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FUNDAMENTAL SOLUTION IN THE THEORY OF VISCOELASTIC MIXTURES

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In the first part of the paper, we derive a linear theory of thermoviscoelastic binary mixtures. Then, the fundamental solution of the system of linear coupled partial differential equations of steady oscillations (steady vibrations) of the theory of viscoelastic binary mixtures is constructed in terms of elementary functions, and basic properties are established.

1. Introduction

The theory of mixtures was introduced to describe the mechanics of materials in which two or more constituents coexist. The crucial factor that differentiates this theory from other classical approaches is the matter of scale. The theory of mixtures is intended to study the behavior of a material at the small scale of its inhomogeneities and the states of its individual constituents. The great abstraction that a material can be modeled as a single homogeneous continuum is avoided. In contrast with approaches that use averaging producers, the theory of mixtures permits to define the motion, mass density, stress tensor, internal energy, temperature, entropy and other relevant physical quantities, for every single constituent of the continuum. In the case of the diffusion of a fluid through a porous solid or of one solid through another, this information is critical. The theory of mixtures overcomes the inadequacy of classical theories which cannot predict the stresses in the solid in a diffusion process. Moreover, the theory allows for the possibility of studying another two distinct physical phenomena: chemical reactions and multiple temperatures. These issues are important in the mechanics of geological and biological materials. For the history of the problem and the analysis of the results we refer to [Bowen 1976; Atkin and Craine 1976b; 1976a; Bedford and Drumheller 1983; Samohyl 1987; Rajagopal and Tao 1995]. Starting from the origin of the modern formulation of the theory a variety of mathematical models have been developed in order to study mixtures exhibiting complex mechanical behaviors.

In the last three decades there has been interest in the formulation of thermomechanical theories of viscoelastic mixtures. There exist various continuum theories of viscoelastic composites [Marinov 1978; McCarthy and Tiersten 1983; Hills and Roberts 1987; 1988; Aboudi 2000; Iesan and Quintanilla 2002]. A nonlinear theory of heat-conducting viscoelastic mixtures in a Lagrangian description was presented by Iesan [2004]. In this theory the mixture consists of two constituents: a porous elastic solid and a viscous fluid. A linear variant of this theory was developed by Quintanilla [2005], and the existence and exponential decay of solutions are proved.

Iesan and Nappa [2008] introduced a nonlinear theory of heat-conducting mixtures where the individual components are modelled as Kelvin–Voigt viscoelastic materials. The basic equations are obtained using a Lagrangian description (in contrast with mixtures of fluids), which naturally yields an Eulerian

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description. We remark that material and spatial descriptions lead to different theories with different meaning of displacement vector field. Moreover the latter does not allow us to consider boundary conditions in the reference configuration.

For investigating boundary value problems of the theory of elasticity and thermoelasticity with the potential method (boundary integral method) it is necessary to construct fundamental solutions of systems of partial differential equations and establish their basic properties [Kupradze 1965; Kupradze et al. 1979].

Several methods are known for constructing fundamental solutions of differential equations of the classical theory of elasticity and thermoelasticity [Gurtin 1972, Chapter 9; Hetnarski and Ignaczak 2004, Chapter 7; Kupradze et al. 1979, Chapter 2]. The fundamental solutions of equations of the linear theory of binary mixtures for elastic solids have been constructed by Svanadze [1988; 1990]. Useful information on fundamental solutions of differential equations is contained in [Hörmander 1983, Chapters 10 and 12; Lopatinsky 1951].

In the first part of this paper (Sections 2 and 3), we derive a linear theory of thermoviscoelastic mixtures, assuming that the constituents have a common temperature and that the mixture is subjected to a thermodynamical process that satisfies the Clausius–Duhem inequality. The intended applications of the theory are to viscoelastic composite materials, to viscoelastic mixtures of two compatible polymers, or to cortical bone. For a review of the literature on viscoelastic properties of cortical bone the reader is referred to [Lakes 2001]. As in [Iesan and Nappa 2008], with the aim to specify the boundary conditions in the reference configuration, a Lagrangian description is adopted. The constitutive equations are derived independently from nonlinear theory. In contrast with the theory of mixtures of fluids we find that the diffusive force depends on both relative displacement and relative velocity. This constitutive relation generalizes Darcy's law and is frame-independent. We recall that, as observed by Wilmanski [2003], Darcy's law is frame-dependent.

In the second part of this paper (Sections 4 and 5), the fundamental solution of the system of linear coupled partial differential equations of steady vibrations of the theory of viscoelastic binary mixtures is constructed in terms of elementary functions, and basic properties are established.

2. Basic equations

The mixtures under consideration consist of two interacting continua s_1 and s_2. We assume that at time t_0 the body occupies the region B of Euclidean three-dimensional space E^3 and is bounded by piecewise smooth surface ∂B. In describing the motion of the body, we refer to the configuration at time t_0 and to a fixed system of rectangular Cartesian axes. We use vector and Cartesian tensor notation with Latin indices having the values $1, 2, 3$. Greek indices are understood to range over the integers $1, 2$ and a summation convention is not used for these indices. Bold letters denote vectors and tensors.

In the following $\boldsymbol{X}$ and $\boldsymbol{Y}$ are the positions of typical particles of s_1 and s_2 in the reference positions. Following Bedford and Stern [1972], we assume that $\boldsymbol{X}=\boldsymbol{Y}$, so that the particles occupy the same position in the reference configuration. The motion of the mixture is given by

$$\boldsymbol{x}=\boldsymbol{x}(\boldsymbol{X},t), \quad \boldsymbol{y}=\boldsymbol{y}(\boldsymbol{X},t), \quad (\boldsymbol{X},t)\in B\times I, \tag{2-1}$$

where $I=[t_0,\infty)$.

We consider arbitrary material regions P_α of each constituent which coincide at time t_0 with a region P. We postulate an energy balance in the form [Green and Naghdi 1965; 1969; 1972]

$$\frac{d}{dt}\sum_{\alpha=1}^{2}\int_P \rho_\alpha\big(e+\tfrac{1}{2}\boldsymbol{v}^{(\alpha)}\cdot\boldsymbol{v}^{(\alpha)}\big)\,dv=\sum_{\alpha=1}^{2}\bigg(\int_P \rho_\alpha\big(\boldsymbol{f}^{(\alpha)}\cdot\boldsymbol{v}^{(\alpha)}+r\big)\,dv-\int_{\partial P}\big(\tilde{\boldsymbol{t}}^{(\alpha)}\cdot\boldsymbol{v}^{(\alpha)}+q^{(\alpha)}\big)\,da\bigg), \tag{2-2}$$

where e is the internal energy of the mixture per unit mass, $\boldsymbol{v}^{(\alpha)}$ is the velocity vector field associated with the constituent s_α, $\boldsymbol{f}^{(\alpha)}$ is the body force per unit mass acting on the constituent s_α, $\tilde{\boldsymbol{t}}^{(\alpha)}$ is the partial stress vector, r is the external volume supply per unit mass per unit time, $q^{(\alpha)}$ is the heat flux per unit area per unit time associated with the constituent s_α, and ρ_α is the mass density of the constituent s_α.

Let $\boldsymbol{u}^{(\alpha)}$ be the displacement vector field associated with the constituent s_α. In the linear theory we assume that $\boldsymbol{u}^{(\alpha)}=\varepsilon\boldsymbol{u}'^{(\alpha)}$, with ε being a constant small enough for squares and higher powers to be neglected, and with $\boldsymbol{u}'^{(\alpha)}$ independent of ε. The functions (2-1) can be expressed in the form

$$\boldsymbol{x}=\boldsymbol{X}+\boldsymbol{u}^{(1)},\qquad \boldsymbol{y}=\boldsymbol{X}+\boldsymbol{u}^{(2)}. \tag{2-3}$$

By (2-2) and (2-3) we get

$$\sum_{\alpha=1}^{2}\int_P \big(\rho_\alpha\dot{e}+\rho_\alpha\dot{\boldsymbol{u}}^{(\alpha)}\cdot\ddot{\boldsymbol{u}}^{(\alpha)}\big)\,dv=\sum_{\alpha=1}^{2}\bigg(\int_P \big(\rho_\alpha\boldsymbol{f}^{(\alpha)}\cdot\dot{\boldsymbol{u}}^{(\alpha)}+\rho r\big)\,dv-\int_{\partial P}\big(\tilde{\boldsymbol{t}}^{(\alpha)}\cdot\dot{\boldsymbol{u}}^{(\alpha)}+q^{(\alpha)}\big)\,da\bigg). \tag{2-4}$$

Following Green and Naghdi [1965], (2-4) is also true when $\dot{\boldsymbol{u}}^{(\alpha)}$ is replaced by $\boldsymbol{u}^{(\alpha)}+\boldsymbol{c}$, with $\boldsymbol{c}$ an arbitrary constant vector, so that by subtraction we have

$$\sum_{\alpha=1}^{2}\bigg(\int_P \rho_\alpha\big(\ddot{\boldsymbol{u}}^{(\alpha)}-\boldsymbol{f}^{(\alpha)}\big)dv-\int_{\partial P}\tilde{\boldsymbol{t}}^{(\alpha)}da\bigg)=\boldsymbol{0}. \tag{2-5}$$

From (2-5) we obtain

$$\tilde{\boldsymbol{t}}^{(1)}+\tilde{\boldsymbol{t}}^{(2)}=\big(\boldsymbol{t}^{(1)}+\boldsymbol{t}^{(2)}\big)^T\boldsymbol{n}, \tag{2-6}$$

where $\boldsymbol{t}^{(\alpha)^T}$ is the transpose of the stress tensor $\boldsymbol{t}^{(\alpha)}$ associated with the constituent s_α, and $\boldsymbol{n}$ is the unit outward normal vector to the surface ∂P. It follows from (2-5) and (2-6) that

$$\sum_{\alpha=1}^{2}\big(\operatorname{div}\boldsymbol{t}^{(\alpha)^T}+\rho_\alpha\boldsymbol{f}^{(\alpha)}-\rho_\alpha\ddot{\boldsymbol{u}}^{(\alpha)}\big)=\boldsymbol{0}. \tag{2-7}$$

