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AN ANISOTROPIC HETEROGENEOUS BEAM WITH OVERSPECIFIED BOUNDARY CONDITIONS

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In this paper a heterogeneous anisotropic cylindrical beam with a rigidly fixed base is considered as an alternative to the relaxed Saint-Venant's problem. The rigidly fixed base results in a problem with overspecified boundary conditions for which a proof of existence is given. The results of this paper indicate that the relaxed Saint-Venant's problem, for loads independent of the axial coordinate, ignores the dependence of the stress field on the axial coordinate. Dependence of the stress field on the axial coordinate could result in warping of transverse cross sections and nonzero in-plane stresses, which is significant for understanding the behavior of natural structures such as wood and mammalian bone.

1. Introduction

Two examples of natural structures that can be represented as heterogeneous anisotropic cylinders are the bole of a tree and long mammalian bones. Bodig and Jayne [1993] describe a cylindrical section of a tree as being an orthotropic material with cylindrical anisotropy, where the axes of symmetry are in the radial direction ($\mathbf{e}_R$), the tangential direction ($\mathbf{e}_T$), and the long direction ($\mathbf{e}_L$) which is directed up the tree. The same paper reports that the compressive strength of wood is weaker in the $\mathbf{e}_R$ and $\mathbf{e}_T$ directions than it is in the $\mathbf{e}_L$ direction. Barrett et al. [1981] notes the fracture toughness of wood is an order of magnitude lower where normal stresses in the $\mathbf{e}_R$ and $\mathbf{e}_T$ directions can open the crack. Kennedy and Carter [1985] note that various material models have been used for long mammalian bones and these include isotropic, transversely isotropic, and cylindrically orthotropic. Taylor et al. [2002] used an orthotropic model to compare elastic constants for the human femur measured by ultrasound to those predicted through a finite element model. Peterlik et al. [2006] found the fracture toughness of bone is greater in the $\mathbf{e}_R$ and $\mathbf{e}_T$ directions than the $\mathbf{e}_L$ direction. Norman et al. [1996] found in the $\mathbf{e}_L$ direction that bone was an order of magnitude stronger in Mode 2 fracture as opposed to Mode 1 fracture, where tension in the $\mathbf{e}_R$ and $\mathbf{e}_T$ directions can open the crack.

Iesan [1987] describes Saint-Venant's problem as determining an equilibrium displacement field for a cylinder loaded by surface forces distributed over its plane ends, and the relaxed Saint-Venant's problem as replacing the distributed surface forces on one end with equivalent resultant loads. To consider a cantilever beam as a relaxed Saint-Venant's problem, the prescribed displacements on the base of the beam are replaced with a stress field that maintains equilibrium; however, this changes the problem from a mixed problem to a traction problem. In solving the relaxed Saint-Venant's problem various constants of integration are developed if the equilibrium displacement field is desired. Sokolnikoff [1956] and Lyons et al. [2002] found these constants of integration could be determined by reintroducing certain

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elements of the fixed boundary condition; however, these elements are on specific points or lines and in general the equilibrium displacement field is not able to satisfy the fixed boundary conditions over the complete base of the cylinder. When considering an anisotropic cylindrical cantilever beam as a relaxed Saint-Venant's problem, authors such as Lekhnitskii [1981], Iesan [1987], and Lyons [2002] found that the stresses are zero in the plane defined by $\mathbf{e}_R$ and $\mathbf{e}_T$. An important question is whether this result is a function of posing the problem as a relaxed Saint-Venant's problem or whether it is a general result.

The fixed boundary conditions considered by Sokolnikoff [1956] and Lyons et al. [2002] include both prescribed displacements and prescribed first derivatives of the displacements at points on the base of the cylinder. Gao and Mura [1991] consider these types of boundary conditions to be overspecified and there is the risk that if they are selected arbitrarily then they could be inconsistent. There are additional conditions imposed on the displacement vector when considering elastodynamic problems and these can be used to better understand the implications of using overspecified boundary conditions.

