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AXISYMMETRIC INDENTATION OF A RIGID CYLINDER ON A LAYERED COMPRESSIBLE AND INCOMPRESSIBLE HALFSPACE

LUIGI LA RAGIONE, FRANCESCO MUSCEO AND ALFREDO SOLLAZZO

We propose a solution for an elastic, axisymmetric, indentation problem. The indenter is a rigid cylinder on an elastic layer in contact with an elastic substrate. The goal is to provide a contact law between the applied force and the displacement of the coating in two cases: frictionless interaction and perfect binding between the coating and the substrate. As examples we have considered situations in which the substrate is either softer, similarly stiff, or stiffer than the coating, both for compressible or incompressible materials.

1. Introduction

Problems concerning the investigation of strain and stress in elastic bodies in contact are the goal of researches devoted to theoretical models and applications in the industry; see, for example, [Li and Chou 1997; Johnson and Sridhar 2001; Wang et al. 2004; Sburlati 2006].

A relative recent interest in contact mechanics (an exhaustive treatise of contact problems can be found in [Johnson 1985]) has focused on indentation problems upon layered solids with coating different from the substrate. This is of interest, for example, in the measurement of mechanical properties, such as hardness and elastic moduli, of surface films in not destructive experimental tests, also on the micro- or nanoscale.

In this paper we focus our attention on the case of a rigid cylinder with circular section indenting a layered body formed by an isotropic halfspace with an isotropic surface coating having different mechanical characteristic from the substrate: this is an axisymmetric problem for compressible or incompressible layers.

From the general case, in which no restriction is made on the elastic properties of the bodies in contact, we deduce a limit condition for the case where the substrate is stiffer than the surface coating. (An example of a solution in the case of a rigid foundation can be found in [Matthewson 1981; Yang 1998; 2003].)

Our solution is based on Hankel integral transforms, developed by Harding and Sneddon [1945] and applied by Sneddon [1946] to the case of half space, that leads to a second kind Fredholm integral equation numerically solvable.

We make the following hypotheses to approach the problem: the friction influences in a negligible way the normal stress distribution on the interface between the layer and the cylinder (see [Johnson 1985], for example); and we consider only the limit contact cases between the layer and the elastic substrate, that is, either perfect bonding or absence of friction.

Keywords: contact mechanics, elasticity, indentation, Hankel transform.

We believe that the model allows to highlight characteristics of solution that are unlikely to be noticed through numerical simulations alone (as in [Komvopoulos 1988]), especially in the inherent case of a rigid foundation where we obtain a solution normalized by all mechanical parameters.

2. Problem formulation

We study the quasistatic axisymmetric indentation of a rigid circular cylinder with radius a, in the context of small deformation. The cylinder produces a normal force F upon an elastic body made by an isotropic surface layer with moduli E_1 and ν_1 and thickness h, and a semi-indefinite isotropic substrate with moduli E_2 and ν_2 (see Figure 1). For the interface between the layer and the elastic substrate we consider two limit conditions: perfect bonding and absence of friction. The contact area does not vary with the loading and it is a circle with radius a. The normal acting force is fixed and the displacement for all points of the contact surface, δ, is the same.

We take two cylindrical frames of reference: the triplet (r, ϑ, z) refers to the surface layer, with r and ϑ belonging to the upper surface of the coating and the z axis that coincides with the axis of symmetry; the triplet (r', ϑ', z') refers to the substrate with r' and ϑ' belonging to the interface layer-substrate surface and the z' axis superimposed on z. That is, the two frames of reference differ for a translation h in the positive z direction.

Because of the symmetry, the problem can be simplified by focusing only on the positive quadrants (O, r, z), $r \geq 0$, $z \geq 0$, and (O, r', z'), $r' \geq 0$, $z' \geq 0$.

We first consider Mitchell's theory [Sneddon 1951] for isotropic bodies deformed in axisymmetry condition where stress and strain can be expressed through a single function. We refer to the generic frame of reference (O, r, ϑ, z) and we introduce a potential $\Phi(r, z)$ related to the nonzero components of the strain such that

$$u_r(r, z) = -\frac{(1+\nu)}{E}\frac{\partial}{\partial r}\frac{\partial \Phi(r, z)}{\partial z}, \tag{1}$$

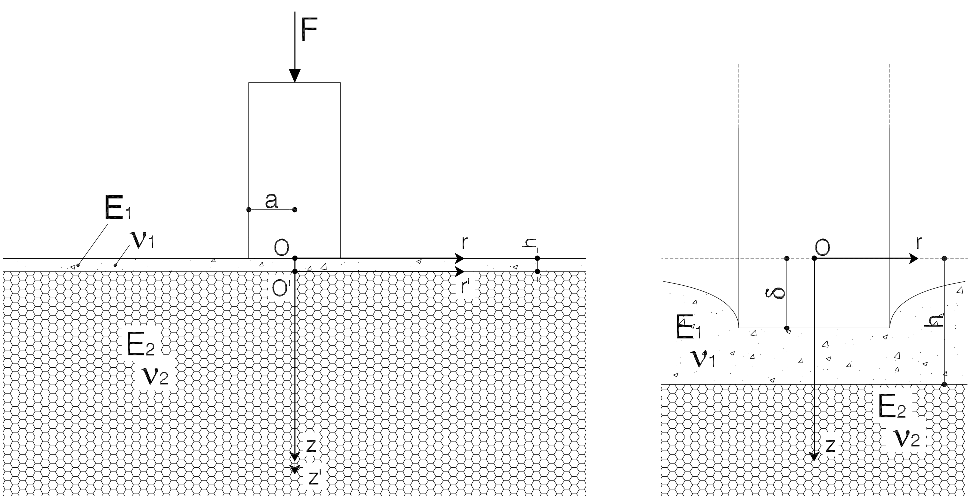


Figure 1. Cylindrical indenter on layered halfspace.

$$u_z(r,z) = \frac{2(1-\nu^2)}{E}\nabla^2\Phi(r,z) - \frac{(1+\nu)}{E}\frac{\partial^2\Phi(r,z)}{\partial z^2}. \tag{2}$$

It is then straightforward to obtain the following expressions for the component of stress:

$$\sigma_z(r,z) = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\Phi(r,z) - \frac{\partial^2\Phi(r,z)}{\partial z^2}\right], \tag{3}$$

$$\sigma_r(r,z) = \frac{\partial}{\partial z}\left[\nu\nabla^2\Phi(r,z) - \frac{\partial^2\Phi(r,z)}{\partial r^2}\right], \tag{4}$$

$$\sigma_\vartheta(r,z) = \frac{\partial}{\partial z}\left[\nu\nabla^2\Phi(r,z) - \frac{1}{r}\frac{\partial\Phi(r,z)}{\partial r}\right], \tag{5}$$

$$\tau_{zr}(r,z) = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\Phi(r,z) - \frac{\partial^2\Phi(r,z)}{\partial z^2}\right], \tag{6}$$

with

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{\partial z^2}.$$

If we apply Hankel transform theory [Sneddon 1951] to (1)–(6) indicating the transform with $\widetilde{\ }$, and introduce the parameter ξ instead of r, all the listed quantities can be written as function of the zeroth order Hankel transform of the potential $\widetilde{\Phi}_0(\xi,z)$:

$$u_r(r,z) = \int_0^\infty \frac{(1+\nu)}{E}\xi^2\frac{d\widetilde{\Phi}_0(\xi,z)}{dz}J_1(\xi r)\,d\xi, \tag{7}$$

$$u_z(r,z) = \int_0^\infty \xi\left[\frac{(1-2\nu)(1+\nu)}{E}\frac{d^2\widetilde{\Phi}_0(\xi,z)}{dz^2} - \frac{2(1-\nu^2)}{E}\xi^2\widetilde{\Phi}_0(\xi,z)\right]J_0(\xi r)\,d\xi, \tag{8}$$

$$\sigma_z(r,z) = \int_0^\infty \xi\left[(1-\nu)\frac{d^3\widetilde{\Phi}_0(\xi,z)}{dz^3} - (2-\nu)\xi^2\frac{d\widetilde{\Phi}_0(\xi,z)}{dz}\right]J_0(\xi r)\,d\xi, \tag{9}$$

$$\sigma_r(r,z) = \int_0^\infty \xi\left(\nu\frac{d^3\widetilde{\Phi}_0(\xi,z)}{dz^3} + (1-\nu)\xi^2\frac{d\widetilde{\Phi}_0(\xi,z)}{dz}\right)J_0(\xi r)\,d\xi - \frac{1}{r}\int_0^\infty \xi^2\frac{d\widetilde{\Phi}_0(\xi,z)}{dz}J_1(\xi r)\,d\xi, \tag{10}$$

$$\sigma_\vartheta(r,z) = \int_0^\infty \xi\nu\left(\frac{d^3\widetilde{\Phi}_0(\xi,z)}{dz^3} - \xi^2\frac{d\widetilde{\Phi}_0(\xi,z)}{dz}\right)J_0(\xi r)\,d\xi + \frac{1}{r}\int_0^\infty \xi^2\frac{d\widetilde{\Phi}_0(\xi,z)}{dz}J_1(\xi r)\,d\xi, \tag{11}$$

$$\tau_{zr}(r,z) = \int_0^\infty \xi^2\left[\nu\frac{d^2\widetilde{\Phi}_0(\xi,z)}{dz^2} + (1-\nu)\xi^2\widetilde{\Phi}_0(\xi,z)\right]J_1(\xi r)\,d\xi. \tag{12}$$

where $J_m(r\xi)$ represents the m order Bessel function of first kind, in the variable $r\xi$.