On taking into account (2-6) and (2-7), (2-4) can be written in the form

$$\begin{aligned}\int_P\Big(\rho\dot{e}&+\tfrac{1}{2}\big(\rho_1\ddot{\boldsymbol{u}}^{(1)}-\rho_2\ddot{\boldsymbol{u}}^{(2)}\big)\cdot\big(\dot{\boldsymbol{u}}^{(1)}-\dot{\boldsymbol{u}}^{(2)}\big)-\tfrac{1}{2}\big(\rho_1\boldsymbol{f}^{(1)}-\rho_2\boldsymbol{f}^{(2)}\big)\cdot\big(\dot{\boldsymbol{u}}^{(1)}-\dot{\boldsymbol{u}}^{(2)}\big)\\&-\tfrac{1}{2}\big(\boldsymbol{t}^{(1)}+\boldsymbol{t}^{(2)}\big)^T\cdot\big(\dot{\boldsymbol{H}}^{(1)}+\dot{\boldsymbol{H}}^{(2)}\big)-\rho r\Big)\,dv=\int_{\partial P}\Big(\tfrac{1}{2}\big(\tilde{\boldsymbol{t}}^{(1)}-\tilde{\boldsymbol{t}}^{(2)}\big)\cdot\big(\dot{\boldsymbol{u}}^{(1)}-\dot{\boldsymbol{u}}^{(2)}\big)+q\Big)\,da,\end{aligned} \tag{2-8}$$

where

$$\boldsymbol{H}^{(\alpha)}=\nabla\boldsymbol{u}^{(\alpha)},\qquad q=q^{(1)}+q^{(2)},\qquad \rho=\rho_1+\rho_2.$$

With an argument similar to that used in obtaining (2-6) from (2-5), we obtain

$$\tfrac{1}{2}\big(\tilde{\boldsymbol{t}}^{(1)}-\tilde{\boldsymbol{t}}^{(2)}-(\boldsymbol{t}^{(1)}-\boldsymbol{t}^{(2)})^T\boldsymbol{n}\big)\cdot(\dot{\boldsymbol{u}}^{(1)}-\dot{\boldsymbol{u}}^{(2)})+q-\boldsymbol{q}\cdot\boldsymbol{n}=0, \tag{2-9}$$

where $\boldsymbol{q}$ is the heat flux vector. We introduce the notation

$$\boldsymbol{p}=\tfrac{1}{2}\big(\operatorname{div}(\boldsymbol{t}^{(1)}-\boldsymbol{t}^{(2)})^T+\rho_1\boldsymbol{f}^{(1)}-\rho_2\boldsymbol{f}^{(2)}-(\rho_1\ddot{\boldsymbol{u}}^{(1)}-\rho_2\ddot{\boldsymbol{u}}^{(2)})\big). \tag{2-10}$$

Introducing (2-9) and (2-10) into Equation (2-8) and applying the resulting equation to an arbitrary region P, we obtain

$$\rho\dot{e}=\boldsymbol{t}^{(1)^T}\cdot\dot{\boldsymbol{H}}^{(1)}+\boldsymbol{t}^{(2)^T}\cdot\dot{\boldsymbol{H}}^{(2)}+\boldsymbol{p}\cdot\dot{\boldsymbol{d}}+\rho r+\operatorname{div}\boldsymbol{q}, \tag{2-11}$$

where

$$\boldsymbol{d}=\boldsymbol{u}^{(1)}-\boldsymbol{u}^{(2)}. \tag{2-12}$$

From (2-10) and (2-11) we get the motion equations of the mixture:

$$\operatorname{div}\boldsymbol{t}^{(1)^T}-\boldsymbol{p}+\rho_1\boldsymbol{f}^{(1)}=\rho_1\ddot{\boldsymbol{u}}^{(1)},\qquad \operatorname{div}\boldsymbol{t}^{(2)^T}+\boldsymbol{p}+\rho_2\boldsymbol{f}^{(2)}=\rho_2\ddot{\boldsymbol{u}}^{(2)}. \tag{2-13}$$

As in [Green and Naghdi 1965], we now consider motions of the mixture which are such that the velocities differ from those of the given motion only by a superposed uniform rigid body angular velocity, the continua occupying the same position at time t. In this case $\dot{\boldsymbol{H}}^{(1)}$ and $\dot{\boldsymbol{H}}^{(2)}$ are replaced by $\dot{\boldsymbol{H}}^{(1)}+\boldsymbol{\Omega}$ and $\dot{\boldsymbol{H}}^{(2)}+\boldsymbol{\Omega}$, respectively, and $\dot{\boldsymbol{d}}$ is replaced by $\dot{\boldsymbol{d}}+\boldsymbol{\Omega}\boldsymbol{d}$, where $\boldsymbol{\Omega}$ is an arbitrary skew symmetric tensor. Equation (2-13) implies that

$$\boldsymbol{t}^{(1)}+\boldsymbol{t}^{(2)}=(\boldsymbol{t}^{(1)}+\boldsymbol{t}^{(2)})^T. \tag{2-14}$$

Now we assume that the constituents have a common temperature and adopt the following entropy production inequality [Green and Naghdi 1965; 1972]:

$$\frac{d}{dt}\int_P\rho\eta\,dv-\int_P\frac{1}{\theta}\rho r\,dv-\int_{\partial P}\frac{1}{\theta}q\,da\geq 0, \tag{2-15}$$

where η is the entropy per unit mass of the mixture, and $\theta(>0)$ is the absolute temperature. If we get $q=\boldsymbol{q}\cdot\boldsymbol{n}$ the inequality (2-15) reduces to

$$\rho\theta\dot{\eta}-\rho r-\operatorname{div}\boldsymbol{q}+\frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{\Theta}\geq 0, \tag{2-16}$$

where $\boldsymbol{\Theta}=\nabla\theta$. Introducing the Helmotz free energy $\psi=e-\eta\theta$, the energy Equation (2-11) takes the form

$$\rho(\dot{\psi}+\dot{\theta}\eta+\theta\dot{\eta})=\boldsymbol{t}^{(1)^T}\cdot\dot{\boldsymbol{H}}^{(1)}+\boldsymbol{t}^{(2)^T}\cdot\dot{\boldsymbol{H}}^{(2)}+\boldsymbol{p}\cdot\dot{\boldsymbol{d}}+\rho r+\operatorname{div}\boldsymbol{q}. \tag{2-17}$$

Taking (2-17) into account, the inequality (2-16) becomes

$$\boldsymbol{t}^{(1)^T}\cdot\dot{\boldsymbol{H}}^{(1)}+\boldsymbol{t}^{(2)^T}\cdot\dot{\boldsymbol{H}}^{(2)}+\boldsymbol{p}\cdot\dot{\boldsymbol{d}}-\dot{\sigma}-\rho\eta\dot{\theta}+\frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{\Theta}\geq 0, \tag{2-18}$$

where $\sigma=\rho\psi$.

3. Constitutive equations

In what follows we assume that the constituents s_α are viscoelastic materials of Kelvin–Voigt type. We consider materials characterized by the following set of independent constitutive variables

$$S = (\boldsymbol{H}^{(1)}, \boldsymbol{H}^{(2)}, \boldsymbol{d}, \dot{\boldsymbol{H}}^{(1)}, \dot{\boldsymbol{H}}^{(2)}, \dot{\boldsymbol{d}}, \theta, \boldsymbol{\Theta}; \boldsymbol{X}).$$

The constitutive equations take the form

$$\sigma = \sigma(S), \quad \boldsymbol{t}^{(\alpha)} = \boldsymbol{t}^{(\alpha)}(S), \quad \boldsymbol{p} = \boldsymbol{p}(S), \quad \eta = \eta(S), \quad \boldsymbol{q} = \boldsymbol{q}(S), \tag{3-1}$$

where the response functionals are assumed to be sufficiently smooth. We assume that there are not internal constraints. In order to satisfy the axiom of material-frame indifference, the functionals (3-1) must be expressible in the form

$$\sigma = \tilde{\sigma}(S^0), \quad \boldsymbol{t}^{(\alpha)} = \tilde{\boldsymbol{t}}^{(\alpha)}(S^0), \quad \boldsymbol{p} = \tilde{\boldsymbol{p}}(S^0), \quad \eta = \tilde{\eta}(S^0), \quad \boldsymbol{q} = \tilde{\boldsymbol{q}}(S^0), \tag{3-2}$$

where

$$S^0 = (\boldsymbol{E}, \boldsymbol{G}, \boldsymbol{d}, \dot{\boldsymbol{E}}, \dot{\boldsymbol{G}}, \dot{\boldsymbol{d}}, \theta, \boldsymbol{\Theta}; \boldsymbol{X}), \tag{3-3}$$

and

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{H}^{(1)} + \boldsymbol{H}^{(1)^T}), \qquad \boldsymbol{G} = \boldsymbol{H}^{(1)^T} + \boldsymbol{H}^{(2)}. \tag{3-4}$$

In view of (3-2), (3-3), and (3-4), the inequality (2-18) implies that σ is independent by $\dot{\boldsymbol{E}}$, $\dot{\boldsymbol{G}}$, $\dot{\boldsymbol{d}}$ and $\boldsymbol{\Theta}$, that is

$$\sigma = U(\boldsymbol{E}, \boldsymbol{G}, \boldsymbol{d}, \theta; \boldsymbol{X}). \tag{3-5}$$

