids: Goodman approximation. In this section, the partial slip problem for elastically dissimilar wavy surfaces with Coulomb friction is solved with the Goodman approximation.

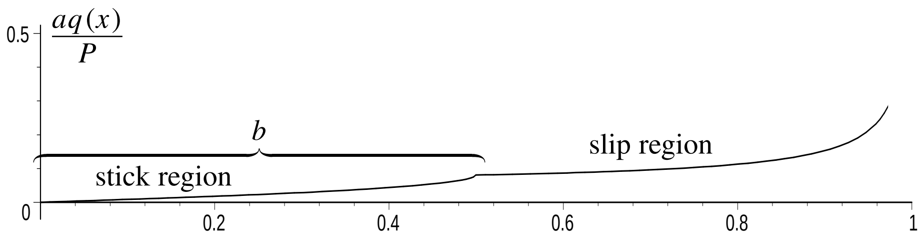

Figure 11. Partial slip of a periodic rigid flat punch with $f = 0.215$, $\nu = 0.3$, $L = 2\pi$ for $a = 1$.

The shear tractions are again divided into stick and slip regions. Thus, substituting (67) into (64) for a sinusoidal profile gives

$$\frac{2}{\pi}\int_0^\gamma \frac{uq(u)}{v^2-u^2}\,du = G(v), \qquad |v| < \gamma, \tag{73}$$

with consistency condition

$$\int_0^\gamma \frac{G(v)}{\sqrt{\gamma^2-v^2}}\,dv = 0, \tag{74}$$

where

$$G(v) = \frac{2fp_0}{\pi\sqrt{1+\alpha^2}}\int_\gamma^\alpha \frac{u\sqrt{\alpha^2-u^2}}{(1+u^2)(v^2-u^2)}\,du + \frac{\beta p_0\sqrt{\alpha^2-v^2}}{\sqrt{1+\alpha^2}(1+v^2)} + \frac{2\Delta C}{A}\frac{|v|}{1+v^2}, \tag{75}$$

and C is an unknown constant. To solve for C, change the order of integration

$$\int_\gamma^\alpha \frac{s\sqrt{\alpha^2-s^2}\,ds}{(1+s^2)}\int_0^\gamma \frac{dv}{(v^2-s^2)\sqrt{\gamma^2-v^2}} = -\frac{\pi}{2}\int_\gamma^\alpha \frac{\sqrt{\alpha^2-s^2}\,ds}{(1+s^2)\sqrt{s^2-\gamma^2}}. \tag{76}$$

Equation (75) is then substituted into (74) in light of (76), resulting in

$$\frac{fp_0}{\sqrt{1+\alpha^2}}\int_\gamma^\alpha \frac{\sqrt{\alpha^2-v^2}\,dv}{(1+v^2)\sqrt{v^2-\gamma^2}} - \frac{\beta p_0}{\sqrt{1+\alpha^2}}\int_0^\gamma \frac{\sqrt{\alpha^2-v^2}\,dv}{(1+v^2)\sqrt{\gamma^2-v^2}} + \frac{2C\Delta}{A}\int_0^\gamma \frac{v\,dv}{(1+v^2)\sqrt{\gamma^2-v^2}} = 0. \tag{77}$$

The integrals in (77) can now all be evaluated analytically. Returning now to the integral equation, the shear tractions are found by first substituting (75) into (73). Then, using partial fractions and (77), $q(u)$ becomes

$$q(u) = -\frac{2u}{\pi}\frac{\sqrt{\gamma^2-u^2}}{(1+u^2)}\Bigg[\frac{-fp_0}{\sqrt{1+\alpha^2}}\int_\gamma^\alpha \frac{\sqrt{\alpha^2-v^2}\,dv}{(v^2-u^2)\sqrt{v^2-\gamma^2}} + \frac{\beta p_0}{\sqrt{1+\alpha^2}}\int_0^\gamma \frac{\sqrt{\alpha^2-v^2}\,dv}{(v^2-u^2)\sqrt{\gamma^2-v^2}} + \frac{2\Delta C}{A}\int_0^\gamma \frac{v\,dv}{(v^2-u^2)\sqrt{\gamma^2-v^2}}\Bigg].$$

Evaluating the integrals gives

$$q(u) = \left[\frac{fp_0}{(1+u^2)}\frac{\sqrt{\alpha^2-u^2}}{\sqrt{1+\alpha^2}}\frac{F(\phi,\gamma/\alpha)}{K(\gamma/\alpha)} + \frac{\Delta C u}{\pi A(1+u^2)}\ln\left(\frac{\alpha+\sqrt{\alpha^2-u^2}}{\alpha-\sqrt{\alpha^2-u^2}}\right)\right]\operatorname{sgn}(\beta u),$$