On this basis, the indefinite balance equations are automatically satisfied, while compatibility gives the equation

$$\left(\frac{d^2}{dz^2} - \xi^2\right)^2 \widetilde{\Phi}_0(\xi,z) = 0\,,$$

whose solution is

$$\widetilde{\Phi}_0(\xi,z) = [L(\xi) + M(\xi)z]\sinh(\xi z) + [N(\xi) + P(\xi)z]\cosh(\xi z)\,, \tag{13}$$

in the unknown functions $L(\xi)$, $M(\xi)$, $N(\xi)$ e $P(\xi)$.

Note that the potential function introduced agrees with the one considered in [Timoshenko and Goodier 1970], and differs from that indicated in [Love 1944] and in [Sneddon 1951] by a term $(1+\nu)\,(1-2\nu)/E$. The former form allows us to determine a solution for the special case of incompressible materials ($\nu = 1/2$).

As the layered halfspace is axisymmetric deformed, the present theory is applicable to both upper layer and substrate.

We label with the apex $^{(c)}$ terms related to the surface coating, and with apex $^{(s)}$ those related to the substrate. The zeroth order Hankel transform for the potential of the coating, in the frame of reference (O, r, z), is given by:

$$\widetilde{\Phi}_0^{(c)}(\xi, z) = [A(\xi) + B(\xi)z]\cosh(\xi z) + [C(\xi) + D(\xi)z]\sinh(\xi z), \tag{14}$$

where $A(\xi)$, $B(\xi)$, $C(\xi)$ and $D(\xi)$ are unknowns. In the frame of reference (O, r', z'), the potential for the substrate is given by

$$\widetilde{\Phi}_0^{(s)}(\xi, z') = -\left[S(\xi)z' + T(\xi)\right]\sinh(\xi z') + \left[S(\xi)z' + T(\xi)\right]\cosh(\xi z'), \tag{15}$$

in the unknowns $S(\xi)$ and $T(\xi)$ where we have imposed the vanishing of stresses and displacements as z' goes to ∞. The equilibrium equation along the z axis can be phrased as

$$2\pi \int_0^a \left[\sigma_z\right]_{z=0}^{(c)} r\,dr = -F. \tag{16}$$

The boundary condition at the external surface of the layer are

$$[\tau_{zr}]_{z=0}^{(c)} = 0 \quad \text{for } r > 0, \tag{17}$$

$$[u_z]_{z=0}^{(c)} = \delta \quad \text{for } 0 \le r \le a, \tag{18}$$

$$[\sigma_z]_{z=0}^{(c)} = 0 \quad \text{for } r > a. \tag{19}$$

The interaction between the layer and the substrate, keeping in mind the relation $z = z' + h$, can be expressed, in the case of a perfect bond, through the equations

$$[u_z]_{z=h}^{(c)} = [u_z]_{z'=0}^{(s)}, \tag{20}$$

$$[u_r]_{z=h}^{(c)} = [u_r]_{z'=0}^{(s)}, \tag{21}$$

$$[\tau_{zr}]_{z=h}^{(c)} = [\tau_{zr}]_{z'=0}^{(s)}, \tag{22}$$

$$[\sigma_z]_{z=h}^{(c)} = [\sigma_z]_{z'=0}^{(s)} \quad . \tag{23}$$

For the frictionless case we replace the conditions (21) and (22), that refer to the continuity of the radial displacement and stress, with

$$[\tau_{zr}]_{z=h}^{(c)} = 0, \qquad [\tau_{zr}]_{z'=0}^{(s)} = 0,$$

which accounts for the absence of tangential traction between the layer and the substrate.

2.1. ***Solution.*** With the boundary conditions above, except for equations (16), (18), (19) and with equations (14), (15), we express $B(\xi)$, $C(\xi)$, $D(\xi)$, $S(\xi)$, $T(\xi)$ as functions of $A(\xi)$ by means of Hankel transforms (see Appendix A). Therefore we write

$$D(\xi) = -\frac{\xi A(\xi)}{2\nu_1}, \tag{24}$$

$$S(\xi) = \frac{S(\xi)_{(N)}}{S(\xi)_{(D)}}, \quad T(\xi) = \frac{T(\xi)_{(N)}}{T(\xi)_{(D)}}, \quad B(\xi) = \frac{B(\xi)_{(N)}}{B(\xi)_{(D)}}, \quad C(\xi) = \frac{C(\xi)_{(N)}}{C(\xi)_{(D)}}, \tag{25}$$

where, in the case of a perfect bond, we obtain

$$\begin{aligned}
S(\xi)_{(N)} &= 2A(\xi)E_2\xi(1-\nu_1^2)\big\{\big(E_2(1+\nu_1)(-2+h\xi+2\nu_1)-hE_1\xi(1+\nu_2)\big)\cosh(\xi h)\\
&\qquad +\big(-E_2(1+h\xi-2\nu_1)(1+\nu_1)+E_1(-1+h\xi)(1+\nu_2)\big)\sinh(\xi h)\big\},\\
T(\xi)_{(N)} &= 2A(\xi)E_2(1-\nu_1^2)\big\{\big(4E_2(-1+\nu_1^2)\nu_2-h\xi[E_1+E_2+E_2\nu_1+\nu_2(E_1-2E_2(1+\nu_1))]\big)\cosh(\xi h)\\
&\qquad +\big(E_1(1+\nu_2)\,[-1+2h\xi(-1+2\nu_2)]+E_2(1+\nu_1)\,[1-2(1+h\xi)\nu_2+\nu_1(-2+4\nu_2)]\big)\sinh(\xi h)\big\},\\
B(\xi)_{(N)} &= \tfrac{1}{2}A(\xi)\xi\big\{[E_2(1+\nu_1)-E_1(1+\nu_2)][-E_2(1+\nu_1)+E_1(-3+\nu_2+4\nu_2^2)]\\
&\qquad +\big[E_2^2(1+\nu_1)^2(-3+4\nu_1)+E_1^2(1+\nu_2)^2(-3+4\nu_2)-2E_1E_2(-1+\nu_1+2\nu_1^2)(-1+\nu_2+2\nu_2^2)\big]\cosh(2\xi h)\\
&\qquad -8E_1E_2(-1+\nu_1^2)(-1+\nu_2^2)\sinh(2\xi h)\big\},\\
C(\xi)_{(N)} &= A(\xi)\,\big\{E_2^2(1+\nu_1)^2\,\big[2+h^2\xi^2+\nu_1(-5+4\nu_1)\big]+E_1^2(h^2\xi^2+\nu_1)(1+\nu_2)^2(-3+4\nu_2)\\
&\qquad -2E_1E_2(1+\nu_1)(-1+h^2\xi^2+2\nu_1)(-1+\nu_2+2\nu_2^2)\\
&\qquad +\nu_1\big[-E_2^2(1+\nu_1)^2(-3+4\nu_1)-E_1^2(1+\nu_2)^2(-3+4\nu_2)+2E_1E_2(-1+\nu_1+2\nu_1^2)(-1+\nu_2+2\nu_2^2)\big]\cosh(2\xi h)\\
&\qquad +8E_1E_2\nu_1(-1+\nu_1^2)(-1+\nu_2^2)\sinh(2\xi h)\big\},\\
T(\xi)_{(D)} &= B(\xi)_{(D)} = C(\xi)_{(D)} = S(\xi)_{(D)},\\
S(\xi)_{(D)} &= \nu_1\big\{2h\xi[-E_2(1+\nu_1)+E_1(1+\nu_2)][-E_2(1+\nu_1)+E_1(-3+\nu_2+4\nu_2^2)]\\
&\qquad -8E_1E_2(-1+\nu_1^2)(-1+\nu_2^2)\cosh(2\xi h)\\
&\qquad +\big[E_2^2(1+\nu_1)^2(-3+4\nu_1)+E_1^2(1+\nu_2)^2(-3+4\nu_2)-2E_1E_2(-1+\nu_1+2\nu_1^2)\,(-1+\nu_2+2\nu_2^2)\big]\sinh(2h\xi)\big\},
\end{aligned}$$

while the frictionless case leads to

$$\begin{aligned}
S(\xi)_{(N)} &= A(\xi)\xi E_2(-1+\nu_1^2)\,[h\xi\cosh(h\xi)+\sinh(\xi h)]\,,\\
T(\xi)_{(N)} &= A(\xi)2E_2\nu_2(-1+\nu_1^2)\,[h\xi\cosh(h\xi)+\sinh(\xi h)]\,,\\
B(\xi)_{(N)} &= A(\xi)\xi\sinh(\xi h)\,\big[E_2(-1+\nu_1^2)\cosh(h\xi)+E_1(-1+\nu_2^2)\sinh(\xi h)\big]\,,\\
C(\xi)_{(N)} &= A(\xi)\,\big[-E_2\xi h(-1+\nu_1^2)+E_1\xi^2h^2(-1+\nu_2^2)+E_1\nu_1(-1+\nu_2^2)\\
&\qquad -E_1\nu_1(-1+\nu_2^2)\cosh(2h\xi)-E_2\nu_1(-1+\nu_1^2)\sinh(2\xi h)\big]\,,\\
T(\xi)_{(D)} &= B(\xi)_{(D)} = C(\xi)_{(D)} = S(\xi)_{(D)},\\
S(\xi)_{(D)} &= \nu_1\,\big[E_2(1-\nu_1^2)-2\xi hE_1(1-\nu_2^2)-E_2(1-\nu_1^2)\cosh(2h\xi)-E_1(1-\nu_2^2)\sinh(2h\xi)\big]\,.
\end{aligned}$$

Conditions (18) and (19), which refer to parts of the domain, represent the dual integral equation that allows us to determine the unknown function $A(\xi)$.