Moreover we have

$$\rho\eta = -\frac{\partial U}{\partial \theta}. \tag{3-6}$$

Using (3-5) and (3-6) the inequality (2-18) reduces to

$$\left(\boldsymbol{t}^{(1)^T} - \frac{\partial U}{\partial \boldsymbol{E}} - \left(\frac{\partial U}{\partial \boldsymbol{G}}\right)^T\right) \cdot \dot{\boldsymbol{H}}^{(1)} + \left(\boldsymbol{t}^{(2)^T} - \frac{\partial U}{\partial \boldsymbol{G}}\right) \cdot \dot{\boldsymbol{H}}^{(2)} + \left(\boldsymbol{p} - \frac{\partial U}{\partial \boldsymbol{d}}\right) \cdot \dot{\boldsymbol{d}} + \frac{1}{\theta}\boldsymbol{q} \cdot \boldsymbol{\Theta} \geq 0. \tag{3-7}$$

We introduce the notations

$$\boldsymbol{\tau}^{(1)} = \boldsymbol{t}^{(1)} - \frac{\partial U}{\partial \boldsymbol{E}} - \frac{\partial U}{\partial \boldsymbol{G}}, \quad \boldsymbol{\tau}^{(2)} = \boldsymbol{t}^{(2)} - (\frac{\partial U}{\partial \boldsymbol{G}})^T, \quad \boldsymbol{\pi} = \boldsymbol{p} - \frac{\partial U}{\partial \boldsymbol{d}}. \tag{3-8}$$

The functions $\boldsymbol{\tau}^{(\alpha)}$ and $\boldsymbol{\pi}$ are the dissipative parts of $\boldsymbol{t}^{(\alpha)}$ and $\boldsymbol{p}$. The inequality (3-7) may be written in the form

$$\boldsymbol{\tau}^{(1)^T} \cdot \dot{\boldsymbol{H}}^{(1)} + \boldsymbol{\tau}^{(2)^T} \cdot \dot{\boldsymbol{H}}^{(2)} + \boldsymbol{\pi} \cdot \dot{\boldsymbol{d}} + \frac{1}{\theta}\boldsymbol{q} \cdot \boldsymbol{\Theta} \geq 0. \tag{3-9}$$

Let us introduce the functions $\boldsymbol{\Gamma}$ and $\boldsymbol{\Lambda}$ by

$$\boldsymbol{\tau}^{(1)} = \boldsymbol{\Gamma}(S^0) + \boldsymbol{\Lambda}(S^0), \qquad \boldsymbol{\tau}^{(2)} = \boldsymbol{\Lambda}^T(S^0). \tag{3-10}$$

From (2-14) we deduce that

$$\boldsymbol{\tau}^{(1)} + \boldsymbol{\tau}^{(2)} = (\boldsymbol{\tau}^{(1)} + \boldsymbol{\tau}^{(2)})^T, \qquad \boldsymbol{\Gamma} = \boldsymbol{\Gamma}^T. \tag{3-11}$$

In view of (3-10) and (3-11) the dissipation inequality (3-9) becomes

$$\boldsymbol{\Gamma}\cdot\dot{\boldsymbol{E}}+\boldsymbol{\Lambda}\cdot\dot{\boldsymbol{G}}+\boldsymbol{\pi}\cdot\dot{\boldsymbol{d}}+\frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{\Theta}\geq 0. \tag{3-12}$$

The inequality (3-12) implies that

$$\boldsymbol{\Gamma}(S^*)=\mathbf{0},\quad \boldsymbol{\Lambda}(S^*)=\mathbf{0},\quad \boldsymbol{\pi}(S^*)=\mathbf{0},\quad \boldsymbol{q}(S^*)=\mathbf{0}, \tag{3-13}$$

where

$$S^*=(\boldsymbol{E},\boldsymbol{G},\boldsymbol{d},\mathbf{0},\mathbf{0},\mathbf{0},\theta,\mathbf{0};\boldsymbol{X}).$$

With the help of (3-6), (3-8), and (3-10), the energy balance reduces to

$$\rho\theta\dot{\eta}=\boldsymbol{\Gamma}\cdot\dot{\boldsymbol{E}}+\boldsymbol{\Lambda}\cdot\dot{\boldsymbol{G}}+\boldsymbol{\pi}\cdot\dot{\boldsymbol{d}}+\rho r+\operatorname{div}\boldsymbol{q}. \tag{3-14}$$

Let us denote

$$\theta=T+T_0,\qquad T=\varepsilon T^{'},\qquad \varepsilon^n\cong 0\quad \text{for } n\geq 2, \tag{3-15}$$

where T_0 is the constant absolute temperature of the body in the reference configuration and $T^{'}$ is independent of ε. In what follows we consider the case of centrosymmetric materials. We assume that U has the form

$$U=\tfrac{1}{2}\boldsymbol{E}\cdot\boldsymbol{A}\boldsymbol{E}+\boldsymbol{E}\cdot\boldsymbol{B}\boldsymbol{G}+\tfrac{1}{2}\boldsymbol{G}\cdot\boldsymbol{C}\boldsymbol{G}+\tfrac{1}{2}\boldsymbol{d}\cdot\boldsymbol{a}\,\boldsymbol{d}-(\boldsymbol{\beta}^{(1)}\cdot\boldsymbol{E}+\boldsymbol{\beta}^{(2)}\cdot\boldsymbol{G})T-\tfrac{1}{2}a_0T^2, \tag{3-16}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are fourth order tensors, $\boldsymbol{a}$, $\boldsymbol{\beta}^{(1)}$ and $\boldsymbol{\beta}^{(2)}$ are second order tensors, and a_0 is a constant. The constitutive coefficients have the symmetries

$$A_{ijrs}=A_{jirs}=A_{rsij},\quad B_{ijrs}=B_{jirs},\quad C_{ijrs}=C_{rsij},\quad a_{ij}=a_{ji},\quad \beta^{(1)}_{ij}=\beta^{(1)}_{ji}. \tag{3-17}$$

Letting be $\boldsymbol{M}$ a fourth order tensor, the transpose of $\boldsymbol{M}$ is the unique tensor $\boldsymbol{M}^T$ with the property

$$\boldsymbol{M}\boldsymbol{p}\cdot\boldsymbol{q}=\boldsymbol{p}\cdot\boldsymbol{M}^T\boldsymbol{q},$$

where $\boldsymbol{p}$ and $\boldsymbol{q}$ are second order tensors. Consequently by (3-17) we have $\boldsymbol{A}=\boldsymbol{A}^T$ and $\boldsymbol{C}=\boldsymbol{C}^T$. From (3-6), (3-8) and (3-16) we obtain

$$\begin{gathered}\boldsymbol{t}^{(1)}=(\boldsymbol{A}+\boldsymbol{B}^T)\boldsymbol{E}+(\boldsymbol{B}+\boldsymbol{C})\boldsymbol{G}-(\boldsymbol{\beta}^{(1)}+\boldsymbol{\beta}^{(2)})T+\boldsymbol{\tau}^{(1)},\\ \boldsymbol{t}^{(2)^T}=\boldsymbol{B}^T\boldsymbol{E}+\boldsymbol{C}\boldsymbol{G}-\boldsymbol{\beta}^{(2)}T+\boldsymbol{\tau}^{(2)^T},\\ \boldsymbol{p}=\boldsymbol{a}\boldsymbol{d}+\boldsymbol{\pi},\qquad \rho\eta=\boldsymbol{\beta}^{(1)}\cdot\boldsymbol{E}+\boldsymbol{\beta}^{(2)}\cdot\boldsymbol{G}+a_0T.\end{gathered} \tag{3-18}$$

The relations (3-13) leads to

$$\boldsymbol{\Gamma}=\boldsymbol{A}^*\dot{\boldsymbol{E}}+\boldsymbol{C}^*\dot{\boldsymbol{G}},\quad \boldsymbol{\Lambda}=\boldsymbol{B}^*\dot{\boldsymbol{E}}+\boldsymbol{D}^*\dot{\boldsymbol{G}},\quad \boldsymbol{\pi}=\boldsymbol{a}^*\dot{\boldsymbol{d}}+\boldsymbol{b}^*\nabla T,\quad \boldsymbol{q}=\boldsymbol{k}\nabla T+\boldsymbol{f}\dot{\boldsymbol{d}}, \tag{3-19}$$

where $\boldsymbol{A}^*$, $\boldsymbol{B}^*$, $\boldsymbol{C}^*$, and $\boldsymbol{D}^*$ are fourth order tensors and $\boldsymbol{a}^*$, $\boldsymbol{b}^*$, $\boldsymbol{k}$, and $\boldsymbol{f}$ are second order tensors. Using (3-19) the relations (3-10) can be put in the form

$$\boldsymbol{\tau}^{(1)}=(\boldsymbol{A}^*+\boldsymbol{B}^*)\dot{\boldsymbol{E}}+(\boldsymbol{C}^*+\boldsymbol{D}^*)\dot{\boldsymbol{G}},\qquad \boldsymbol{\tau}^{(2)^T}=\boldsymbol{B}^*\dot{\boldsymbol{E}}+\boldsymbol{D}^*\dot{\boldsymbol{G}}. \tag{3-20}$$