This paper considers a cylinder that is rigidly fixed at the base and subject to a time dependent forcing function on the free end. The cylinder is composed of a material that is heterogeneous and anisotropic, with elastic coefficients that are independent of the axial coordinate. In this paper the objective of Section 3 is to confirm that the stresses and strains are identically equal to zero on the base of the cylinder when the base is rigidly fixed, the objective of Sections 4 and 5 is to prove the existence of a solution which will indicate the overspecified boundary conditions are consistent, and the objective of Section 6 is to compare specific attributes of the stress field found in this paper to the stress field found when posing the cylinder as a relaxed Saint-Venant's problem.

2. Problem statement

Consider a cylindrical cantilever beam with constant cross sections (Figure 1), where the displacement of the free end combined with the length of the beam is such that geometric nonlinear effects are not significant. Let the region B refer to the interior of the cylinder, ∂B is the boundary, and $\bar{B} = \partial B \cup B$. Let Σ_1 be the open cross section at $x_3 = 0$, let Σ_2 be the open cross section at $x_3 = h$. The lateral surface of the cylinder is Π. In the following Greek indices range from 1 to 2, while Latin indices range from 1

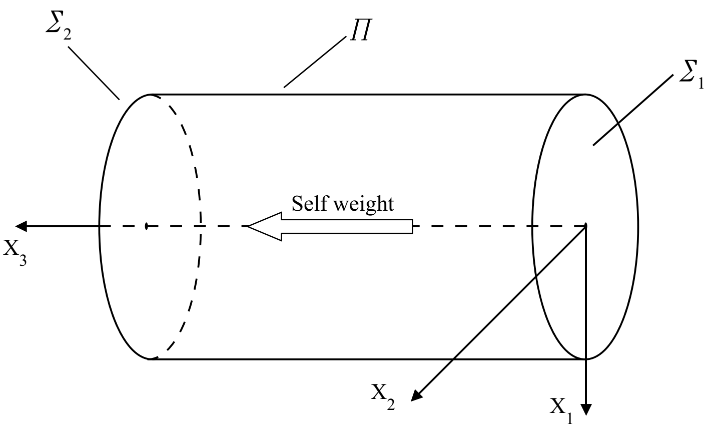

Figure 1. Cylindrical beam in Cartesian coordinates.

to 3. Summation notation is used for repeated indices and a comma followed by a subscript will indicate a partial derivative with respect to the subscript. In addition, the Kronecker delta function (δ_{ij}) will be used.

The beam considered in this paper will be rigidly fixed on Σ_2. Eringen and Suhubi [1974] give the necessary condition for a body to be locally rigid $d_{ij}(\mathbf{x}, t) = \frac{1}{2}(v_i(\mathbf{x}, t),_j + v_j(\mathbf{x}, t),_i) = 0$. Here, d_{ij} is the rate of deformation tensor, and v_i is the velocity vector.

The boundary conditions for the problem considered in this paper are

$$\begin{aligned} \mathbf{u}(\Sigma_2, t) &= 0 && \text{on } \Sigma_2, \\ d_{ij}(\Sigma_2, t) &= 0 && \text{on } \Sigma_2, \\ \mathbf{s}(\Pi, t) &= 0 && \text{on } \Pi, \\ \mathbf{s}(\Sigma_1, t) &= \mathbf{k}(\Sigma_1)\Phi(t) && \text{on } \Sigma_1. \end{aligned} \tag{1}$$

Here $\mathbf{u}(\Sigma_2, t)$ is the prescribed displacement vector on Σ_2, $s_i(\mathbf{x}, t) = S_{ij}(\mathbf{x}, t)n_j$, where n_j is the unit vector normal to the surface of interest and $S_{ij}(\mathbf{x}, t)$ is Cauchy's stress tensor, $\mathbf{k}(\Sigma_1)$ is a vector that is a function of the coordinates on Σ_1, and $\Phi(t)$ is a scalar function of time.