In terms of Hankel transforms, equations (18) and (19) can be written as

$$\int_0^\infty \xi\,\big[\widetilde{u}_{z0}\big]^{(c)}_{z=0} J_0(\xi r)\,d\xi = \delta \quad \text{for } 0 \le r \le a, \tag{26}$$

$$\int_0^\infty \xi\,\big[\widetilde{\sigma}_{z0}\big]^{(c)}_{z=0} J_0(\xi r)\,d\xi = 0 \quad \text{for } r > a, \tag{27}$$

with $\big[\widetilde{u}_{z0}\big]^{(c)}$ and $\big[\widetilde{\sigma}_{z0}\big]^{(c)}$ function of $\widetilde{\Phi}^{(c)}_0(\xi, z)$ as specified in Appendix A.

If we write

$$\big[\widetilde{u}_{z0}\big]^{(c)}_{z=0} = -\xi^2 A(\xi) k_u, \tag{28}$$

$$\big[\widetilde{\sigma}_{z0}\big]^{(c)}_{z=0} = \xi^3 A(\xi) k_s, \tag{29}$$

where

$$k_u = \frac{(1-\nu_1^2)}{E_1\nu_1} \qquad \text{and} \qquad k_s(\xi) = \frac{k_{s(N)}}{k_{s(D)}}, \tag{30}$$

with, in the case of a perfect bond

$$\begin{aligned} k_{s(N)} = -\big\{&E_2^2(1+\nu_1)^2\big(5+2h^2\xi^2+4\nu_1(-3+2\nu_1)\big) - E_1^2(1+2h^2\xi^2)(1+\nu_2)^2(-3+4\nu_2)\\ &+2E_1E_2(1+\nu_1)(-1+2h^2\xi^2+2\nu_1)(-1+\nu_2+2\nu_2^2)\\ &+\big[E_2^2(1+\nu_1)^2(-3+4\nu_1)+E_1^2(1+\nu_2)^2(-3+4\nu_2)-2E_1E_2(-1+\nu_1+2\nu_1^2)(-1+\nu_2+2\nu_2^2)\big]\cosh(2h\xi)\\ &-8E_1E_2(-1+\nu_1^2)(-1+\nu_2^2)\sinh(2h\xi)\big\},\end{aligned}$$

$$k_{s(D)} = 2S(\xi)_{(D)},$$

and, in the frictionless case,

$$k_{s(N)} = -2\xi h E_2(1-\nu_1^2) - E_1(1+2\xi^2h^2)(1-\nu_2^2) - E_1(1-\nu_2^2)\cosh(2h\xi) - E_2(1-\nu_1^2)\sinh(2h\xi),$$

$$k_{s(D)} = 2S(\xi)_{(D)},$$

then equations (26) and (27) become

$$\int_0^\infty -\xi^3 A(\xi) k_u J_0(\xi r)\,d\xi = \delta \quad \text{for } 0 \le r \le a, \tag{31}$$

$$\int_0^\infty \xi^4 A(\xi) k_s(\xi) J_0(\xi r)\,d\xi = 0 \quad \text{for } r > a. \tag{32}$$

Equations (31) and (32) represent the dual integral equation that solves the problem. At this point we search for a numerical solution instead of treating the problem analytically. From (31) and (32) we can find solutions in the case of an incompressible layer; this special case is not treatable if we adopt Sneddon's expression (for $\nu = 1/2$ we have $k_s(\xi) = \infty$).

In Appendix B we show that if the thickness of the coating tends to infinite the potential of surface layer converges to the one related to the indefinite isotropic halfspace with moduli E_1 and ν_1, while when the thickness vanishes the potential of substrate converge to that related to the indefinite isotropic halfspace with moduli E_2 and ν_2.

3. Numerical solutions

The dual integral equation (31)–(32), for the Erdélyi–Sneddon solution method [Sneddon 1966], can be converted in a second kind Fredholm equation whose kernels, in our cases, have only singularity in the integration limits.

With

$$A^*(\xi) = \xi^4 A(\xi) k_s(\xi), \quad \bar{\xi} = a\xi, \quad \bar{r} = r/a,$$
$$\frac{\varrho}{k_s(\bar{\xi})} = 1 + I(\bar{\xi}), \tag{33}$$
$$\overline{A^*}(\bar{\xi}) = A^*(\bar{\xi}) \frac{k_u}{\varrho\delta},$$

the dual integral equation (31)–(32) becomes

$$\int_0^\infty \xi^{-1} \overline{A^*}(\bar{\xi}) \left[1 + I(\bar{\xi})\right] J_0(\bar{\xi}\bar{r})\, d\bar{\xi} = -1 \quad 0 < \bar{r} < 1, \tag{34}$$

$$\int_0^\infty \overline{A^*}(\bar{\xi}) J_0(\bar{\xi}\bar{r})\, d\bar{\xi} = 0 \quad \bar{r} > 1. \tag{35}$$

The parameter ϱ has to be chosen so that the Fourier cosine transform of function $I(t)$ exists with the auxiliary parameters $|x-u|$ e $|x+u|$; that is, the following integrals converge

$$\int_0^\infty I(t) \cos(t\,|x-u|)\, dt + \int_0^\infty I(t) \cos(t\,|x+u|)\, dt.$$

This condition implies that

$$\lim_{\bar{\xi}\to\infty} \frac{\varrho}{k_s(\xi)} = 1,$$

which means, either for perfect bond or frictionless case, that $\varrho = 1/2\nu_1$.

The solution of (34)–(35) is

$$\overline{A^*}(\bar{\xi}) = \frac{\bar{\xi}}{\sqrt{\pi}} \int_0^1 h_1(t) \cos\left(\bar{\xi}t\right) dt, \tag{36}$$

where $h_1(t)$ is the solution of the following Fredholm integral equation [Sneddon 1966]

$$h_1(x) + \int_0^1 h_1(u) k(x,u)\, du = H(x), \tag{37}$$

with

$$k(x,u) = \frac{u}{x\sqrt{2\pi}} \left(\sqrt{\frac{2}{\pi}} \int_0^\infty I(t) \cos(t\,|x-u|)\, dt + \sqrt{\frac{2}{\pi}} \int_0^\infty I(t) \cos(t\,|x+u|)\, dt \right) \tag{38}$$

and

$$H(x) = -\frac{2}{x\sqrt{\pi}}.$$

Once we have solved Equation (37) numerically, together with (36), we obtain the dimensionless normal stress distribution inside the contact area

$$\overline{\left[\sigma_z\left(\bar{r}\right)\right]}^{(c)}_{z=0} = \left[\sigma_z\left(\bar{r}\right)\right]^{(c)}_{z=0}\frac{ak_u}{\varrho\delta} = \int_0^\infty \overline{A^*}(\bar{\xi})\,J_0(\bar{\xi}\bar{r})\,d\bar{\xi} \quad [0 < \bar{r} < 1]. \tag{39}$$

Next we impose the translation balance, Equation (16), and obtain the dimensionless force-displacement relation

$$\bar{F} = F\frac{k_u}{\varrho\delta a} = -2\pi\int_0^1 \overline{[\sigma_z(\bar{r})]}^{(c)}_{z=0}\,\bar{r}\,d\bar{r}, \tag{40}$$

which allows us to obtain a symbolic solution in E_1 and ν_1 in the case of elastic layer, with no friction, on a rigid substrate. In order to solve Equation (38) numerically we use the Newton–Cotes method, with the exclusion of the integration limits (see [Davis and Rabinowitz 1984], for example). We replace the integral in (37) by a series by dividing the range [0, 1] into N parts, so that

$$h_1(x_i) + \sum_{j=1}^{N-1} h_1(u_j)k(x_i, u_j)w_j = H\left(x_i\right), \tag{41}$$

where w_j represents the weight to be considered for the chosen integration method. Varying x_i and u_j with the same step, Equation (41) is equivalent to a linear system of equations in the variable $h_1(x_i)$:

$$(K + V)\,H_1 = H,$$

where H_1 is the variable array $h_1\left(x_i\right)$, K the coefficients matrix $k\left(x_i, u_j\right)w_j$, V the $(N-1)$th order identity matrix and H the known terms array $H\left(x_i\right)$. When we get $h_1\left(x_i\right)$ we can then evaluate numerically the expressions (36), (39) and (40).