Taking into account (3-15) the energy Equation (3-14) reduces to

$$\rho T_0 \dot{\eta} = \rho r + \operatorname{div} \boldsymbol{q}. \tag{3-21}$$

The basic equations of linear viscoelastic mixtures are: the equations of motion (2-13), the equation of energy (3-21), the constitutive equations (3-18) and (3-20), and the geometric equations (2-12) and (3-4). We remark that the relation $(3\text{-}18)_3$ generalizes Darcy's law. This relation has been obtained from constitutive assumptions and is frame-independent. In the case of isotropy, the constitutive equations (3-18) and (3-20) take the form

$$\begin{aligned}
\boldsymbol{t}^{(1)} = 2(\mu+\zeta)\boldsymbol{E} + (\lambda+\nu)(\operatorname{tr}\boldsymbol{E})\boldsymbol{I} + (2\kappa+\zeta)\boldsymbol{G} + (2\gamma+\zeta)\boldsymbol{G}^T \\
+ (\alpha+\nu)(\operatorname{tr}\boldsymbol{G})\boldsymbol{I} - (\beta^{(1)}+\beta^{(2)})T\boldsymbol{I} + 2(\mu^*+\zeta_1^*)\dot{\boldsymbol{E}} \\
+ (\lambda^*+\nu_1^*)(\operatorname{tr}\dot{\boldsymbol{E}})\boldsymbol{I} + (2\kappa^*+\zeta^*)\dot{\boldsymbol{G}} + (2\gamma^*+\zeta^*)\dot{\boldsymbol{G}}^T + (\alpha^*+\nu^*)(\operatorname{tr}\dot{\boldsymbol{G}})\boldsymbol{I}, \\
\boldsymbol{t}^{(2)} = 2\zeta\boldsymbol{E} + \nu(\operatorname{tr}\boldsymbol{E})\boldsymbol{I} + 2\kappa\boldsymbol{G}^T + 2\gamma\boldsymbol{G} + \alpha(\operatorname{tr}\boldsymbol{G})\boldsymbol{I} - \beta^{(2)}T\boldsymbol{I} \\
+2\zeta_1^*\dot{\boldsymbol{E}} + \nu_1^*(\operatorname{tr}\dot{\boldsymbol{E}})\boldsymbol{I} + 2\gamma^*\dot{\boldsymbol{G}} + 2\kappa^*\dot{\boldsymbol{G}}^T + \alpha^*(\operatorname{tr}\dot{\boldsymbol{G}})\boldsymbol{I}, \\
\boldsymbol{p} = a\boldsymbol{d} + a^*\dot{\boldsymbol{d}} + b^*\nabla T, \quad \rho\eta = \beta^{(1)}\operatorname{tr}\boldsymbol{E} + \beta^{(2)}\operatorname{tr}\boldsymbol{G} + a_0 T, \quad \boldsymbol{q} = k\nabla T + f\dot{\boldsymbol{d}},
\end{aligned} \tag{3-22}$$

where $\lambda, \mu, \gamma, \dots, f$ are constitutive coefficients, $\boldsymbol{I} = (\delta_{jl})_{3\times 3}$ is the unit matrix, δ_{lj} is the Kronecker delta, $\boldsymbol{u}^{(1)} = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)})$, $\boldsymbol{u}^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)})$.

We introduce the notations

$$\begin{aligned}
&\alpha_1 = \mu+2\kappa+2\zeta, && \alpha_2 = \lambda+\mu+\alpha+2\nu+2\gamma+2\zeta,\\
&\alpha_3 = 2\gamma+\zeta, && \alpha_4 = \alpha+\nu+2\kappa+\zeta,\\
&\alpha_5 = 2\kappa, && \alpha_6 = \alpha+2\gamma,\\
&\alpha_1^* = \mu^*+2\kappa^*+\zeta^*+\zeta_1^*, && \alpha_2^* = \lambda^*+\mu^*+\alpha^*+\nu^*+\nu_1^*+2\gamma^*+\zeta^*+\zeta_1^*,\\
&\alpha_3^* = 2\gamma^*+\zeta^*, && \alpha_4^* = \alpha^*+\nu^*+2\kappa^*+\zeta^*,\\
&\alpha_5^* = 2\gamma^*+\zeta_1^*, && \alpha_6^* = \alpha^*+\nu_1^*+2\kappa^*+\zeta_1^*,\\
&\alpha_7^* = 2\kappa^*, && \alpha_8^* = \alpha^*+2\gamma^*,\\
&\beta_1 = \beta^{(1)}+\beta^{(2)}+b^*, && \beta_2 = \beta^{(2)}-b^*,\\
&\beta_3 = T_0(\beta^{(1)}+\beta^{(2)}), && \beta_4 = T_0\,\beta^{(2)}.
\end{aligned} \tag{3-23}$$

From (2-12), (2-13), (3-4), and (3-21)–(3-23) we have

$$\begin{aligned}
&\alpha_1\Delta\boldsymbol{u}^{(1)} + \alpha_2\nabla\operatorname{div}\boldsymbol{u}^{(1)} + \alpha_3\Delta\boldsymbol{u}^{(2)} + \alpha_4\nabla\operatorname{div}\boldsymbol{u}^{(2)} - a(\boldsymbol{u}^{(1)}-\boldsymbol{u}^{(2)})\\
&+\alpha_1^*\Delta\dot{\boldsymbol{u}}^{(1)} + \alpha_2^*\nabla\operatorname{div}\dot{\boldsymbol{u}}^{(1)} + \alpha_3^*\Delta\dot{\boldsymbol{u}}^{(2)} + \alpha_4^*\nabla\operatorname{div}\dot{\boldsymbol{u}}^{(2)} - a^*(\dot{\boldsymbol{u}}^{(1)}-\dot{\boldsymbol{u}}^{(2)}) - \beta_1\nabla T + \rho_1\boldsymbol{f}^{(1)} = \rho_1\ddot{\boldsymbol{u}}^{(1)},\\
&\alpha_3\Delta\boldsymbol{u}^{(1)} + \alpha_4\nabla\operatorname{div}\boldsymbol{u}^{(1)} + \alpha_5\Delta\boldsymbol{u}^{(2)} + \alpha_6\nabla\operatorname{div}\boldsymbol{u}^{(2)} + a(\boldsymbol{u}^{(1)}-\boldsymbol{u}^{(2)})\\
&+\alpha_5^*\Delta\dot{\boldsymbol{u}}^{(1)} + \alpha_6^*\nabla\operatorname{div}\dot{\boldsymbol{u}}^{(1)} + \alpha_7^*\Delta\dot{\boldsymbol{u}}^{(2)} + \alpha_8^*\nabla\operatorname{div}\dot{\boldsymbol{u}}^{(2)} + a^*(\dot{\boldsymbol{u}}^{(1)}-\dot{\boldsymbol{u}}^{(2)}) - \beta_2\nabla T + \rho_2\boldsymbol{f}^{(2)} = \rho_2\ddot{\boldsymbol{u}}^{(2)},\\
&\qquad k\Delta T - a_0 T_0\,\dot{T} - \operatorname{div}(\beta_3\dot{\boldsymbol{u}}^{(1)} + \beta_4\dot{\boldsymbol{u}}^{(2)}) + f\operatorname{div}(\dot{\boldsymbol{u}}^{(1)}-\dot{\boldsymbol{u}}^{(2)}) + \rho r = 0.
\end{aligned} \tag{3-24}$$

The system (3-24) can be written as

$$
\begin{aligned}
&\hat{a}_1 \Delta \boldsymbol{u}^{(1)} + \hat{a}_2 \nabla \operatorname{div} \boldsymbol{u}^{(1)} + \hat{a}_3 \Delta \boldsymbol{u}^{(2)} + \hat{a}_4 \nabla \operatorname{div} \boldsymbol{u}^{(2)} - \hat{\xi}(\boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}) + \beta_1 \nabla T + \rho_1 \boldsymbol{f}^{(1)} = \rho_1 \ddot{\boldsymbol{u}}^{(1)},\\
&\hat{a}_5 \Delta \boldsymbol{u}^{(1)} + \hat{a}_6 \nabla \operatorname{div} \boldsymbol{u}^{(1)} + \hat{a}_7 \Delta \boldsymbol{u}^{(2)} + \hat{a}_8 \nabla \operatorname{div} \boldsymbol{u}^{(2)} + \hat{\xi}(\boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}) + \beta_2 \nabla T + \rho_2 \boldsymbol{f}^{(2)} = \rho_2 \ddot{\boldsymbol{u}}^{(2)},\\
&\quad k \Delta T - a_0 T_0 \dot{T} - (T_0 \beta_1 - T_0 b^* - f) \operatorname{div} \dot{\boldsymbol{u}}^{(1)} - (T_0 \beta_2 + T_0 b^* + f) \operatorname{div} \dot{\boldsymbol{u}}^{(2)} + \rho r = 0,
\end{aligned}
\tag{3-25}
$$

where

$$
\hat{a}_j = \alpha_j + \alpha_j^* \frac{\partial}{\partial t}, \quad \hat{a}_l = \alpha_{l-2} + \alpha_l^* \frac{\partial}{\partial t}, \quad \hat{\xi} = a + a^* \frac{\partial}{\partial t}, \quad j = 1, 2, 3, 4, \quad l = 5, 6, 7, 8.
$$

In the isothermal case from (3-25) we obtain the following system of equations of motion in the linear theory of viscoelastic mixtures:

$$
\begin{aligned}
&\hat{a}_1 \Delta \boldsymbol{u}^{(1)} + \hat{a}_2 \nabla \operatorname{div} \boldsymbol{u}^{(1)} + \hat{a}_3 \Delta \boldsymbol{u}^{(2)} + \hat{a}_4 \nabla \operatorname{div} \boldsymbol{u}^{(2)} - \hat{\xi}(\boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}) + \rho_1 \boldsymbol{f}^{(1)} = \rho_1 \ddot{\boldsymbol{u}}^{(1)},\\
&\hat{a}_5 \Delta \boldsymbol{u}^{(1)} + \hat{a}_6 \nabla \operatorname{div} \boldsymbol{u}^{(1)} + \hat{a}_7 \Delta \boldsymbol{u}^{(2)} + \hat{a}_8 \nabla \operatorname{div} \boldsymbol{u}^{(2)} + \hat{\xi}(\boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}) + \rho_2 \boldsymbol{f}^{(2)} = \rho_2 \ddot{\boldsymbol{u}}^{(2)}.
\end{aligned}
\tag{3-26}
$$

If the body forces $\boldsymbol{f}^{(1)}$ and $\boldsymbol{f}^{(2)}$ are assumed to be absent, and the partial displacement vectors $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(1)}$ are postulated to have a harmonic time variation, that is,

$$
\boldsymbol{u}^{(j)}(\boldsymbol{x}, t) = \operatorname{Re}[\boldsymbol{w}^{(j)}(\boldsymbol{x}) e^{-i\omega t}], \qquad j = 1, 2,
$$

then from system of equations of motion (3-26) we obtain the following system of equations of steady vibration in the linear theory of viscoelastic mixtures:

$$
\begin{aligned}
&a_1 \Delta \boldsymbol{w}^{(1)} + a_2 \nabla \operatorname{div} \boldsymbol{w}^{(1)} + a_3 \Delta \boldsymbol{w}^{(2)} + a_4 \nabla \operatorname{div} \boldsymbol{w}^{(2)} + \xi_1 \boldsymbol{w}^{(1)} + \xi \boldsymbol{w}^{(2)} = \boldsymbol{0},\\
&a_5 \Delta \boldsymbol{w}^{(1)} + a_6 \nabla \operatorname{div} \boldsymbol{w}^{(1)} + a_7 \Delta \boldsymbol{w}^{(2)} + a_8 \nabla \operatorname{div} \boldsymbol{w}^{(2)} + \xi \boldsymbol{w}^{(1)} + \xi_2 \boldsymbol{w}^{(2)} = \boldsymbol{0},
\end{aligned}
\tag{3-27}
$$

where ω is the oscillation frequency ($\omega > 0$), and

$$
a_j = \alpha_j - i\omega\alpha_j^*, \quad a_l = \alpha_{l-2} - i\omega\alpha_l^*, \quad j = 1, 2, 3, 4, \quad l = 5, 6, 7, 8,
$$

$$
\xi = a - i\omega a^*, \quad \xi_s = \rho_s \omega^2 - \xi, \quad s = 1, 2.
$$

In the second part of this paper (Sections 4 and 5) the fundamental solution of the system (3-27) is constructed in terms of elementary functions, and basic properties are established.