At $t = 0$ the beam is subject to body loads due to gravity alone. Therefore, the initial conditions are

$$\mathbf{u}(\mathbf{x}, 0) = -\mathbf{q}(\mathbf{x}) \text{ on } \bar{B} \quad \text{and} \quad \mathbf{u}(\mathbf{x}, 0),_t = \zeta(\mathbf{x}) \text{ on } \bar{B}. \tag{2}$$

Here $-\mathbf{q}(\mathbf{x})$ is the displacement vector due to gravity alone, and $\zeta(\mathbf{x})$ is the initial velocity vector.

The constitutive equation is

$$S_{ij} = C_{ijkl}(x_\alpha)E_{kl}. \tag{3}$$

Here S_{ij} is Cauchy's stress tensor, E_{ij} is the infinitesimal strain tensor, and C_{ijkl} is the elasticity tensor which is independent of the axial coordinate x_3, symmetric, and positive definite.

The problem considered in this paper is one of linear elasticity, where nonlinear terms in the strain tensor, velocity vector, and acceleration vector are discarded as terms of higher order. In addition, when considering a spatial description of acceleration, the convective components are discarded as terms of higher order. Achenbach [1973] notes that once the problem is completely linearized the distinction between the material and spatial descriptions of motion vanish; therefore, either the notation for the material description or the spatial description can be used. Material coordinates are typically used in solid mechanics; however, in the problem considered by this paper the velocities are prescribed on Σ_2 and thus a spatial description can be useful. This paper uses a material description except for the derivation involving the continuity equation, and Equations (7)–(10); however, given the observations by Achenbach [1973] a new coordinate system is not introduced.

3. Considering stress and strain as x_3 approaches h

Theorem 3.1. Let $\mathbf{u}(\mathbf{x}, t) \in C^{1,2}(\bar{B}) \cap C^{2,2}(B)$. Given Equation (2) and the first two equations of (1), then, $\lim_{x_3 \to h} u_i(\mathbf{x}, t),_j = 0$.

Proof. The boundary values of the field equations are defined as follows. Let $\mathbf{x}_0$ be some point on ∂B and let $\mathbf{u}(\mathbf{x}, t)$ tend to a definite limit as the point $\mathbf{x}$ approaches $\mathbf{x}_0$ from B. Recall Greek indices range from 1 to 2, while Latin indices range from 1 to 3. Given the first equation of (1), and that x_i and t are independent, then

$$\lim_{x_3\to h} \mathbf{u}(\mathbf{x},t)_{,\alpha} = 0, \tag{4}$$

$$\lim_{x_3\to h} \mathbf{v}(\mathbf{x},t) = 0, \tag{5}$$

$$\lim_{x_3\to h} \mathbf{v}(\mathbf{x},t)_{,\alpha} = 0. \tag{6}$$

Consider the density of the material in $\bar{B}$ to be a function of position and time

$$\rho = \rho(\mathbf{x},t). \tag{7}$$

The equation of continuity is

$$\frac{D\rho(\mathbf{x},t)}{Dt} + \rho(\mathbf{x},t)v_j(\mathbf{x},t)_{,j} = 0. \tag{8}$$

Expand the material derivative in (8), take the limit as x_3 approaches h, and take into account Equation (5), then

$$\lim_{x_3\to h} \frac{D\rho(\mathbf{x},t)}{Dt} = \lim_{x_3\to h} \rho(\mathbf{x},t)_{,t}. \tag{9}$$

Take the limit of (8) as x_3 approaches h, take into account Equation (9) and the second equation of (1) with $i = j$, then

$$\lim_{x_3\to h} \rho(\mathbf{x},t)_{,t} = 0. \tag{10}$$

Integrating (10) with respect to time it can be seen that the density of the material in B is independent of time as x_3 approaches h.

$$\lim_{x_3\to h} \rho(\mathbf{x},t) = \lim_{x_3\to h} \rho(\mathbf{x}). \tag{11}$$