4. Applications

We note that with $k_s(\xi)$ in (30) in both cases, perfect bond and frictionless condition, we do not have a symbolic solution. Therefore we take some values for all mechanical properties of the two elastic bodies. As examples of our activity we consider four cases of interacting materials; in particular we take a steel layer above a polystyrene substrate such that $E_1 = 210\,\text{GPa}$, $\nu_1 = 0.3$ and $E_2 = 2\,\text{GPa}$, $\nu_2 = 0.4$. This represents an application for a case in which the substrate is softer than the coating. Then we take a steel layer above a glass substrate such that $E_1 = 210\,\text{GPa}$, $\nu_1 = 0.3$ and $E_2 = 70\,\text{GPa}$, $\nu_2 = 0.22$; the opposite case is also treated such that $E_1 = 70\,\text{GPa}$, $\nu_1 = 0.22$ and $E_2 = 210\,\text{GPa}$, $\nu_2 = 0.3$. These represent an application for a case in which the substrate and the coating have similar stiffness. As last case, in the next section, we deal with the case of substrate much stiffer than the upper layer that can be treated as a limit case of rigid foundation.

In Figure 2 we plot the dimensionless force $\bar{F}$ for different values of the ratio $\bar{h} = h/a$, in the case of a perfect bond of a steel layer above a polystyrene substrate (top) and in the frictionless case (bottom). Figure 3 deals with the case of a steel layer above a glass substrate, and Figure 4 with that of a glass layer above a steel substrate.

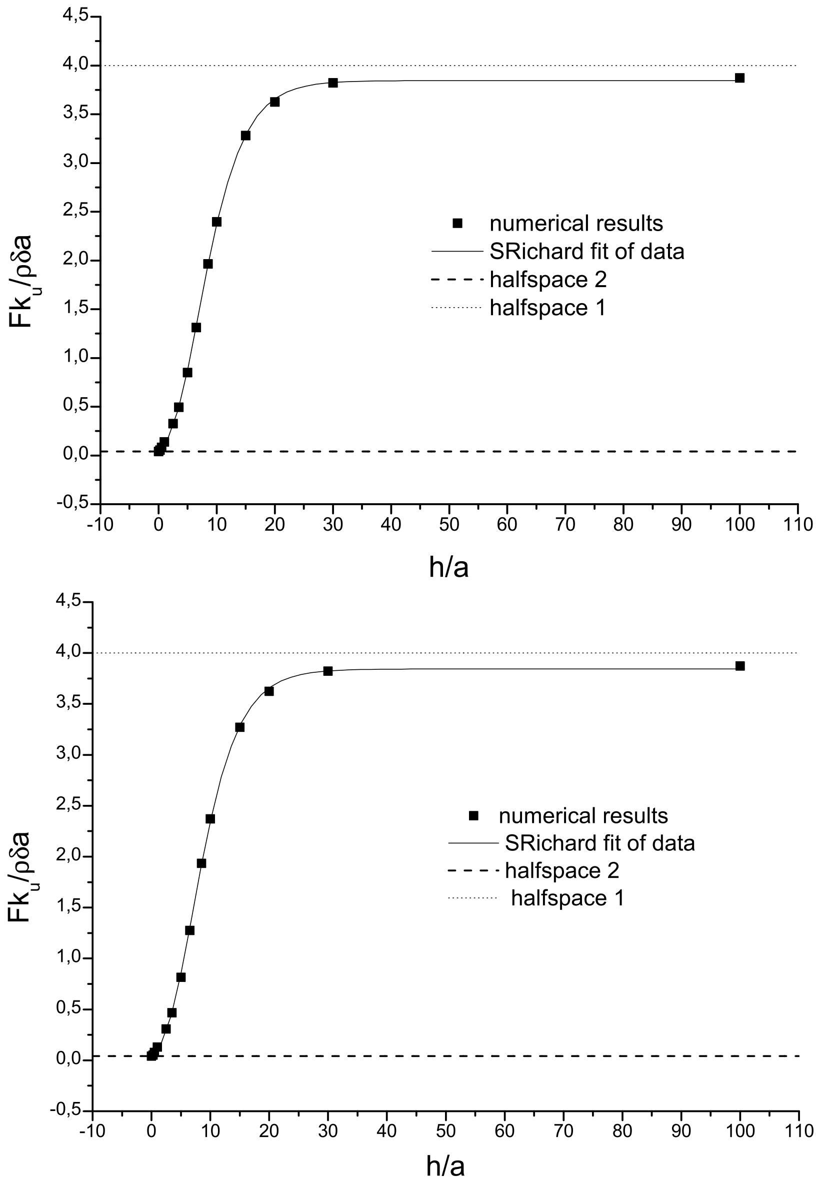


Figure 2. Elastic substrate (steel on polystyrene): contact law for perfect-bond (top) and frictionless case (bottom). Parameters: $E_1 = 210$ GPa, $\nu_1 = 0.3$, $E_2 = 2$ GPa, $\nu_2 = 0.4$).

In all figures we have also considered the limit values of $\bar{F}$ related to the elastic halfspace 1, with moduli E_1 and ν_1, and to the elastic halfspace 2, with moduli E_2 and ν_2. (For the half space with moduli E and ν, we have

$$\bar{F} = \frac{2}{\delta a} \frac{\left(1 - \nu_1^2\right)}{E_1} \left[\frac{2\delta E a}{(1 - \nu^2)} \right].$$

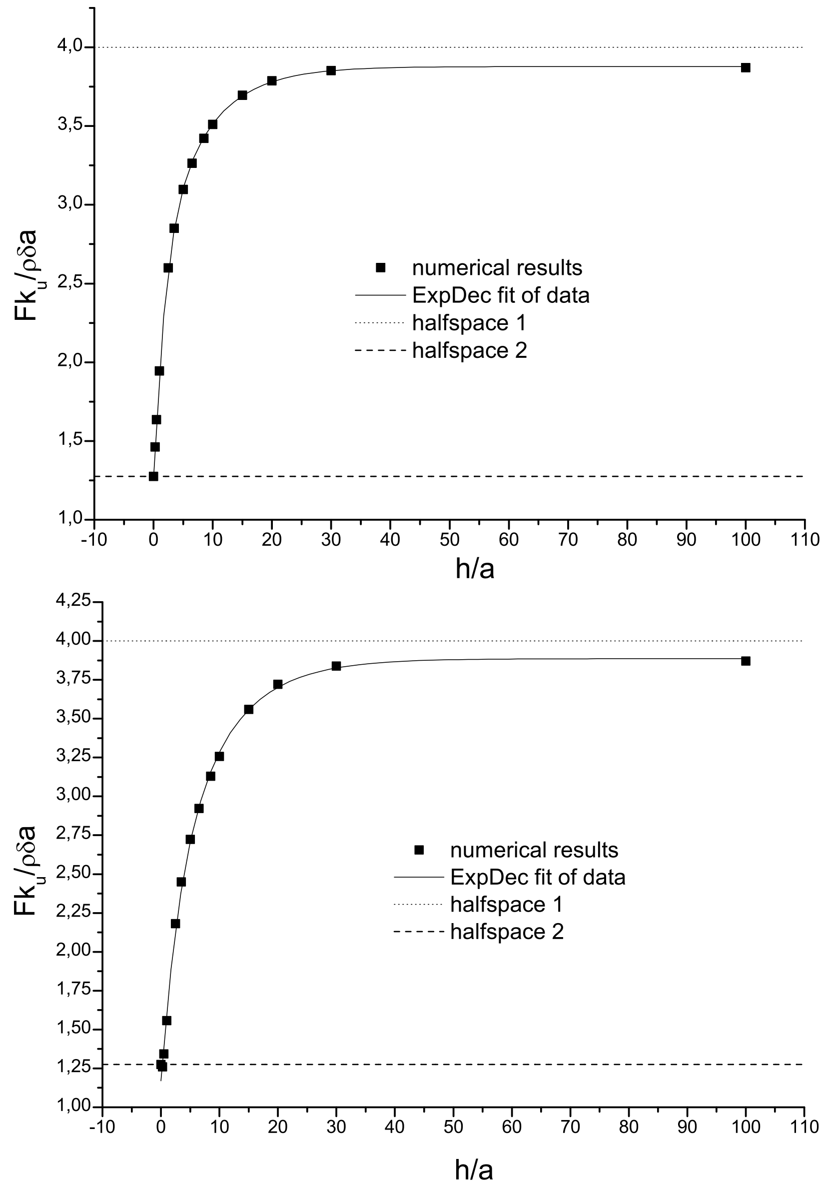


Figure 3. Elastic substrate (steel on glass): contact law for perfect-bond (top) and frictionless case (bottom). Parameters: $E_1 = 210\,\text{GPa}$, $\nu_1 = 0.3$, $E_2 = 70\,\text{GPa}$, $\nu_2 = 0.22$).

For the halfspace with moduli E_1 and ν_1, we have $\bar{F}_1 = 4$. For the halfspace with moduli E_2 and ν_2 we have

$$\bar{F}_2 = 4\frac{E_2}{E_1}\frac{\left(1-\nu_1^2\right)}{\left(1-\nu_2^2\right)}.$$

Therefore $\bar{F}_2 = 0.04127$ for steel on polystyrene, $\bar{F}_2 = 1.275$ for steel on glass and $\bar{F}_2 = 12.549$ for glass on steel.)