4. Fundamental solution of the system of equations of steady vibration

We introduce the matrix differential operator

$$
\boldsymbol{R}(\boldsymbol{D}_{\boldsymbol{x}}) = (R_{mn}(\boldsymbol{D}_{\boldsymbol{x}}))_{6\times 6},
$$

where

$$
\begin{aligned}
R_{lj}(\boldsymbol{D}_{\boldsymbol{x}}) &= (a_1 \Delta + \xi_1)\delta_{lj} + a_2 \frac{\partial^2}{\partial x_l \partial x_j}, & R_{l;j+3}(\boldsymbol{D}_{\boldsymbol{x}}) &= (a_3 \Delta + \xi)\delta_{lj} + a_4 \frac{\partial^2}{\partial x_l \partial x_j},\\
R_{l+3;j}(\boldsymbol{D}_{\boldsymbol{x}}) &= (a_5 \Delta + \xi)\delta_{lj} + a_6 \frac{\partial^2}{\partial x_l \partial x_j}, & R_{l+3;j+3}(\boldsymbol{D}_{\boldsymbol{x}}) &= (a_7 \Delta + \xi_2)\delta_{lj} + a_8 \frac{\partial^2}{\partial x_l \partial x_j},
\end{aligned}
$$

$$\boldsymbol{x} = (x_1, x_2, x_3), \quad \boldsymbol{D}_{\boldsymbol{x}} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), \quad \xi_s = \rho_s \omega^2 - \xi, \quad s = 1, 2, \quad l, j = 1, 2, 3.$$

The system (3-27) can be written as

$$\boldsymbol{R}(\boldsymbol{D}_{\boldsymbol{x}})\boldsymbol{W}(\boldsymbol{x}) = \boldsymbol{0},$$

where $\boldsymbol{W} = (\boldsymbol{w}^{(1)}, \boldsymbol{w}^{(2)})$ is a six-component vector function on E^3.

We assume that the constitutive coefficients satisfy the condition

$$(a_1a_7 - a_3a_5)[(a_1 + a_2)(a_7 + a_8) - (a_3 + a_4)(a_5 + a_6)] \neq 0. \tag{4-1}$$

Definition. The fundamental solution of the system (3-27) (the fundamental matrix of operator $\boldsymbol{R}(\boldsymbol{D}_{\boldsymbol{x}})$) is the matrix $\boldsymbol{\Psi}(\boldsymbol{x}) = (\Psi_{lj}(\boldsymbol{x}))_{6\times 6}$ satisfying the condition [Hörmander 1983, Chapter 10; Lopatinsky 1951]

$$\boldsymbol{R}(\boldsymbol{D}_{\boldsymbol{x}})\boldsymbol{\Psi}(\boldsymbol{x}) = \delta(\boldsymbol{x})\boldsymbol{J}, \tag{4-2}$$

where δ is the Dirac delta, $\boldsymbol{J} = (\delta_{lj})_{6\times 6}$ is the unit matrix, and $\boldsymbol{x} \in E^3$.

We consider the system of equations

$$\begin{aligned} &a_1\Delta \boldsymbol{w}^{(1)} + a_2\nabla \operatorname{div} \boldsymbol{w}^{(1)} + a_5\Delta \boldsymbol{w}^{(2)} + a_6\nabla \operatorname{div} \boldsymbol{w}^{(2)} + \xi_1 \boldsymbol{w}^{(1)} + \xi \boldsymbol{w}^{(2)} = \boldsymbol{F}', \\ &a_3\Delta \boldsymbol{w}^{(1)} + a_4\nabla \operatorname{div} \boldsymbol{w}^{(1)} + a_7\Delta \boldsymbol{w}^{(2)} + a_8\nabla \operatorname{div} \boldsymbol{w}^{(2)} + \xi \boldsymbol{w}^{(1)} + \xi_2 \boldsymbol{w}^{(2)} = \boldsymbol{F}'', \end{aligned} \tag{4-3}$$

where $\boldsymbol{F}'$ and $\boldsymbol{F}''$ are three-component vector functions on E^3.

As one may easily verify, the system (4-3) can be written in the form

$$\boldsymbol{R}^T(\boldsymbol{D}_{\boldsymbol{x}})\boldsymbol{W}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}), \tag{4-4}$$

where $\boldsymbol{R}^T$ is the transpose of matrix $\boldsymbol{R}$, $\boldsymbol{F} = (\boldsymbol{F}', \boldsymbol{F}'')$, and $\boldsymbol{x} \in E^3$.

Applying the operator div to $(4\text{-}3)_1$ and $(4\text{-}3)_2$ we have

$$\begin{aligned} &[(a_1 + a_2)\Delta + \xi_1] \operatorname{div} \boldsymbol{w}^{(1)} + [(a_5 + a_6)\Delta + \xi] \operatorname{div} \boldsymbol{w}^{(2)} = \operatorname{div} \boldsymbol{F}', \\ &[(a_3 + a_4)\Delta + \xi] \operatorname{div} \boldsymbol{w}^{(1)} + [(a_7 + a_8)\Delta + \xi_2] \operatorname{div} \boldsymbol{w}^{(2)} = \operatorname{div} \boldsymbol{F}''. \end{aligned} \tag{4-5}$$

The system (4-5) may be written in matrix form:

$$\boldsymbol{Q}(\Delta)\boldsymbol{V} = \boldsymbol{f}, \tag{4-6}$$

where $\boldsymbol{V} = (\operatorname{div} \boldsymbol{w}^{(1)}, \operatorname{div} \boldsymbol{w}^{(2)})$, $\boldsymbol{f} = (f_1, f_2) = (\operatorname{div} \boldsymbol{F}', \operatorname{div} \boldsymbol{F}'')$, and

$$\boldsymbol{Q}(\Delta) = (Q_{lj}(\Delta))_{2\times 2} = \begin{pmatrix} (a_1 + a_2)\Delta + \xi_1 & (a_5 + a_6)\Delta + \xi \\ (a_3 + a_4)\Delta + \xi & (a_7 + a_8)\Delta + \xi_2 \end{pmatrix}_{3\times 3}.$$

System (4-6) implies

$$\Lambda_1(\Delta)\boldsymbol{V} = \Phi, \tag{4-7}$$

where

$$\Phi=(\Phi_1,\Phi_2),\quad \Phi_j=\frac{1}{d_1}\sum_{l=1}^{2}Q'_{lj}\,f_l,\quad \Lambda_1(\Delta)=\frac{1}{d_1}\det \boldsymbol{Q}(\Delta),$$
$$d_1=(a_1+a_2)(a_7+a_8)-(a_3+a_4)(a_5+a_6),\qquad j=1,2, \tag{4-8}$$

and Q'_{lj} is the cofactor of the element Q_{lj} of the matrix $\boldsymbol{Q}$,

$$Q^*_{11}=(a_7+a_8)\Delta+\xi_2,\qquad Q^*_{12}=-((a_3+a_4)\Delta+\xi),$$
$$Q^*_{21}=-((a_5+a_6)\Delta+\xi),\qquad Q^*_{22}=(a_1+a_2)\Delta+\xi_1.$$

It is easily seen that

$$\Lambda_1(\Delta)=(\Delta+k_1^2)(\Delta+k_2^2),$$

where k_1^2 and k_2^2 are the roots of the equation $\Lambda_1(-\chi)=0$ (with respect to χ).