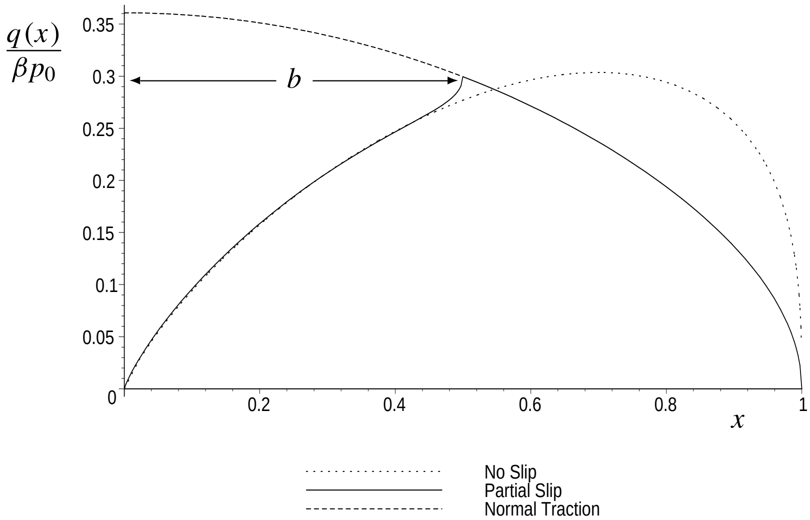

Figure 12. Partial slip of a sinusoidal surface on an elastic half-space for $a = 1$ with $f = 0.215$, $\nu = 0.3$, and $L = 2\pi$.

where $\phi = \sin^{-1}(u/\gamma)$ and $F(\phi, k)$ is an incomplete elliptic integral of first kind. Returning to original variables, the shear tractions are

$$q(x) = \left(\frac{f p_0}{\sqrt{2}} \cos\frac{x}{2} \sqrt{\cos x - \cos a}\, \frac{F\big(\phi,\ \tan(b/2)/\tan(a/2)\big)}{K\big(\tan(b/2)/\tan(a/2)\big)} + \frac{C p_0 \sin(x/2)}{2\pi} \ln \frac{\tan(b/2) + \sqrt{\tan(b/2)^2 - \tan(x/2)^2}}{\tan(b/2) - \sqrt{\tan(b/2)^2 - \tan(x/2)^2}} \right) \operatorname{sgn}\Big(\beta \tan\frac{x}{2}\Big), \quad |x| < \tan\frac{b}{2},$$

where C is determined from (77) with (71) and $\phi = \sin^{-1}(\tan(x/2)/\tan(b/2))$.

The partial slip contact of an elastically dissimilar wavy surface and a half-space is compared with the complete stick solution, (61), and normal traction solution, (21), in Figure 12. Again, the coefficient of friction is chosen such that the stick and slip regions are equal. The dotted line above represents the normal traction solution multiplied by the coefficient of friction, meaning Coulomb friction, whereas the dash-dot line represents the no-slip solution.

For the partial slip solution, the transition of the shear traction from the stick region to the slip region is much more drastic than that for the flat punch for the same parameters. When the contact length approaches the wavelength, the Goodman approximation is no longer valid. Not surprisingly, increasing the coefficient of friction can also have the same effect, since greater shear tractions are required to maintain the large contact region. The no slip solution is indistinguishable from the partial slip curve initially, but later overshoots the normal traction plot, as was the case for the single indenter [Johnson 1985b]. The complete stick solution however does not violate the Coulomb friction law as the contact region approaches the wavelength, although the curve no longer is approximately equivalent to the partial slip curve for small x.

Conclusion

The contact mechanics solutions derived in the paper may prove useful for validating computational approaches and are convenient for benchmarking numerical calculations or analyzing experimental results. Because of the systematic nature of the theory presented, more complicated problems can also be solved starting with the fundamental integral equations of contact.

The findings of the paper are summarized below:

(1) Based on a periodic array of Flamant solutions, the contact integral equations were derived. The equilibrium equations were also provided for periodic profiles.

(2) Previously known solutions for the frictionless contact of a sinusoidal profile, flat punch and inclined punch were derived from the integral equations of contact in order to demonstrate the validity of the method.

(3) New closed-form results for sliding contact with Coulomb friction for evenly-spaced flat punches and sinusoidal profiles were derived.

(4) Partial slip results for elastically similar materials were rederived using the contact integral equations and extended to periodic power law indenters.

(5) The first solution for the complete stick of periodic flat punches using the coupled integral equation was derived; the Goodman approximation was applied to complete stick and partial slip problems for both periodic flat punches and a sinusoidal profile to obtain new closed-form solutions and its applicability commented upon.

7. Acknowledgments

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Appendix A: Integral equation inversions

Hills et al. [1993] provide straightforward solutions of Cauchy singular integral equations of the first and second kind. Since these solution techniques are regularly used throughout the paper and in view of the limited availability of the book, the results are reproduced exactly in this appendix, for the reader's convenience.