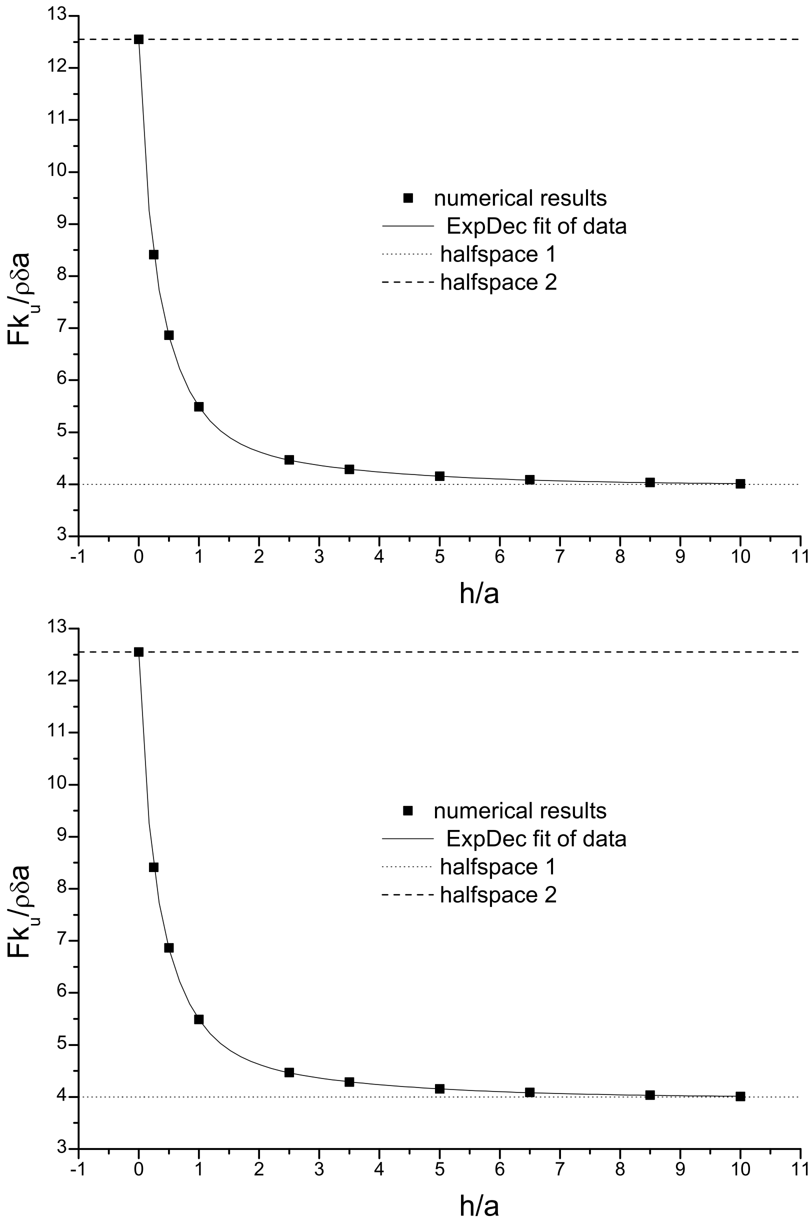


Figure 4. Elastic substrate (glass on steel): contact law for perfect-bond (top) and frictionless case (bottom). Parameters: $E_1 = 70\,\text{GPa}$, $\nu_1 = 0.22$, $E_2 = 210\,\text{GPa}$, $\nu_2 = 0.3$).

The numerical results have been fitted by means of the SRichards curve or ExpDec curve (see Appendix C). We also provide in Appendix C the numerical results related to Figures 2–4 in order to show more in detail the differences that emerge in the perfect-bond and frictionless cases.

We observe that for the coating's thickness $h < 0.5a$ the solution is close to the halfspace with the same characteristic of the substrate, while, when the ratio $\bar{h} = h/a$ increases it tends to the case of the halfspace with moduli E_1, ν_1, such that, for $h > 10a$, we can assume that the substrate does not influence the force-displacement relation.

With fixed thickness, $\bar{F}$ related to the frictionless case is lower than that related to perfect bond; that is, in the latter case the layer results less capable of being deformed.

5. Rigid foundation

As already underlined, if the substrate is much stiffer than the surface coating the problem can be approximated to the case of an elastic layer lying on a rigid foundation [Matthewson 1981; Yang 2003; Yang 1998]. The solution can be obtained directly by the general case developed previously by taking the limit $E_2 \to \infty$.

In the case of a perfect bond we have

$$\lim_{E_2\to\infty} B(\xi) = -\xi A(\xi)\frac{1+(3-4\nu_1)\nu_1\cosh(2\xi h)}{4h\xi\nu_1+2\nu_1(-3+4\nu_1)\sinh(2\xi h)}, \tag{42}$$

$$\lim_{E_2\to\infty} C(\xi) = A(\xi)\frac{2+h^2\xi^2+\nu_1(-5+4\nu_1)+(3-4\nu_1)\nu_1\cosh(2\xi h)}{2h\xi\nu_1+\nu_1(-3+4\nu_1)\sinh(2\xi h)}, \tag{43}$$

$$\lim_{E_2\to\infty} D(\xi) = -\frac{\xi A(\xi)}{2\nu_1}, \tag{44}$$

$$\lim_{E_2\to\infty} \left[\widetilde{u}_{z0}\right]_{z=0}^{(c)} = -\xi^2 A(\xi)\frac{(1-\nu_1^2)}{E_1\nu_1}, \tag{45}$$

$$\lim_{E_2\to\infty} \left[\widetilde{\sigma}_{z0}\right]_{z=0}^{(c)} = -\xi^3 A(\xi)\frac{5+2h^2\xi^2+4\nu_1(-3+2\nu_1)+(3-4\nu_1)\cosh(2\xi h)}{4h\xi\nu_1+2\nu_1(-3+4\nu_1)\sinh(2\xi h)}. \tag{46}$$

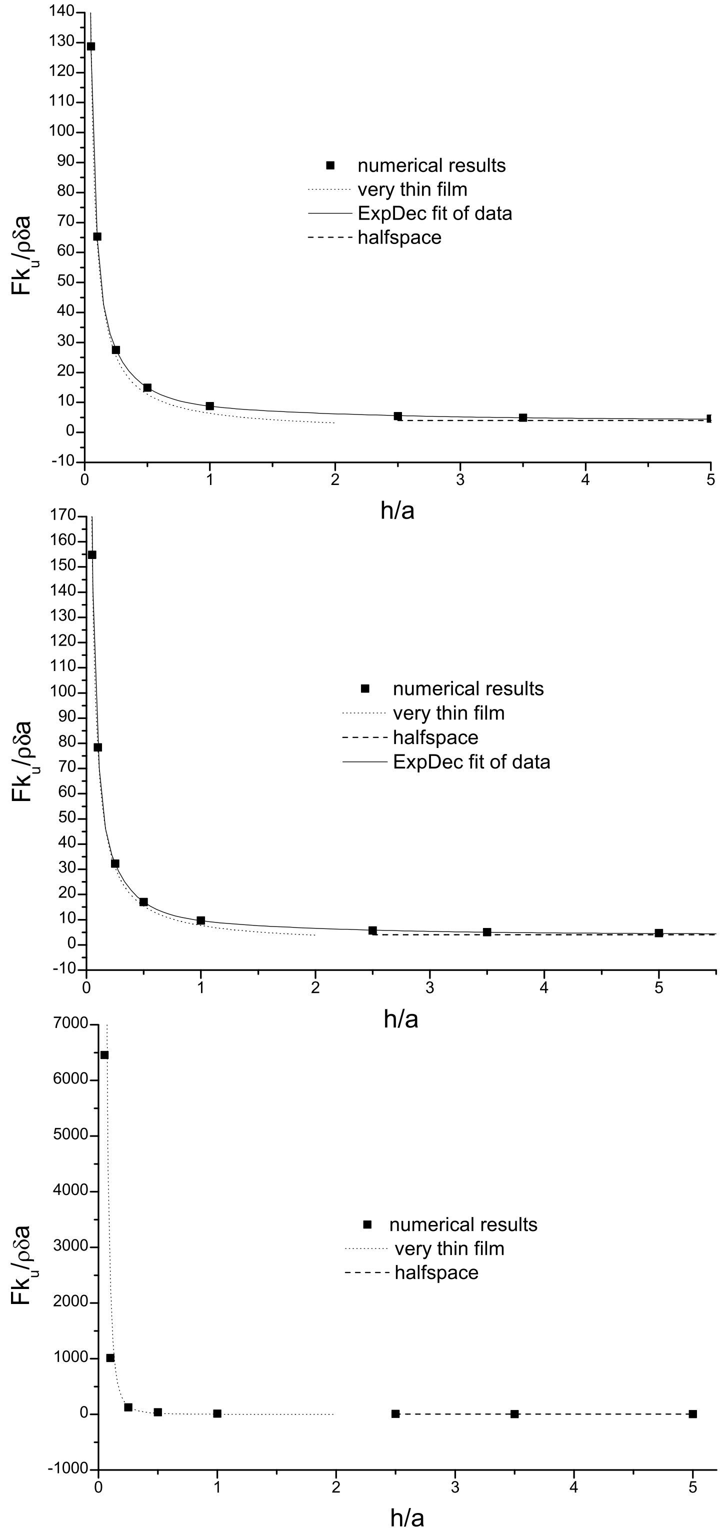
The dual integral equation (31)–(32) seems not to be analytically solvable; however the numerical solution, for equations (45) and (46), is symbolic in the parameter E_1. We plot in Figure 5 the dimensionless force-displacement diagrams for $\nu_1 = 0.1$, 0.3 and 0.5. The numerical results have been fitted by means of the ExpDec curve for the compressible layer (see Appendix C). Note that, for a given thickness, when the Poisson ratio increases we need a greater force to produce the same displacement.