Applying the operator $\Lambda_1(\Delta)$ to (4-3)$_1$ and (4-3)$_2$, and taking (4-7) into account, we obtain

$$\begin{aligned}\Lambda_1(\Delta)((a_1\Delta+\xi_1)\boldsymbol{w}^{(1)}+(a_5\Delta+\xi)\boldsymbol{w}^{(2)})&=\boldsymbol{F}_1,\\ \Lambda_1(\Delta)((a_3\Delta+\xi)\boldsymbol{w}^{(1)}+(a_7\Delta+\xi_2)\boldsymbol{w}^{(2)})&=\boldsymbol{F}_2,\end{aligned} \tag{4-9}$$

where

$$\boldsymbol{F}_1=\Lambda_1(\Delta)\boldsymbol{F}'-\nabla\,(a_2\Phi_1+a_6\Phi_2),\qquad \boldsymbol{F}_2=\Lambda_1(\Delta)\boldsymbol{F}''-\nabla\,(a_4\Phi_1+a_8\Phi_2). \tag{4-10}$$

From system (4-9) we have

$$\Lambda_1(\Delta)\Lambda_2(\Delta)\boldsymbol{w}^{(1)}=\boldsymbol{H}_1,\qquad \Lambda_1(\Delta)\Lambda_2(\Delta)\boldsymbol{w}^{(2)}=\boldsymbol{H}_2, \tag{4-11}$$

where

$$\Lambda_2(\Delta)=\frac{1}{d_2}\det \boldsymbol{Z}(\Delta),\qquad d_2=a_1a_7-a_3a_5,\qquad \boldsymbol{Z}(\Delta)=\begin{pmatrix}a_1\Delta+\xi_1 & a_5\Delta+\xi\\ a_3\Delta+\xi & a_7\Delta+\xi_2\end{pmatrix}_{2\times 2},$$

and

$$\boldsymbol{H}_j=\frac{1}{d_2}\sum_{l=1}^{2}Z'_{lj}(\Delta)\boldsymbol{F}_l,\qquad j=1,2. \tag{4-12}$$

Z'_{lj} is the cofactor of the element Z_{lj} of the matrix $\boldsymbol{Z}$.

$$Z'_{11}=a_7\Delta+\xi_2,\qquad Z'_{12}=-a_3\Delta-\xi,$$
$$Z'_{21}=-a_5\Delta-\xi,\qquad Z'_{22}=a_1\Delta+\xi_1.$$

Obviously, $\Lambda_2(\Delta)=(\Delta+k_4^2)(\Delta+k_5^2)$, where k_4^2 and k_5^2 are the roots of the equation $\Lambda_2(-\chi)=0$ (with respect to χ).

On the basis of (4-11) we get

$$\tilde{\boldsymbol{\Lambda}}(\Delta)\boldsymbol{W}(\boldsymbol{x})=\boldsymbol{H}(\boldsymbol{x}), \tag{4-13}$$

where $\boldsymbol{H}=(\boldsymbol{H}_1,\boldsymbol{H}_2)$ and $\tilde{\boldsymbol{\Lambda}}(\Delta)$ is the diagonal matrix

$$\tilde{\boldsymbol{\Lambda}}(\Delta)=(\tilde{\Lambda}_{lj}(\Delta))_{6\times 6},\quad \tilde{\Lambda}_{11}(\Delta)=\tilde{\Lambda}_{22}(\Delta)=\cdots=\tilde{\Lambda}_{66}(\Delta)=\Lambda_1(\Delta)\Lambda_2(\Delta),$$
$$\tilde{\Lambda}_{mn}(\Delta)=0,\qquad m,n=1,2,\dots,6,\qquad m\neq n.$$

In what follows we use the notations

$$m_{l1}(\Delta)=\frac{1}{d_1}\big(a_2 Q'_{l1}(\Delta)+a_6 Q'_{l2}(\Delta)\big),\qquad m_{l2}(\Delta)=\frac{1}{d_1}\big(a_4 Q'_{l1}(\Delta)+a_8 Q'_{l2}(\Delta)\big),$$
$$n_{lj}(\Delta)=-\frac{1}{d_2}\sum_{p=1}^{2} m_{lp} Z'_{pj},\qquad l,j=1,2. \tag{4-14}$$

In view of (4-8), (4-10), and (4-14), from (4-12) we have

$$\boldsymbol{H}_j=\Big(\frac{1}{d_2}Z'_{1j}(\Delta)\Lambda_1(\Delta)\boldsymbol{I}+n_{1j}(\Delta)\,\nabla\mathrm{div}\Big)\boldsymbol{F}'+\Big(\frac{1}{d_2}Z'_{2j}(\Delta)\Lambda_1(\Delta)\boldsymbol{I}+n_{2j}(\Delta)\,\nabla\,\mathrm{div}\Big)\boldsymbol{F}'',\qquad j=1,2, \tag{4-15}$$

where $\boldsymbol{I}=(\delta_{lj})_{3\times3}$ is the unit matrix.

Thus, from (4-15) we obtain

$$\boldsymbol{H}(\boldsymbol{x})=\boldsymbol{L}^T(\boldsymbol{D}_{\mathbf{x}})\boldsymbol{F}(\boldsymbol{x}), \tag{4-16}$$

where

$$\boldsymbol{L}=(L_{lj})_{6\times6}=\begin{pmatrix}\boldsymbol{L}^{(1)} & \boldsymbol{L}^{(2)}\\ \boldsymbol{L}^{(3)} & \boldsymbol{L}^{(4)}\end{pmatrix}_{6\times6},\qquad \boldsymbol{L}^{(m)}=(L_{lj}^{(m)})_{3\times3},$$
$$\boldsymbol{L}^{(q)}(\boldsymbol{D}_{\mathbf{x}})=\frac{1}{d_2}Z'_{1q}(\Delta)\Lambda_1(\Delta)\boldsymbol{I}+n_{1q}(\Delta)\nabla\,\mathrm{div},$$
$$\boldsymbol{L}^{(q+2)}(\boldsymbol{D}_{\boldsymbol{x}})=\frac{1}{d_2}Z'_{2q}(\Delta)\Lambda_1(\Delta)\boldsymbol{I}+n_{2q}(\Delta)\nabla\,\mathrm{div},$$
$$q=1,2,\qquad m=1,2,3,4. \tag{4-17}$$

By virtue of (4-4) and (4-16), from (4-13) it follows that $\tilde{\boldsymbol{\Lambda}}\boldsymbol{W}=\boldsymbol{L}^T\boldsymbol{R}^T\boldsymbol{W}$. It is obvious that $\boldsymbol{L}^T\boldsymbol{R}^T=\tilde{\boldsymbol{\Lambda}}$ and, hence,

$$\boldsymbol{R}(\boldsymbol{D}_{\boldsymbol{x}})\boldsymbol{L}(\boldsymbol{D}_{\boldsymbol{x}})=\tilde{\boldsymbol{\Lambda}}(\Delta). \tag{4-18}$$

We assume that $k_l^2\neq k_j^2\neq 0$, that $l,j=1,2,3,4$, and that $l\neq j$.

Let

$$\boldsymbol{Y}(\boldsymbol{x})=(Y_{mn}(\boldsymbol{x}))_{6\times6},\quad Y_{11}(\boldsymbol{x})=Y_{22}(\boldsymbol{x})=\cdots=Y_{66}(\boldsymbol{x})=\sum_{j=1}^{4}\eta_j h_j(\boldsymbol{x}),\quad Y_{mn}(\boldsymbol{x})=0,$$
$$m,n=1,2,\dots,6,\qquad m\neq n, \tag{4-19}$$

where

$$h_j(\boldsymbol{x})=-\frac{1}{4\pi\,|\boldsymbol{x}|}e^{ik_j|\boldsymbol{x}|},\qquad \eta_j=\prod_{l\neq j,\,l=1}^{4}(k_l^2-k_j^2)^{-1},\qquad j=1,2,3,4. \tag{4-20}$$

Lemma 4.1. *The matrix* $\boldsymbol{Y}$ *is the fundamental matrix of operator* $\tilde{\boldsymbol{\Lambda}}(\Delta)$, *that is*

$$\tilde{\boldsymbol{\Lambda}}(\Delta)\boldsymbol{Y}(\boldsymbol{x})=\delta(\boldsymbol{x})\boldsymbol{J}. \tag{4-21}$$

Proof. It is sufficient to show that

$$\Lambda_1(\Delta)\Lambda_2(\Delta)Y_{11}(\boldsymbol{x})=\delta(\boldsymbol{x}).$$

Taking into account the equalities

$$\sum_{j=1}^{4} \eta_j = 0, \qquad \sum_{j=2}^{4} \eta_j (k_1^2 - k_j^2) = 0, \qquad \sum_{j=3}^{4} \eta_j (k_1^2 - k_j^2)(k_2^2 - k_j^2) = 0,$$
$$\eta_4 (k_1^2 - k_4^2)(k_2^2 - k_4^2)(k_3^2 - k_4^2) = 1,$$
$$(\Delta + k_l^2) h_j(\boldsymbol{x}) = \delta(\boldsymbol{x}) + (k_l^2 - k_j^2) h_j(\boldsymbol{x}), \qquad l, j = 1, 2, 3, 4, \qquad \boldsymbol{x} \in E^3,$$

from (4-19) we have

$$\begin{aligned}\Lambda_1(\Delta)\Lambda_2(\Delta) Y_{11}(\boldsymbol{x}) &= (\Delta + k_2^2)\Lambda_2(\Delta) \sum_{j=1}^{4} \eta_j \big(\delta(\boldsymbol{x}) + (k_1^2 - k_j^2) h_j(\boldsymbol{x})\big) \\ &= \Lambda_2(\Delta) \sum_{j=2}^{4} \eta_j (k_1^2 - k_j^2) \big(\delta(\boldsymbol{x}) + (k_2^2 - k_j^2) h_j(\boldsymbol{x})\big) \\ &= (\Delta + k_4^2) \sum_{j=3}^{4} \eta_j (k_1^2 - k_j^2)(k_2^2 - k_j^2) \big(\delta(\boldsymbol{x}) + (k_3^2 - k_j^2) h_j(\boldsymbol{x})\big) \\ &= (\Delta + k_4^2) h_4(\boldsymbol{x}) = \delta(\boldsymbol{x}). \end{aligned}$$ □

We introduce the matrix

$$\boldsymbol{\Psi}(\boldsymbol{x}) = \boldsymbol{L}(\boldsymbol{D_x}) \boldsymbol{Y}(\boldsymbol{x}). \tag{4-22}$$

Using identity (4-18) from (4-21) and (4-22) we obtain

$$\boldsymbol{R}(\boldsymbol{D_x})\boldsymbol{\Psi}(\boldsymbol{x}) = \boldsymbol{R}(\boldsymbol{D_x})\boldsymbol{L}(\boldsymbol{D_x})\boldsymbol{Y}(\boldsymbol{x}) = \tilde{\boldsymbol{\Lambda}}(\Delta)\boldsymbol{Y}(\boldsymbol{x}) = \delta(\boldsymbol{x})\boldsymbol{J}.$$

Hence, $\boldsymbol{\Psi}(\boldsymbol{x})$ is a solution to (4-2). We have thereby proved:

Theorem 4.2. *The matrix $\boldsymbol{\Psi}(x)$ defined by (4-22) is the fundamental solution of system (3-27).*

Obviously, the matrix $\boldsymbol{\Psi}(x)$ can be written in the form

$$\boldsymbol{\Psi} = (\Psi_{mn})_{6\times 6} = \begin{pmatrix} \boldsymbol{\Psi}^{(1)} & \boldsymbol{\Psi}^{(2)} \\ \boldsymbol{\Psi}^{(3)} & \boldsymbol{\Psi}^{(4)} \end{pmatrix}_{6\times 6},$$

where

$$\boldsymbol{\Psi}^{(j)}(\boldsymbol{x}) = \boldsymbol{L}^{(j)}(\boldsymbol{D_x}) Y_{11}(\boldsymbol{x}), \qquad j = 1, 2, 3, 4. \tag{4-23}$$

5. Basic properties of the matrix $\boldsymbol{\Psi}(\boldsymbol{x})$

Theorem 4.2 leads to the following results.