The density at time t is related to the reference density by

$$\rho(\mathbf{x},t) = \frac{1}{J(\mathbf{x},t)}\rho(\mathbf{x},0).$$

Here, $J(\mathbf{x},t)$ is the Jacobian determinant, which represents the dilation of an infinitesimal volume at the material point $\mathbf{x}$. However, given (11)

$$\lim_{x_3\to h} \rho(\mathbf{x}) = \lim_{x_3\to h} \frac{1}{J(\mathbf{x},t)}\rho(\mathbf{x}). \tag{12}$$

Recall for infinitesimal strains that

$$J(\mathbf{x},t) = 1 + div\,\mathbf{u}(\mathbf{x},t). \tag{13}$$

Substitute (13) into (12), then it can be seen that

$$\lim_{x_3\to h} div\,\mathbf{u}(\mathbf{x},t) = \lim_{x_3\to h} u_j(\mathbf{x},t)_{,j} = 0. \tag{14}$$

Substitute (4) into (14) and take the derivative of this with time, then

$$\lim_{x_3\to h} u_3(\mathbf{x},t)_{,3} = 0 \quad \text{and} \quad \lim_{x_3\to h} v_3(\mathbf{x},t)_{,3} = 0. \tag{15}$$

Set $i = \alpha$ and $j = 3$ in the second equation of (1) and take into account Equation (6), then

$$\lim_{x_3 \to h} v_\alpha(\mathbf{x}, t)_{,3} = 0. \tag{16}$$

Integrate (16) with respect to time and take into account the first equation of Equation (2), then

$$\lim_{x_3 \to h} u_\alpha(\mathbf{x}, t),_3 = -\lim_{x_3 \to h} q_\alpha(\mathbf{x}),_3. \tag{17}$$

Equation (17) indicates the shear strains are independent of time as x_3 approaches h. This is an unlikely result given the time dependent forcing function being applied to Σ_1. Therefore, as will be demonstrated in Section 4, the only nontrivial solution is to let $\lim_{x_3 \to h} q_\alpha(\mathbf{x})_{,3} = 0$, then

$$\lim_{x_3 \to h} u_\alpha(\mathbf{x}, t)_{,3} = 0. \tag{18}$$

Given equations (4)–(6), (15), (16), and (18) the velocities and the derivatives of the displacements and velocities with respect to the coordinates as x_3 approaches h can be summarized as follows

$$\lim_{x_3 \to h} u_i(\mathbf{x}, t),_j = 0, \quad \lim_{x_3 \to h} v_i(\mathbf{x}, t) = 0, \quad \text{and} \quad \lim_{x_3 \to h} v_i(\mathbf{x}, t),_j = 0. \tag{19}$$

Equation (19) completes the proof of Theorem 3.1. □

Consider the constitutive Equation (3), note the material coefficients are independent ofx_3, and take the limit as x_3 approaches h.

$$\lim_{x_3 \to h} S_{ij}(\mathbf{u}) = \frac{C_{ijkl}}{2}\Big(\lim_{x_3 \to h} u_k,_l + \lim_{x_3 \to h} u_l,_k\Big). \tag{20}$$

Substitute the first equation of (19) into (20), then

$$\lim_{x_3 \to h} S_{ij}(\mathbf{u}) = 0. \tag{21}$$

Note the results, Equation (19) and (21), follow from assuming infinitesimal strains; in general these results do not hold if finite strains are allowed.

4. Displacement functions

Theorem 4.1. Let $\Phi(t) = at$, for $0 \le t < t_L$ where a is a constant and t_L is the time when the strains are no longer infinitesimal; given equations (1), (2), and (19), then $u_i(\mathbf{x}, t) = t\zeta_i(\mathbf{x}) - q_i(\mathbf{x})$ and $u_i(\mathbf{x}, t)_{,tt} = 0$.