The same Figure 5 also shows the dimensionless force-displacement relation for the halfspace with moduli E_1 and ν_1 and for very thin layer; in this last case we have introduced in (42)–(46) the series

$$\sinh(h\xi) = \sum_{n=0}^{\infty}\frac{(h\xi)^{2n+1}}{(2n+1)!}, \qquad \cosh(h\xi) = \sum_{n=0}^{\infty}\frac{(h\xi)^{2n}}{(2n)!}, \tag{47}$$

neglecting in each case the higher order terms (details of the approximations are given in Appendix D). With this approximation, we can treat the dual integral Equation (31)–(32) analytically. These equations, in fact, can be reduced to the classic Titchmarsh's form [Sneddon 1966] if the film is incompressible (see Appendix E), while, in the case of compressible films, they can be solved by applying the inversion theorem for Hankel transforms (see Appendix E). For an incompressible film we have

$$A(\xi) = -\frac{\delta a^2}{2\xi^2 k_u} J_2(\xi a),$$



$\nu_1 = 0.1$

$\nu_1 = 0.3$

$\nu_1 = 0.5$

Figure 5. Rigid substrate: contact law for perfect bond case.

and, for (16), we obtain the force-displacement relation

$$\delta = \frac{8Fh^3}{a^4 \pi E_1}. \tag{48}$$

For a compressible film, instead, we have

$$A(\xi) = -\frac{1}{\xi^3}\frac{a\delta}{k_u} J_1(\xi a),$$

and, for (16), we obtain the force-displacement relation

$$\delta = \frac{Fh\,(1+\nu_1)\,(1-2\nu_1)}{\pi a^2 E_1\,(1-\nu_1)}. \tag{49}$$

For very thin films we note a great difference between displacements δ given by (48) and (49): the former depends on h^3 while the latter varies as h.

In the frictionless case we have

$$\lim_{E_2\to\infty} B(\xi) = \xi A(\xi)\frac{\cosh(\xi h)}{2\nu_1 \sinh(\xi h)}, \tag{50}$$

$$\lim_{E_2\to\infty} C(\xi) = -A(\xi)\frac{[h\xi + \nu_1 \sinh(2\xi h)]}{2\nu_1\,[\sinh(\xi h)]^2}, \tag{51}$$

$$\lim_{E_2\to\infty} D(\xi) = -\frac{\xi A(\xi)}{2\nu_1}, \tag{52}$$

$$\lim_{E_2\to\infty} \left[\widetilde{u}_{z0}\right]^{(c)}_{z=0} = -\xi^2 A(\xi)\frac{(1-\nu_1^2)}{E_1\nu_1}, \tag{53}$$

$$\lim_{E_2\to\infty} \left[\widetilde{\sigma}_{z0}\right]^{(c)}_{z=0} = \xi^3 A(\xi)\frac{[2h\xi + \sinh(2\xi h)]}{4\nu_1\,[\sinh(\xi h)]^2}. \tag{54}$$

Unlike the case of a perfect bond, the dual integral equation (53)–(54) has a numerical symbolic solution in both mechanical parameters, E_1 and ν_1, of the surface layer. We plot in Figure 6 the dimensionless force-displacement relation.

Moreover in this case we consider the dimensionless force-displacement relation for the halfspace with moduli E_1 and ν_1 and for very thin layer; we have also plotted the fitting ExpDec curve (see Appendix C). For thin layer we have again replaced in $((50)-(54))$, the series (47) and we do not have a different significant order of infinitesimal for the compressible and incompressible film. So we then obtain a single solution for compressible and incompressible cases. Again equations (31) and (32) are solvable analytically applying the inversion theorem for Hankel transform (see Appendix E) and the solution is

$$A(\xi) = -\frac{1}{\xi^3}\frac{a\delta}{k_u} J_1(\xi a),$$

and, with (16), we obtain the force-displacement relation

$$\delta = \frac{Fh\left(1-\nu_1^2\right)}{\pi a^2 E_1}. \tag{55}$$

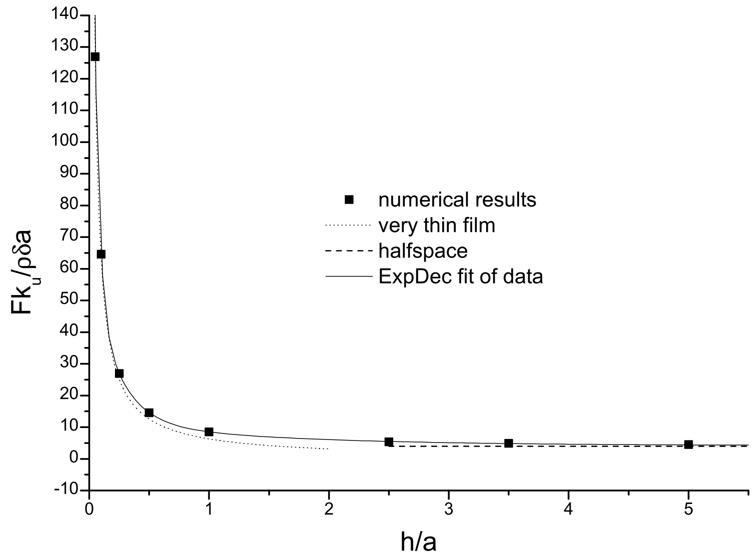


Figure 6. Rigid substrate: contact law for frictionless case.

In case of a compressible or incompressible very thin layer that is free to slide on a rigid foundation, we obtain, therefore, a direct proportionality between δ and h, as in the case of compressible films in perfect bond. Yang [1998; 2003; 2006] obtained the same results, considering the direct case of an elastic layer on a rigid foundation, again in the compressible and incompressible cases. For the latter he formulates the differential problem on the basis of the constitutive relation of incompressible materials.

6. Conclusion

We have considered an elastic, axisymmetric indentation problem under conditions that represent the upper and lower limits of real situations, for which the degree of adhesion between the layers is intermediate between the conditions of complete bond and frictionless.

From the analysis developed for an elastic coating on an elastic substrate is deduced that, varying $\bar{h} = h/a$ between 0.5 and 10, the F-δ law is influenced by all the mechanical properties of materials of the two layers. The same is true for the function $\overline{[\sigma_z\,(\bar{r})]}^{(c)}_{z=0}$ given by (39), for different values of the ration h/a. Therefore we have only provided the dimensionless relations F vs. δ.

A comparison of the results leads us to conclude that a given displacement δ is associated to a smaller force F for the frictionless case than for the perfect bond condition; this indicates that the frictionless on interface makes more deformable the layered body. This difference increases with the reduction of the thickness of the surface layer and when the Poisson ratio increases.

This is evident if we compare the two cases of a compressible film on a rigid foundation: the dimensionless solution in the frictionless case is slightly lower than that of perfect bond related to film with modulus ν_1 minimum among those considered ($\nu_1 = 0.1$).

The contrast is all the clearer for incompressible films, especially if the foundation is rigid and the surface layer is very thin. In fact, formulas (48) and (55), referring to the latter condition respectively for perfect bonding and absence of friction, differ by two orders of magnitude: in one case there appears h^3 and in the other h.

We also pointed out that the choice of analytical solution highlight characteristics of the solution that are difficult to notice through a complete numerical study.