Corollary 5.1. *Each column of the matrix $\boldsymbol{\Psi}(\boldsymbol{x})$ is the solution of system (3-27) at every point $\boldsymbol{x} \in E^3$ except the origin.*

Corollary 5.2. *The fundamental solution of the system*

$$
\begin{aligned}
&a_1\Delta \boldsymbol{w}^{(1)}+a_2\nabla \operatorname{div} \boldsymbol{w}^{(1)}+a_3\Delta \boldsymbol{w}^{(2)}+a_4\nabla \operatorname{div} \boldsymbol{w}^{(2)}=\mathbf{0},\\
&a_5\Delta \boldsymbol{w}^{(1)}+a_6\nabla \operatorname{div} \boldsymbol{w}^{(1)}+a_7\Delta \boldsymbol{w}^{(2)}+a_8\nabla \operatorname{div} \boldsymbol{w}^{(2)}=\mathbf{0},
\end{aligned}
\tag{5-1}
$$

is the matrix

$$
\tilde{\boldsymbol{\Psi}}=(\tilde{\Psi}_{mn})_{6\times 6}=\begin{pmatrix}\tilde{\boldsymbol{\Psi}}^{(1)} & \tilde{\boldsymbol{\Psi}}^{(2)}\\ \tilde{\boldsymbol{\Psi}}^{(3)} & \tilde{\boldsymbol{\Psi}}^{(4)}\end{pmatrix}_{6\times 6},
$$

where

$$
\begin{aligned}
\tilde{\boldsymbol{\Psi}}^{(p)}&=(\tilde{\Psi}^{(p)}_{lj})_{3\times 3},\\
\tilde{\Psi}^{(1)}_{lj}&=\left(\frac{a_7+a_8}{d_1}\frac{\partial^2}{\partial x_l\partial x_j}-\frac{a_7}{d_2}\left(\frac{\partial^2}{\partial x_l\partial x_j}-\Delta\delta_{lj}\right)\right)h_0(\boldsymbol{x}),\\
\tilde{\Psi}^{(2)}_{lj}&=\left(-\frac{a_3+a_4}{d_1}\frac{\partial^2}{\partial x_l\partial x_j}+\frac{a_3}{d_2}\left(\frac{\partial^2}{\partial x_l\partial x_j}-\Delta\delta_{lj}\right)\right)h_0(\boldsymbol{x}),\\
\tilde{\Psi}^{(3)}_{lj}&=\left(-\frac{a_5+a_6}{d_1}\frac{\partial^2}{\partial x_l\partial x_j}+\frac{a_5}{d_2}\left(\frac{\partial^2}{\partial x_l\partial x_j}-\Delta\delta_{lj}\right)\right)h_0(\boldsymbol{x}),\\
\tilde{\Psi}^{(4)}_{lj}&=\left(\frac{a_1+a_2}{d_1}\frac{\partial^2}{\partial x_l\partial x_j}-\frac{a_1}{d_2}\left(\frac{\partial^2}{\partial x_l\partial x_j}-\Delta\delta_{lj}\right)\right)h_0(\boldsymbol{x}),\\
h_0(\boldsymbol{x})&=-\frac{|\boldsymbol{x}|}{8\pi},\qquad l,j=1,2,3,\qquad p=1,2,3,4.
\end{aligned}
\tag{5-2}
$$

Obviously, the relations

$$
\tilde{\Psi}_{mn}(\boldsymbol{x})=O(|\boldsymbol{x}|^{-1}),
\tag{5-3}
$$

and

$$
\frac{\partial^q}{\partial x_1^{q_1}\partial x_2^{q_2}\partial x_3^{q_3}}\tilde{\Psi}_{mn}(\boldsymbol{x})=O(|\boldsymbol{x}|^{-1-q}),
$$

hold in a neighborhood of the origin, where $m,n=1,2,\dots,6,\ q=q_1+q_2+q_3,\ q\geq 1$.

In what follows we shall use the following lemma.

Lemma 5.3. *If condition (4-1) is satisfied, then*

$$
\Delta n_{lj}(\Delta)=\frac{1}{d_1}Q'_{lj}(\Delta)\Lambda_2(\Delta)-\frac{1}{d_2}Z'_{lj}(\Delta)\Lambda_1(\Delta),\qquad l,j=1,2.
\tag{5-4}
$$

Proof. In view of (4-14), we have

$$
d_1d_2\,\Delta\ n_{lj}(\Delta)=-\Delta\big(Z'_{1j}(\Delta)(a_2Q'_{l1}(\Delta)+a_6Q'_{l2}(\Delta))+Z'_{2j}(\Delta)(a_4Q'_{l1}(\Delta)+a_8Q'_{l2}(\Delta))\big).
\tag{5-5}
$$

Taking into account the equalities

$$a_2\Delta Q'_{l1}+a_6\Delta Q'_{l2}=\delta_{1l}\, d_1\, \Lambda_1(\Delta)-((a_1\Delta+\xi_1)Q'_{l1}+(a_5\Delta+\xi)Q'_{l2}),$$
$$a_4\Delta Q'_{l1}+a_8\Delta Q'_{l2}=\delta_{2l}\, d_1\, \Lambda_1(\Delta)-((a_3\Delta+\xi bot)Q'_{l1}+(a_7\Delta+\xi_2)Q'_{l2}),$$
$$(a_1\Delta+\xi_1)Z'_{1j}+(a_3\Delta+\xi)Z'_{2j}=d_2\,\delta_{1j}\,\Lambda_2(\Delta),$$
$$(a_5\Delta+\xi)Z'_{1j}+(a_7\Delta+\xi_2)Z'_{2j}=d_2\,\delta_{2j}\,\Lambda_2(\Delta),$$

from (5-5) we obtain

$$d_1 d_2\Delta n_{lj}(\Delta)=$$
$$-d_1(\delta_{1l}Z'_{1j}+\delta_{2l}Z'_{2j})\Lambda_1(\Delta)+\big((a_1\Delta+\xi_1)Z'_{1j}+(a_3\Delta+\xi)Z'_{2j}\big)Q'_{l1}+\big((a_5\Delta+\xi)Z'_{1j}+(a_7\Delta+\xi_2)Z'_{2j}\big)Q'_{l2}$$
$$=-d_1 Z'_{lj}\Lambda_1(\Delta)+d_2(\delta_{1j}Q'_{l1}+\delta_{2j}Q'_{l2})\Lambda_2(\Delta)=d_2 Q'_{lj}\Lambda_2(\Delta)-d_1 Z'_{lj}\Lambda_1(\Delta). \quad \square$$

In what follows we use the notations

$$d_{pj}=\eta_{1j}\; Q'_{1p}(-k_j^2),\quad d_{p+2;j}=\eta_{1j}\; Q'_{2p}(-k_j^2),\quad d_{pl}=\eta_{1l}\; Z'_{1p}(-k_l^2),\quad d_{p+2;l}=\eta_{1l}\; Z'_{2p}(-k_l^2),$$
$$\eta_{1j}=\frac{(-1)^j}{d_1\,k_j^2\,(k_2^2-k_1^2)},\quad \eta_{1l}=\frac{(-1)^l}{d_1\,k_l^2\,(k_4^2-k_3^2)},\quad p,j=1,2,\quad l=3,4. \tag{5-6}$$

Theorem 5.4. *If $\boldsymbol{x}\in E^3\setminus\{0\}$, then*

$$\Psi^{(p)}_{mq}(\boldsymbol{x})=\frac{\partial^2}{\partial x_m\partial x_q}\sum_{j=1}^{2}d_{pj}\,h_j(\boldsymbol{x})-\Big(\frac{\partial^2}{\partial x_m\partial x_q}-\Delta\,\delta_{mq}\Big)\sum_{l=3}^{4}d_{pl}\,h_l(\boldsymbol{x}), \tag{5-7}$$
$$m,q=1,2,3,\quad p=1,2,3,4.$$

Proof. On the basis of (4-17), (4-19), (5-4) and equality

$$h_j(\boldsymbol{x})=-\frac{1}{k_j^2}\Delta h_j(\boldsymbol{x}),\quad \boldsymbol{x}\in E^3\setminus\{0\},\quad j=1,2,3,4,$$

from (4-23) we obtain

$$\Psi^{(1)}_{mq}(\boldsymbol{x})=\Big(\frac{1}{d_2}Z'_{11}(\Delta)\Lambda_1(\Delta)\delta_{mq}+n_{11}(\Delta)\,\frac{\partial^2}{\partial x_m\partial x_q}\Big)Y_{11}(\boldsymbol{x})$$
$$=-\sum_{j=1}^{4}\frac{\eta_j}{k_j^2}\Big(\frac{1}{d_2}\,Z'_{11}(\Delta)\Lambda_1(\Delta)+\Delta n_{11}(\Delta)\Big)\frac{\partial^2}{\partial x_m\partial x_q}\,h_j(\boldsymbol{x})$$
$$+\sum_{j=1}^{4}\frac{\eta_j}{d_2\,k_j^2}\,Z'_{11}(\Delta)\Lambda_1(\Delta)\Big(\frac{\partial^2}{\partial x_m\partial x_q}-\Delta\,\delta_{mq}\Big)\,h_j(\boldsymbol{x})$$
$$=-\frac{\partial^2}{\partial x_m\partial x_q}\sum_{j=1}^{4}\frac{\eta_j}{d_1\,k_j^2}\,Q'_{11}(-k_j^2)\Lambda_2(-k_j^2)\,h_j(\boldsymbol{x})$$
$$+\Big(\frac{\partial^2}{\partial x_m\partial x_q}-\Delta\,\delta_{mq}\Big)\sum_{j=1}^{4}\frac{\eta_j}{d_2\,k_j^2}\,Z'_{11}(-k_j^2)\Lambda_1(-k_j^2)\,h_j(\boldsymbol{x}). \tag{5-8}$$