Proof. The equation of motion in material description is

$$\rho_o(\mathbf{x})v_i(\mathbf{x}, t),_t = S_{ij}(\mathbf{x}, t),_j + \delta_{i3}\rho_o(\mathbf{x})g. \tag{22}$$

Let $\mathbf{U}(\mathbf{x}, t) \in C^{1,2}(\bar{B}) \cap C^{2,2}(B)$ and

$$\begin{aligned} U_i(\mathbf{x}, t) &= u_i(\mathbf{x}, t) + q_i(\mathbf{x}), \\ S_{ij}(\mathbf{q}(\mathbf{x})),_j &= \delta_{i3}\rho_o g. \end{aligned} \tag{23}$$

Rearrange (23)

$$u_i(\mathbf{x}, t) = U_i(\mathbf{x}, t) - q_i(\mathbf{x}). \tag{24}$$

Substitute Equation (24) into the equations of motion, (22), then

$$\rho_o U_i(\mathbf{x}, t)_{,tt} = S_{ij}(\mathbf{x}, t)_{,j}. \tag{25}$$

Substitute (24) into the initial conditions, (2), then

$$U_i(\mathbf{x}, 0) = 0 \quad \text{and} \quad U_i(\mathbf{x}, 0)_{,t} = \zeta(\mathbf{x}) \tag{26}$$

Substitute (24) into the first equation of (1), the first equation of (19), and the third and fourth equations of (1).

$$U_i(\Sigma_2, t) = q_i(\Sigma_2), \tag{27}$$

$$U_i(\Sigma_2, t)_{,j} = q_i(\Sigma_2)_{,j}, \tag{28}$$

$$S_{ij}\big(\mathbf{U}(\Sigma_1, t)\big)n_j^{(\Sigma_1)} = k_i(\Sigma_1)\Phi(t) + S_{ij}\big(\mathbf{q}(\Sigma_1)\big)n_j^{(\Sigma_1)}, \tag{29}$$

$$S_{ij}\big(\mathbf{U}(\Pi, t)\big)n_j^{(\Pi)} = S_{ij}\big(\mathbf{q}(\Pi)\big)n_j^{(\Pi)}. \tag{30}$$

Let

$$U_i(\mathbf{x}, t) = X_i(\mathbf{x})T(t). \tag{31}$$

Here, $X_i(\mathbf{x})$ is a vector function of $\mathbf{x}$ alone, and $T(t)$ is a scalar function of t alone.

Substitute Equation (31) into (25), then

$$\rho_o X_i(\mathbf{x})T(t)_{,tt} = S_{ij}(\mathbf{X}(\mathbf{x}))_{,j}T(t). \tag{32}$$

Separate the variables in (32), then

$$\frac{T(t)_{,tt}}{T(t)} = \frac{S_{ij}(\mathbf{X}(\mathbf{x}))_{,j}}{\rho_o X_i(\mathbf{x})} = b_i.$$

Here b$_i$ are constant; however, $T(t)$is independent of $X_i(\mathbf{x})$. Therefore, $b_1 = b_2 = b_3 = b$ and

$$T(t)_{,tt} - bT(t) = 0, \tag{33}$$

$$S_{ij}\big(\mathbf{X}(\mathbf{x})\big)_{,j} - b\rho_o X_i(\mathbf{x}) = 0. \tag{34}$$

Substitute (31) into the first equation of (26), then

$$X_i(\mathbf{x})T(0) = 0. \tag{35}$$

The only nontrivial solution to (35) is when

$$T(0) = 0. \tag{36}$$

Substitute Equation (31) into the second equation of (26), then

$$X_i(\mathbf{x})T(0)_{,t} = \zeta_i(\mathbf{x}). \tag{37}$$

The only nontrivial solution to (37) is when

$$T(0)_{,t} = c \quad \text{and} \quad X_i(\mathbf{x}) = c^{-1}\zeta_i(\mathbf{x}). \tag{38}$$

Here c is a constant.