Appendix A. Hankel transforms

If we apply the inversion theorem to equations (1), (2), (3) and (6) we obtain the Hankel transforms of some components of stress and displacement as functions of $\widetilde{\Phi}_0(\xi, z)$ and derivatives of $\widetilde{\Phi}_0(\xi, z)$:

$$\widetilde{u}_{r_1}(\xi, z) = \frac{(1+\nu)}{E}\xi \frac{d\widetilde{\Phi_0}(\xi, z)}{dz}, \tag{56}$$

$$\widetilde{u}_{z_0}(\xi, z) = \frac{(1-2\nu)(1+\nu)}{E}\frac{d^2\widetilde{\Phi}_0(\xi, z)}{dz^2} - \frac{2(1-\nu^2)}{E}\xi^2\widetilde{\Phi}_0(\xi, z), \tag{57}$$

$$\widetilde{\sigma}_{z_0}(\xi, z) = (1-\nu)\frac{d^3\widetilde{\Phi}_0(\xi, z)}{dz^3} - (2-\nu)\xi^2\frac{d\widetilde{\Phi}_0(\xi, z)}{dz}, \tag{58}$$

$$\widetilde{\tau}_{zr_1}(\xi, z) = \xi\left[\nu\frac{d^2\widetilde{\Phi}_0(\xi, z)}{dz^2} + (1-\nu)\xi^2\widetilde{\Phi}_0(\xi, z)\right], \tag{59}$$

Appendix B. Convergence for $h \to \infty$ and for $h \to 0$

The following proof of convergence to halfspace is valid either for perfect bond that for frictionless case. For $h \to \infty$, equations (28), (29) become

$$\lim_{h\to\infty}\left[\widetilde{u}_{z_0}\right]^{(c)}_{z=0} = -\xi^2 A(\xi)\frac{\left(1-\nu_1^2\right)}{E_1\nu_1}, \qquad \lim_{h\to\infty}\left[\widetilde{\sigma}_{z_0}\right]^{(c)}_{z=0} = \xi^3 A(\xi)\frac{1}{2\nu_1}.$$

Consequently the dual integral (31)–(32) can be treated analytically as it reduces to the classical Titchmarsh's form [Sneddon 1966], and give

$$A(\xi) = -\frac{2}{\pi\xi^3}\frac{E_1\nu_1}{\left(1-\nu_1^2\right)}\delta\sin(\xi a). \tag{60}$$

For the other functions $B(\xi)$, $C(\xi)$ and $D(\xi)$ we simply have

$$\lim_{h\to\infty} B(\xi) = \frac{\xi A(\xi)}{2\nu_1}; \qquad \lim_{h\to\infty} C(\xi) = -A(\xi); \qquad \lim_{h\to\infty} D(\xi) = -\frac{\xi A(\xi)}{2\nu_1}, \tag{61}$$

and, by replacing Equation (60) in (61), the potential in the layer, given by equation (14), becomes

$$\widetilde{\Phi}_0^{(c)} = -\frac{\sin(\xi a)}{\xi^4}\frac{E_1\delta}{\pi\left(1-\nu_1^2\right)}\left[2\nu_1 + \xi z\right]e^{(-\xi z)}, \tag{62}$$

which is equivalent to the solution of the halfspace of moduli E_1 e ν_1.

For $h \to 0$, equations (28)–(29) become

$$\lim_{h\to 0}\left[\widetilde{u}_{z_0}\right]^{(c)}_{z=0} = -\xi^2 A(\xi)\frac{\left(1-\nu_1^2\right)}{E_1\nu_1}, \qquad \lim_{h\to 0}\left[\widetilde{\sigma}_{z_0}\right]^{(c)}_{z=0} = \xi^3 A(\xi)\frac{E_2\left(1-\nu_1^2\right)}{2E_1\nu_1\left(1-\nu_2^2\right)}.$$

With these equations it is again the case that the solution of the dual integral (31)–(32) is given by (60), because the right-hand term in (32) vanishes. In the limit the unknown functions $S(\xi)$ and $T(\xi)$ are

$$\lim_{h\to 0} S(\xi) = \frac{E_2\left(1-\nu_1^2\right)}{2E_1\nu_1\left(1-\nu_2^2\right)}\xi A(\xi); \qquad \lim_{h\to 0} T(\xi) = \frac{E_2\nu_2\left(1-\nu_1^2\right)}{E_1\nu_1\left(1-\nu_2^2\right)}A(\xi), \tag{63}$$

and, replacing (60) in (63), the potential in the substrate, given by equation (15), becomes

$$\widetilde{\Phi}_0^{(s)} = -\frac{\sin(\xi a)}{\xi^4}\frac{E_2\delta}{\pi\left(1-\nu_2^2\right)}(2\nu_2+\xi z')e^{(-\xi z')}, \tag{64}$$

that is equivalent to the solution of the halfspace of moduli E_2 e ν_2, because for $h \to 0$ the z' axis has the same origin of z.[1]

Appendix C. Numerical results and fitting curves

The following tables show the numerical results for $\overline{F}$, in the case of perfect bonding (pb) and no friction (nf), as well as the percent difference.

Steel on polystyrene

$\bar{h}$	0.25	0.5	1	2.5	3.5	5	6.5	8.5	10	15	20	30	100
$\overline{F}$ (pb)	.0550	.0815	.1385	.3267	.4936	.8489	1.3118	1.9653	2.3968	3.2809	3.6260	3.8228	3.8717
$\overline{F}$ (nf)	.0522	.0772	.1312	.3089	.4681	.8149	1.2757	1.9347	2.3724	3.2719	3.6227	3.8221	3.8716
$\Delta\overline{F}$ (%)	5.09	5.27	5.27	5.45	5.17	4.00	2.75	1.58	1.02	0.27	0.09	0.02	0.003

Steel on glass

$\bar{h}$	0.25	0.5	1	2.5	3.5	5	6.5	8.5	10	15	20	30	100
$\overline{F}$ (pb)	1.463	1.636	1.945	2.600	2.852	3.097	3.264	3.422	3.511	3.696	3.788	3.853	3.872
$\overline{F}$ (nf)	1.261	1.344	1.557	2.181	2.450	2.723	2.922	3.129	3.257	3.560	3.722	3.838	3.862
$\Delta\overline{F}$ (%)	13.82	17.85	19.92	16.11	14.09	12.07	10.47	8.56	7.25	3.67	1.75	0.38	0.26

Glass on steel

$\bar{h}$	0.25	0.5	1	2.5	3.5	5	6.5	8.5	10
$\overline{F}$ (pb)	9.0541	7.6161	6.1548	4.7986	4.5174	4.3118	4.2043	4.1222	4.0827
$\overline{F}$ (nf)	8.4112	6.8633	5.4886	4.4709	4.2846	4.1551	4.0864	4.0338	4.0084
$\Delta\overline{F}$ (%)	7.10	9.88	10.82	6.83	3.63	3.63	2.80	2.14	1.82

The fitting of curves was carried out based on 13 values of the ratio h/a. SRichards and ExpDec are exponential curves; the general expression for a SRichards curve is

$$\overline{F} = b\left[1+(d-1)\,e^{-c(\bar{h}-h_0)}\right]^{1/(1-d)}$$

[1]The Hankel transform of Sneddon's potential [1951] for the halfspace is equivalent to (62) or (64) times $E^{-1}(1+\nu)(1-2\nu)$, in agreement with the formulation of the problem.

while the expression for the ExpDec curve is

$$\bar{F} = F_0 + b_1 e^{-(\bar{h}-h_0)/c_1} + b_2 e^{-(\bar{h}-h_0)/c_2} + b_3 e^{-(\bar{h}-h_0)/c3}$$

The constants obtained were as follows (pb = perfect bond; nf = no friction):

Steel on polystyrene

	b	d	c	h_0
(pb)	3.8447	1.05499	0.22874	6.88795
(nf)	3.84321	1.06893	0.23112	7.02854

Steel on glass

	F_0	h_0	b_1	b_2	b_3	c_1	c_2	c_3
(pb)	3.87775	0	−1.30211	−1.31437	0	1.88306	7.68174	0
(nf)	3.88583	0	−1.73246	−0.98371	0	8.90297	3.07826	0

Glass on steel

	F_0	h_0	b_1	b_2	b_3	c_1	c_2	c_3
(pb)	3.9879	0.13699	7.44575	1.20125	6.4023	0.12538	2.60411	0.52208
(nf)	4.04243	0.14828	6.62349	1.72188	6.14044	0.12554	2.77703	0.62372

Rigid foundation

	F_0	h_0	b_1	b_2	b_3	c_1	c_2	c_3
(nf)	4.1702	0	264.127	50.342	7.86791	0.03896	0.21493	1.40524
(pb, $\nu_1 = 0.1$)	4.1947	0	270.474	51.108	8.31471	0.03942	0.21365	1.41220
(pb, $\nu_1 = 0.3$)	4.2509	0	323.156	61.805	9.99856	0.03983	0.21263	1.3695

Appendix D. Approximations on the hyperbolic functions for very thin coating

In the case of a very thin coating we have substituted the hyperbolic series (47), neglecting the higher order terms, in order to solve the dual integral equation and obtain the force-displacement law. To this end we follow the approach proposed by Yang [2003; 1998] and we report here some details of the calculation useful to derive the explicit form of the dual integral equation based upon the potentials adopted.

For a compressible very thin coating in perfect bond on a rigid foundation, if we assume

$$\sinh(h\xi) \simeq h\xi, \quad \cosh(h\xi) \simeq 1 + \tfrac{1}{2}(h\xi)^2, \tag{65}$$

use the identities

$$\sinh(2h\xi) = 2\sinh(h\xi), \quad \cosh(2h\xi) = \sinh^2(h\xi) + \cosh^2(h\xi), \tag{66}$$

and substitute formulas (65) and (66) in (42), (43) and (46) we obtain

$$B(\xi) = A(\xi)\,\frac{1-\nu_1}{2h\nu_1(1-2\nu_1)}, \quad C(\xi) = -A(\xi)\,\frac{1-\nu_1}{2h\xi\nu_1(1-2\nu_1)}, \quad [\widetilde{\sigma}_{z0}]^{(c)}_{z=0} = \xi^2 A(\xi)\,\frac{(1-\nu_1)^2}{h\nu_1(1-2\nu_1)}. \tag{67}$$

With this approximation, we can treat the dual integral (31)–(32) analytically. These equations, in fact, can be solved applying the inversion theorem for Hankel transforms.