Using the identities (5-6) and the relations

$$\eta_j\,\Lambda_1(-k_j^2)=\begin{cases}0, & j=1,2,\\ \frac{(-1)^j}{k_3^2-k_4^2}, & j=3,4,\end{cases}\qquad \eta_j\,\Lambda_2(-k_j^2)=\begin{cases}\frac{(-1)^j}{k_1^2-k_2^2}, & j=1,2,\\ 0, & j=3,4,\end{cases}$$

from (5-8) we have

$$\begin{aligned}\Psi_{mq}^{(1)}(\boldsymbol{x})&=\frac{\partial^2}{\partial x_m\partial x_q}\sum_{j=1}^{2}\eta_{1j}\,Q_{11}'(-k_j^2)\,h_j(\boldsymbol{x})-\Big(\frac{\partial^2}{\partial x_m\partial x_q}-\Delta\,\delta_{mq}\Big)\sum_{l=3}^{4}\eta_{1l}\,Z_{11}'(-k_l^2)\,h_l(\boldsymbol{x})\\ &=\frac{\partial^2}{\partial x_m\partial x_q}\sum_{j=1}^{2}d_{1j}\,h_j(\boldsymbol{x})-\Big(\frac{\partial^2}{\partial x_m\partial x_q}-\Delta\,\delta_{mq}\Big)\sum_{l=3}^{4}d_{1l}\,h_l(\boldsymbol{x}).\end{aligned}$$

The other formulae of (5-7) can be proven quite similarly. □

Theorem 5.4 leads to the following result.

Corollary 5.5. *If $\boldsymbol{x}\in E^3\backslash\{0\}$, then each element Ψ_{mn} of the matrix $\boldsymbol{\Psi}(\boldsymbol{x})$ has the form*

$$\Psi_{mn}(\boldsymbol{x})=\sum_{j=1}^{4}\Psi_{mnj}(\boldsymbol{x}),$$

where Ψ_{mnj} satisfies the condition

$$(\Delta+k_j^2)\Psi_{mnj}(\boldsymbol{x})=0,\qquad m,n=1,2,\dots,6,\qquad j=1,2,3,4.$$

Theorem 5.6. *The relations*

$$\Psi_{mn}(\boldsymbol{x})-\tilde{\Psi}_{mn}(\boldsymbol{x})=\text{const}+O(|\boldsymbol{x}|),\qquad \frac{\partial^q}{\partial x_1^{q_1}\partial x_2^{q_2}\partial x_3^{q_3}}(\Psi_{mn}(\boldsymbol{x})-\tilde{\Psi}_{mn}(\boldsymbol{x}))=O(|\boldsymbol{x}|^{1-q}),\tag{5-9}$$

and

$$\Psi_{mn}(\boldsymbol{x})=O(|\boldsymbol{x}|^{-1}),\tag{5-10}$$

hold in a neighborhood of the origin, where $m,n=1,2,\dots,6$, $q=q_1+q_2+q_3$, $q\ge 1$.

Proof. In view of (5-2) and (5-7) we obtain

$$\Psi_{mq}^{(1)}(\boldsymbol{x})-\tilde{\Psi}_{mq}^{(1)}(\boldsymbol{x})=$$

$$\frac{\partial^2}{\partial x_m\partial x_q}\Big(\sum_{j=1}^{2}d_{1j}\,h_j(\boldsymbol{x})-\frac{a_7+a_8}{d_1}\,h_0(\boldsymbol{x})\Big)-\Big(\frac{\partial^2}{\partial x_m\partial x_q}-\Delta\,\delta_{mq}\Big)\Big(\sum_{l=3}^{4}d_{1l}h_l(\boldsymbol{x})-\frac{a_7}{d_2}h_0(\boldsymbol{x})\Big).\tag{5-11}$$

In a neighborhood of the origin, from (4-20) we have

$$h_p(\boldsymbol{x})=-\frac{1}{4\pi\,|\boldsymbol{x}|}\sum_{n=0}^{\infty}\frac{(ik_p|\boldsymbol{x}|)^n}{n!}=h'(\boldsymbol{x})-\frac{ik_p}{4\pi}-k_p^2h_0(\boldsymbol{x})+\tilde{h}_p(\boldsymbol{x}),\tag{5-12}$$

where

$$h'(\boldsymbol{x})=-\frac{1}{4\pi\,|\boldsymbol{x}|},\qquad \tilde{h}_p(\boldsymbol{x})=-\frac{1}{4\pi\,|\boldsymbol{x}|}\sum_{n=3}^{\infty}\frac{(ik_p|\boldsymbol{x}|)^n}{n!},\qquad p=1,2,3,4.$$

Obviously,

$$\tilde{h}_p(\boldsymbol{x}) = O(|\boldsymbol{x}|^2), \qquad \frac{\partial}{\partial x_l}\tilde{h}_p(\boldsymbol{x}) = O(|\boldsymbol{x}|), \qquad \frac{\partial^2}{\partial x_l \partial x_j}\tilde{h}_p(\boldsymbol{x}) = \text{const} + O(|\boldsymbol{x}|),$$
$$l, j = 1, 2, 3, \qquad p = 1, 2, 3, 4. \tag{5-13}$$

On the basis of (5-12) we obtain

$$\sum_{j=1}^{2} d_{1j}\, h_j(\boldsymbol{x}) - \frac{a_7 + a_8}{d_1}\, h_0(\boldsymbol{x})$$
$$= \sum_{j=1}^{2} d_{1j} h'(\boldsymbol{x}) - \Big(\sum_{j=1}^{2} d_{1j} k_j^2 + \frac{a_7 + a_8}{d_1}\Big) h_0(x) + \sum_{j=1}^{2} d_{1j}\Big(-\frac{i k_j}{4\pi} + \tilde{h}_j(\boldsymbol{x})\Big),$$
$$\sum_{l=3}^{4} d_{1l}\, h_l(\boldsymbol{x}) - \frac{a_7}{d_2}\, h_0(\boldsymbol{x}) = \sum_{l=3}^{4} d_{1l} h'(\boldsymbol{x}) - \Big(\sum_{l=3}^{4} d_{1l} k_l^2 + \frac{a_7}{d_2}\Big) h_0(x) + \sum_{l=3}^{4} d_{1l}\Big(-\frac{i k_l}{4\pi} + \tilde{h}_l(\boldsymbol{x})\Big). \tag{5-14}$$

Taking into account the equalities (5-14) and

$$\Delta h'(\boldsymbol{x}) = 0 \quad \text{for } \boldsymbol{x} \neq 0, \qquad \sum_{j=1}^{2} d_{1j} = \sum_{l=3}^{4} d_{1l}, \qquad \sum_{j=1}^{2} d_{1j} k_j^2 + \frac{a_7 + a_8}{d_1} = 0, \qquad \sum_{l=3}^{4} d_{1l} k_l^2 + \frac{a_7}{d_2} = 0,$$

from (5-11) we have

$$\Psi_{mq}^{(1)}(\boldsymbol{x}) - \tilde{\Psi}_{mq}^{(1)}(\boldsymbol{x}) = \frac{\partial^2}{\partial x_m \partial x_q} \sum_{j=1}^{2} d_{1j}\big(h'(\boldsymbol{x}) + \tilde{h}_j(\boldsymbol{x})\big) - \Big(\frac{\partial^2}{\partial x_m \partial x_q} - \Delta\,\delta_{mq}\Big) \sum_{l=3}^{4} d_{1l}\big(h'(\boldsymbol{x}) + \tilde{h}_l(\boldsymbol{x})\big)$$
$$= \frac{\partial^2}{\partial x_m \partial x_q} \sum_{j=1}^{2} d_{1j} \tilde{h}_j(\boldsymbol{x}) - \Big(\frac{\partial^2}{\partial x_m \partial x_q} - \Delta\,\delta_{mq}\Big) \sum_{l=3}^{4} d_{1l} \tilde{h}_l(\boldsymbol{x}).$$

In view of (5-13), we obtain from this the relation $(5\text{-}9)_1$, for $m, n = 1, 2, 3$. The other formulae of (5-9) can be proven in a similar manner.

The relation (5-10) can be obtained easily from $(5\text{-}9)_1$ and (5-3). □

Thus, the fundamental solution $\tilde{\boldsymbol{\Psi}}(\boldsymbol{x})$ of the system (5-1) is the singular part of the matrix $\boldsymbol{\Psi}(\boldsymbol{x})$ in a neighborhood of the origin.

6. Concluding remark

The fundamental solution $\boldsymbol{\Psi}(\boldsymbol{x})$ of the system (3-27) makes it possible to investigate three-dimensional boundary value problems of the linear theory of viscoelastic binary mixtures with the boundary integral method (potential method). The main results obtained in the classical theory of elasticity, thermoelasticity and micropolar theory of elasticity with the potential method are given in [Kupradze et al. 1979]. A wide class of boundary value problems of steady vibration of the linear theory of thermoelasticity of binary mixtures is investigated using the potential method by Burchuladze and Svanadze [2000].

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