Substitute (31) into (27), then

$$X_i(\Sigma_2)T(t) = q_i(\Sigma_2) \tag{39}$$

The only nontrivial solution to (39) is when

$$X_i(\Sigma_2) = q_i(\Sigma_2) = 0. \tag{40}$$

Substitute (31) into (28), then

$$X_i(\Sigma_2)_{,j}T(t) = q_i(\Sigma_2)_{,j}. \tag{41}$$

The only nontrivial solution to (41) is when

$$X_i(\Sigma_2)_{,j} = q_i(\Sigma_2)_{,j} = 0. \tag{42}$$

Substitute Equation (31) into (29), then

$$S_{ij}\big(\mathbf{X}(\Sigma_1)\big)n_j^{(\Sigma_1)}T(t) = k_i(\Sigma_1)\Phi(t) + S_{ij}\big(\mathbf{q}(\Sigma_1)\big)n_j^{(\Sigma_1)}. \tag{43}$$

Given (36) and (38) we have $T(t) = ct$; therefore, let $S_{ij}\big(\mathbf{q}(\Sigma_1)\big)n_j^{(\Sigma_1)} = 0$ in (43) and isolate $T(t)$.

$$T(t) = \frac{k_i(\Sigma_1)\Phi(t)}{S_{ij}\big(\mathbf{X}(\Sigma_1)\big)n_j^{(\Sigma_1)}}, \qquad \text{no sum over } i. \tag{44}$$

By definition $T(t)$ is a scalar function, while the right hand side of (44) is vector valued; therefore, let

$$k_i(\Sigma_1) = S_{ij}\big(\mathbf{X}(\Sigma_1)\big)n_j^{(\Sigma_1)}. \tag{45}$$

Note, from the definition in 4.1 we have $\Phi(t) = at$; therefore, given (44) and (45)

$$T(t) = at. \tag{46}$$

Substitute (31) into (30), then

$$S_{ij}\big(\mathbf{X}(\Pi)\big)n_j^{(\Pi)}T(t) = S_{ij}\big(\mathbf{q}(\Pi)\big)n_j^{(\Pi)}. \tag{47}$$

The only nontrivial solution to (47) is

$$S_{ij}\big(\mathbf{X}(\Pi)\big)n_j^{(\Pi)} = S_{ij}\big(\mathbf{q}(\Pi)\big)n_j^{(\Pi)} = 0. \tag{48}$$

Equations (33), (36), and (38) form an initial value problem that can be solved for three cases of b.

CASE 1. Let $b = 0$, then (33) becomes

$$T(t)_{,tt} = 0. \tag{49}$$

Integrating (49) twice with respect to time, and taking into account Equation (36) and (38), it can be seen that

$$T(t) = ct. \tag{50}$$

Taking Equation (46) into account, (50) becomes $T(t) = at$. **CASE 2.** Let $b > 0$, and let $b = \lambda^2$, where $\lambda > 0$. Then (33) becomes

$$T(t)_{,tt} - \lambda^2 T(t) = 0. \tag{51}$$

Let $T(t) = e^{rt}$, then (51) becomes

$$r^2 - \lambda^2 = 0. \tag{52}$$

Factoring (52) it can be seen that $r = \pm\lambda$, therefore, there are two solutions for $T(t)$.

$$T(t) = c_1 e^{\lambda t} + c_2 e^{-\lambda t}. \tag{53}$$

Considering (36) and (38) it can be seen that $c_1 = -c_2$ and $c_1 = c/2\lambda$; therefore, (53) becomes

$$T(t) = \frac{c}{\lambda}\sinh(\lambda t). \tag{54}$$

Considering (54) and (46) it can be seen for $a \neq 0$ and $t > 0$ there are no solutions for $T(t)$ when $b > 0$.

CASE 3. Let $b < 0$, and let $b = -\lambda^2$, where $\lambda > 0$. Then (33) becomes

$$T(t)_{,tt} + \lambda^2 T(t) = 0. \tag{55}$$

Let $T(t) = e^{rt}$, then (55) becomes

$$r^2 + \lambda^2 = 0. \tag{56}$$

Factoring (56) it can be seen that $r = \pm i\lambda$; therefore, there are two solutions for $T(t)$

$$T(t) = c_1 \cos(\lambda t) + c_2 \sin(\lambda t). \tag{57}$$

Considering (36) and (38), it can be seen that $c_1 = 0$ and $c_2 = c/\lambda$; therefore, (57) becomes

$$T(t) = \frac{c}{\lambda}\sin(\lambda t). \tag{58}$$

Considering (58) and (46) it can be seen for $a \neq 0$ and $t > 0$ there are no solutions for $T(t)$ when $b < 0$.