In case of an incompressible coating the expressions (67) become indeterminate and consequently we must include additional terms in the series expansion (47). If we take

$$\sinh(h\xi) \simeq h\xi + \tfrac{1}{6}(h\xi)^3, \quad \cosh(h\xi) \simeq 1 + \tfrac{1}{2}(h\xi)^2, \tag{68}$$

set $\nu = \frac{1}{2}$, and substitute formulas (68), (66) in (42), (43) and (46), we obtain

$$B(\xi) = A(\xi)\frac{3}{2h^3\xi^2}, \quad C(\xi) = -A(\xi)\frac{3}{2h^3\xi^3}, \quad [\widetilde{\sigma}_{z0}]^{(c)}_{z=0} = A(\xi)\frac{3}{2h^3}. \tag{69}$$

With this approximation we can treat the dual integral (31)–(32) analytically because it can be reduced to the classic Titchmarsh form.

For a compressible very thin coating free to slide on a rigid foundation, if we substitute formulas (65) and (66) in (50), (51) and (54) we obtain

$$B(\xi) = A(\xi)\,\frac{1}{2h\nu_1}, \quad C(\xi) = -A(\xi)\,\frac{1+2\nu_1}{2h\xi\nu_1}, \quad [\widetilde{\sigma}_{z0}]^{(c)}_{z=0} = \xi^2 A(\xi)\,\frac{1}{h\nu_1}. \tag{70}$$

In this case the approximation (65) is valid for incompressible upper layer too, and the dual integral (31)–(32) is solvable analytically applying the inversion theorem for Hankel transform.

Appendix E. Analytical solution of the dual integral equation for a very thin coating on a rigid foundation

For a very thin incompressible layer in perfect bond on a rigid foundation we have the transform of the surface normal stress given by $(69)_3$. Consequently, the dual integral equation can be written as

$$\begin{aligned} \int_0^\infty -\xi^3 A(\xi)\frac{3}{2E_1} J_0(\xi r)\,d\xi &= \delta \quad \text{for } 0 < r < a, \\ \int_0^\infty \xi A(\xi)\frac{3}{2h^3} J_0(\xi r)\,d\xi &= 0 \quad \text{for } r > a, \end{aligned} \tag{71}$$

and if we assume

$$\bar{\xi} = \xi a, \quad \bar{r} = \frac{a}{r}, \quad k_u = \frac{3}{2E_1}, \quad \bar{\delta} = -a^4\frac{\delta}{k_u}, \quad \overline{A(\bar{\xi})} = \bar{\xi} A(\bar{\xi}),$$

we have

$$\begin{aligned} \int_0^\infty \bar{\xi}^2 \overline{A(\bar{\xi})} J_0(\bar{\xi}\bar{r})\,d\bar{\xi} &= \bar{\delta} \quad \text{for } 0 < \bar{r} < 1, \\ \int_0^\infty \overline{A(\bar{\xi})} J_0(\bar{\xi}\bar{r})\,d\bar{\xi} &= 0 \quad \text{for } \bar{r} > 1. \end{aligned} \tag{72}$$

The dual integral equation in the form

$$\int_0^\infty \bar{\xi}^{-2\alpha} A(\bar{\xi}) J_m(\bar{\xi}\bar{r})\, d\bar{\xi} = f(\bar{r}) \quad \text{for } 0 < \bar{r} < 1,$$
$$\int_0^\infty A(\bar{\xi}) J_m(\bar{\xi}\bar{r})\, d\bar{\xi} = 0 \qquad \text{for } \bar{r} > 1,$$

with $\alpha < 0$, was solved by Titchmarsh; the solution is

$$A(\bar{\xi}) = \frac{\left(2\bar{\xi}\right)^{1+\alpha}}{\Gamma(-\alpha)} \int_0^1 t^{1-\alpha} J_{m-\alpha}(\bar{\xi}t) \left(\int_0^1 \left(1-s^2\right)^{-1-\alpha} s^{m+1} f(ts) ds \right) dt.$$

Therefore, the solution for the dual (72) is

$$\overline{A(\bar{\xi})} = \int_0^1 \bar{r}^2 J_1(\bar{\xi}\bar{r}) \left(\int_0^1 s\bar{\delta}\, ds \right) d\bar{r} = -\frac{1}{\bar{\xi}} a^4 \frac{\delta}{2k_u} J_2(\bar{\xi}),$$

and then we have

$$A(\xi) = -\frac{\delta a^2}{2\xi^2 k_u} J_2(\xi a). \tag{73}$$

If we substitute (73) in (71) we deduce the normal stress distribution over the contact area

$$[\sigma_z(r,z)]_{z=0} = \int_0^\infty \frac{-3a^2\delta}{4h^3\xi k_u} J_2(\xi a) J_0(\xi r)\, d\xi = \begin{cases} 0 & \text{for } r > a, \\ -\dfrac{\delta E_1}{4h^3}(a^2 - r^2) & \text{for } 0 < r < a, \end{cases}$$

while, with Equation (16), we have the force-displacement law

$$\delta = \frac{8Fh^3}{a^4 \pi E_1}.$$

For a very thin compressible layer in perfect bond on a rigid foundation we have the transform of the surface normal stress given by $(67)_3$. The dual integral equation can be written as

$$\begin{aligned} \int_0^\infty -\xi^3 A(\xi) \frac{1-\nu_1^2}{E_1\nu_1} J_0(\xi r)\, d\xi = \delta \quad & \text{for } 0 < r < a, \\ \int_0^\infty \xi^3 A(\xi) \frac{(1-\nu_1)^2}{h\nu_1(1-2\nu_1)} J_0(\xi r)\, d\xi = 0 \quad & \text{for } r > a, \end{aligned} \tag{74}$$

and then

$$\int_0^\infty -\xi^3 A(\xi) J_0(\xi r)\, d\xi = \begin{cases} 0 & \text{for } r > a, \\ \dfrac{E_1\nu_1}{1-\nu_1^2} & \text{for } \delta \quad 0 < r < a, \end{cases}$$

By means of the inversion theorem for the Hankel transform, we obtain

$$A(\xi) = -\frac{1}{\xi^2} \int_0^a r \frac{\delta E_1 \nu_1}{\left(1-\nu_1^2\right)} J_0(\xi r)\, dr = -\frac{1}{\xi^3} \frac{a\delta E_1\nu_1}{\left(1-\nu_1^2\right)} J_1(\xi a).$$

Consequently we have

$$
\begin{aligned}
\left[\sigma_z(r,z)\right]_{z=0} &= \int_0^\infty -\frac{a\delta E_1\nu_1}{\left(1-\nu_1^2\right)}\frac{(1-\nu_1)^2}{h\nu_1\,(1-2\nu_1)}J_1(\xi a)J_0(\xi r)\,d\xi \\
&= \begin{cases} 0 & \text{for } r > a, \\ -\delta\,\dfrac{E_1(1-\nu_1)}{h(1+\nu_1)(1-2\nu_1)} & \text{for } 0 < r < a, \end{cases}
\end{aligned}
\tag{75}
$$

and with (16) we derive

$$
\delta = \frac{Fh\,(1+\nu_1)\,(1-2\nu_1)}{\pi a^2 E_1\,(1-\nu_1)}.
$$

For a very thin compressible or incompressible layer free to slide on a rigid foundation we have the transform of the surface normal stress given by $(70)_3$. The dual integral equation can be written as

$$
\begin{aligned}
\int_0^\infty -\xi^3 A(\xi)\frac{\left(1-\nu_1^2\right)}{E_1\nu_1}J_0(\xi r)\,d\xi &= \delta \quad 0 < r < a, \\
\int_0^\infty \xi^3 A(\xi)\frac{1}{h\nu_1}J_0(\xi r)\,d\xi &= 0 \quad r > a,
\end{aligned}
\tag{76}
$$

and then

$$
\int_0^\infty \xi^3 A(\xi)J_0(\xi r)\,d\xi = \begin{cases} 0 & \text{for } r > a, \\ -\delta\,\dfrac{E_1\nu_1}{1-\nu_1^2} & \text{for } 0 < r < a, \end{cases}
\tag{77}
$$

By means of the inversion theorem for the Hankel transform, we obtain

$$
A(\xi) = -\frac{1}{\xi^2}\int_0^a r\frac{\delta E_1\nu_1}{\left(1-\nu_1^2\right)}J_0(\xi r)\,dr = -\frac{1}{\xi^3}\frac{a\delta E_1\nu_1}{\left(1-\nu_1^2\right)}J_1(\xi a).
\tag{78}
$$

Consequently

$$
\left[\sigma_z(r,z)\right]_{z=0} = \int_0^\infty -\frac{a\delta E_1\nu_1}{\left(1-\nu_1^2\right)}\frac{1}{h\nu_1}J_1(\xi a)J_0(\xi r)\,d\xi = 0 = \begin{cases} 0 & \text{for } r > a, \\ -\delta\dfrac{E_1}{h\left(1-\nu_1^2\right)} & \text{for } 0 < r < a. \end{cases}
\tag{79}
$$

Using (16), we then obtain

$$
\delta = \frac{Fh\left(1-\nu_1^2\right)}{\pi a^2 E_1}.
$$

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