A.1 *Cauchy singular integral equations of the first kind*

Equation to be solved: $$\frac{1}{\pi}\int_{-a}^{a}\frac{f(s)}{x-s}ds = g(x), \qquad |x| < a$$

Solution: $$f(x) = -\frac{w(x)}{\pi}\int_{-a}^{a}\frac{g(s)}{w(s)(s-x)}ds + Cw(x),$$

where the solution is required to have the following characteristics:

Singular at both ends: $w(x) = 1/\sqrt{a^2 - x^2}, \quad C \neq 0$

Nonsingular at $x = a$: $w(x) = \sqrt{(a-x)/(a+x)}, \quad C \neq 0$

Nonsingular at $x = -a$: $w(x) = \sqrt{(a+x)/(a-x)}, \quad C = 0$

Nonsingular at $x = \pm a$: $w(x) = \sqrt{a^2 - x^2}, \quad C = 0$, with the consistency condition $\displaystyle\int_{-a}^{a} \frac{g(s)}{w(s)} ds = 0$

A.2 *Cauchy singular integral equations of the second kind*

Equation to be solved: $$Rf(x) + \frac{P}{\pi} \int_{-a}^{a} \frac{f(s)}{s-x} ds = g(x), \qquad |x| < a$$

Solution: $$f(x) = \frac{1}{R^2 + P^2}\left(Rg(x) - \frac{P}{\pi} w(x) \int_{-a}^{a} \frac{g(s)}{w(s)(s-x)} ds \right) + Cw(x),$$

assuming the functions involved are wholly real and that $\tan \pi B = -P/R$.

The solution is required to have the following characteristics:

Singular at both ends: $w(x) = (a-x)^{B-1}(a+x)^{-B}, \quad C \neq 0$

Nonsingular at $x = a$: $w(x) = (a-x)^{B}(a+x)^{-B}, \quad C = 0$

Nonsingular at $x = -a$: $w(x) = (a-x)^{B-1}(a+x)^{1-B}, \quad C = 0$

Nonsingular at $x = \pm a$: $w(x) = (a-x)^{B}(a+x)^{1-B}, \; C = 0$, with consistency condition $\displaystyle\int_{-a}^{a} \frac{g(s)}{w(s)} ds = 0$

Appendix B: Integrals of importance

Below are a few of the complex integrals that were derived to solve periodic problems in plane elasticity. Elliptic integrals are also provided since their solutions prove useful as well.

Complex integrals for $0 < m < 1$ *and* $\tan \phi = 1/a$

$$\int_{-a}^{a} \frac{(a-u)^{m-1}(a+u)^{-m}}{1+u^2} du = \frac{\pi}{\sqrt{1+\alpha^2}} \frac{\sin[\pi m - \phi(2m-1)]}{\sin \pi m}$$

$$\int_{-a}^{a} \frac{u(a-u)^{m-1}(a+u)^{-m}}{1+u^2} du = \frac{\pi}{\sqrt{1+\alpha^2}} \frac{\cos[\pi m - \phi(2m-1)]}{\sin \pi m}$$

$$\int_{-a}^{a} \frac{(a-u)^{m}(a+u)^{1-m}}{1+u^2} du = \frac{\pi}{\sin \pi m} \{\sqrt{1+a^2} \sin[\pi m - \phi(2m-1)] - 1\}$$

$$\int_{-a}^{a} \frac{u(a-u)^{m}(a+u)^{1-m}}{1+u^2} du = \frac{\pi}{\sin \pi m} \{\sqrt{1+a^2} \cos[\pi m - \phi(2m-1)] - a(2m-1)\}$$

$$\int_{-a}^{a} \frac{(a-u)^{m}(a+u)^{1-m}}{(1+u^2)^2} du = \frac{\alpha\pi(2m-1)\sqrt{1+a^2}}{\sin \pi m} \cos[\pi m - \phi(2m-1)]$$

$$\int_{-a}^{a} \frac{u(a-u)^{m}(a+u)^{1-m}}{(1+u^2)^2} du = -\frac{\pi\sqrt{1+a^2}}{\sin \pi m} \cos[\pi m - \phi(2m-1)]$$

Elliptic integrals of the first, second and third kinds

Incomplete integrals:

$$F(\phi,k)=\int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \qquad E(\phi,k)=\int_0^{\phi}\sqrt{1-k^2\sin^2\theta}\,d\theta \qquad \Pi(n;\phi,k)=\int_0^{\phi}\frac{d\theta}{(1-n^2\theta)\sqrt{1-k^2\sin^2\theta}}$$

Complete integrals: $K(k)=F(\pi/2,k)$ $\quad E(k)=E(\pi/2,k)$ $\quad \Pi(n,k)=\Pi(n;\pi/2,k)$

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