Combining the solutions for $T(t)$ from the three cases of b results in

$$T(t) = at. \tag{59}$$

Equations (34), (40), (42), (45), and (48) form a boundary value problem defining $\mathbf{X}(\mathbf{x})$ that can be solved for three cases of b. From the analysis of $T(t)$ it was found that $b = 0$; therefore, the equilibrium problem that defines $\mathbf{X}(\mathbf{x})$ is

$$\begin{aligned} S_{ij}\big(\mathbf{X}(\mathbf{x})\big)_{,j} &= 0, \\ X_i(\Sigma_2) &= 0, \\ X_i(\Sigma_2)_{,j} &= 0, \\ S_{ij}\big(\mathbf{X}(\Sigma_1)\big)n_j^{(\Sigma_1)} &= k_i(\Sigma_1), \\ S_{ij}\big(\mathbf{X}(\Pi)\big)n_j^{(\Pi)} &= 0. \end{aligned} \tag{60}$$

Substitute Equation (59) into the first equation of (38), then from the second equation of (38)

$$X_i(\mathbf{x}) = a^{-1}\zeta_i(\mathbf{x}). \tag{61}$$

Substitute (61) and (59) into (31), and the resulting function into (24), then

$$u_i(\mathbf{x}, t) = t\zeta_i(\mathbf{x}) - q_i(\mathbf{x}) \quad \text{and} \quad u_i(\mathbf{x}, t)_{,tt} = 0 \tag{62}$$

Equation (62) completes the proof of Theorem 4.1. □

5. Nonexistence

Ericksen [1963] gives the main result of nonexistence theorems as nonuniqueness implies nonexistence. Ericksen [1965] notes that it is tacitly hoped in elastostatics that problems are well posed when uniqueness is obtained; however, this is a weak result since it leaves open the possibility of problems where existence fails even though uniqueness is proven. Therefore, nonuniqueness is used in this section as one possible test to determine if solutions to the problem defined by Equation (60) are nonexistent.

Gurtin [1972] presents a nonexistence theorem for the mixed problem of elastostatics where the elasticity tensor is symmetric. The difference between the elastostatics problem defined by (60) and the one considered by Gurtin [1972] is that the third equation of (60) prescribes the strains on Σ_2 and this could affect the method used to form the null data problem. Let $[\mathbf{w}, \mathbf{E}, \mathbf{S}]$ be the difference between two solutions to (60) that are not equal modulo a rigid displacement, where $\mathbf{w}$ is the difference between displacements, $\mathbf{E}$ is the difference between strains, and $\mathbf{S}$ is the difference between stresses. Note, since $\mathbf{E} \not\equiv \mathbf{0}$ in B then $w_i(x)_{,j} \not\equiv 0$ in B. Recall from the proof of Theorem 3.1 that $\lim_{x_3 \to h} u_i(\mathbf{x}, t)_{,j} = 0$ is solely dependent on the Σ_2 boundary conditions, and this result produced the third equation of (60). Therefore, even though $w_i(x)_{,j} \not\equiv 0$ in B it must still vanish as interior points approach Σ_2. Thus,

$$w_i(\mathbf{x}) = w_i(\mathbf{x})_{,j} = 0 \text{ on } \Sigma_2, \quad \text{and} \quad s_i(\mathbf{x}) = S_{ij}(\mathbf{w}(\mathbf{x}))n_j = 0 \text{ on } \Sigma_1 \cup \Pi. \tag{63}$$

Equation (63) corresponds to a nontrivial solution of (60) with null data. Thus, Gurtin's [1972] nonexistence theorem for mixed problems applies to the problem defined by (60), and the task at hand is to determine if (60) has a unique solution in which case nonexistence of a solution is not proven.

The following proof of uniqueness follows Gurtin's [1981] theorem for elastostatics. Consider the problem defined by (60) and note that $\mathbf{C}$ is positive definite. Let $[\mathbf{X}_1, \mathbf{E}_1, \mathbf{S}_1]$ and $[\mathbf{X}_2, \mathbf{E}_2, \mathbf{S}_2]$ be two solutions to (60), where

$$\mathbf{w}(\mathbf{x}) = \mathbf{X}_1(\mathbf{x}) - \mathbf{X}_2(\mathbf{x}) \quad \text{and} \quad \mathbf{E}(\mathbf{x}) = \mathbf{E}_1(\mathbf{X}_1(\mathbf{x})) - \mathbf{E}_2(\mathbf{X}_2(\mathbf{x})) \tag{64}$$

Here $[\mathbf{w}, \mathbf{E}, \mathbf{S}]$ is an elastic state that satisfies the boundary conditions $\mathbf{w}(\mathbf{x}) = 0$ on Σ_2 and $\mathbf{S}(\mathbf{x})\mathbf{n} = 0$ on $\Sigma_1 \cup \Pi$; therefore,

$$\mathbf{S}(\mathbf{x})\mathbf{n} \cdot \mathbf{w}(\mathbf{x}) = 0 \quad \text{on } \partial B. \tag{65}$$

Substituting (65) into the Theorem of Work and Energy as presented by Gurtin [1981] and noting that the body forces are zero in (60) results in

$$\int_B \mathbf{E} \cdot \mathbf{C}[\mathbf{E}]\, dV = 0. \tag{66}$$

Since $\mathbf{C}$ is positive definite, (66) can only be true if $\mathbf{E}(\mathbf{x}) = 0$, which results in $\mathbf{S}(\mathbf{x}) = 0$. Therefore, given (64) the solution to (60) is unique up to a rigid displacement.

Let $\mathbf{w}(\mathbf{x}) = \alpha + \eta \times \mathbf{x}$. Here α and η are constant vectors. Given (63) α and η must be identically equal to zero; therefore, the solution to (60) is unique. Thus, the Nonexistence Theorem proposed by Gurtin [1972] is unable to detect the nonexistence of a solution to Equation (60).

6. Remarks on the relaxed Saint-Venant's problem

Iesan [1987] found the following stress function that satisfies the equilibrium problem for Extension, Bending, and Torsion posed as a relaxed Saint-Venant's problem for a material defined by Equation (3).

$$S_{ij}(\Lambda) = C_{ij33}(a_\rho x_\rho + a_3) - a_4 C_{ij\alpha 3} e_{\alpha\beta} x_\beta + T_{ij}(x_\alpha). \tag{67}$$

Here Λ is an infinitesimal equilibrium displacement field, a_q are constants with q ranging from 1 to 4, T_{ij} are the generalized plane stresses that are independent of x_3, and $e_{\alpha\beta}$ is the two-dimensional alternator symbol. When considering the problem corresponding to (67), Chiriță [1979] and Lyons [2002] found for certain forms of the elasticity tensor that $S_{\alpha\beta}(\Lambda) = 0$ throughout $\overline{B}$.

Given the third equation of (60) and (3)

$$\lim_{x_3 \to h} S_{ij}(\mathbf{X}) = 0. \tag{68}$$

Iesan [1987] notes the justification for the relaxed Saint-Venant's problem is based on Saint-Venant's principle, which assumes the effects of posing the problem as a relaxed Saint-Venant's problem are negligible except possibly near the ends of the cylinder. Note in Equation (67) that S_{ij} are independent of x_3 throughout $\overline{B}$; however, for Equation (68) to hold and for the stresses to be nonzero in some region of $\overline{B}$ there must be a x_3 dependence. Dependence of the field equations on x_3 indicates warping of the transverse cross sections with the possible result that $S_{\alpha\beta} \neq 0$ throughout $\overline{B}$. This is an important consideration for anisotropic materials such as trees and bone that are significantly weaker in the x_α directions.

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