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AND THE DYNAMICS OF MATERIALS

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The explicit dependence of material properties on the basic space-time parametrization of continuum mechanics is a most interesting departure from standard behavior. The best framework to formally place these effects in evidence is the so-called material mechanics of materials advocated by the late G. Herrmann. Among the manifestations of these dependencies, material inhomogeneity is the most obvious and frequent one, but others such as volumetric growth or ageing also occur naturally, not to speak of artificial situations where one tries to build dynamic materials whose parts purposefully react differently in space and time at the will of the designer. These different cases are briefly sketched out first on the basis of a Lagrangian formulation and then in the general framework of thermomechanics.

1. Introduction

Long before the works of Kuhn [1962], the French epistemologist Bachelard [1940] wrote a marvelous little book introducing the notion of epistemological fracture (the equivalent of the change of scientific paradigm of Kuhn). Bachelard introduced the notion that many marked scientific progresses are made through the *negation* of previous theories; at least this is what appears in the language used and also in the definition of the object of study. He wrote of the historical introduction of non-Euclidean geometry and of non-Newtonian physics. We could as well speak of non-Riemannian geometry (used in generalizations of Einstein's general relativity theory and the theory of crystal defects which have been of interest to us) and maybe of the more popularized nonlinear sciences. This is more to the point because nonlinearity can only be defined mathematically by a negation of linearity [Maugin et al. 1992, Ch. 1]. The negation or deprivation of a property in science is marked by different prefixes, depending mostly on the predominant influence of Greek or Latin: thus "*non*linear", "*an*isotropic", "*in*homogeneous", "*hetero*geneous", to cite only examples from the mechanics of materials. Sometimes more sophisticated expressions had to be found.

Most of the nineteenth century's phenomenological physics (heat conduction, electromagnetics, early continuum mechanics) is based on the consideration of linear, isotropic, homogeneous, time-independent properties. It is that relative simplicity which allowed for rapid developments thanks to the allied relative mathematical simplicity of associated problems. The success of some mathematical techniques (such as Fourier series and integral transforms) is directly associated to their adequacy in treating these problems. But some "non" properties were soon to appear especially in optics and crystallography where anisotropy plays a fundamental role. With such considerations new mathematical tools had to be developed such as tensor algebra (W. Voigt in crystallography) and then analysis (such as in general relativity), while nonlinearity largely forced the development of perturbation theory and other techniques. In modern

continuum thermomechanics which is our current interest, canons of a somewhat standardized mathe-matically formal presentation were fixed in the second part of the twentieth century. This resulted in the blossoming of treatises on continuum mechanics, a phenomenon best illustrated by the justified success met by the treatises of [Truesdell and Toupin 1960; Truesdell and Noll 1965], the textbooks of Eringen [1962; 1967], and many more, often influenced by the books of these pioneers. Many of the material properties and accompanying "non" properties were defined more precisely in these treatises, and these definitions have been duly accepted and copied by the whole profession. However, even in these very inclusive treatises the "non" properties are only paid lip-service except for nonlinearity (necessary in the framework of finite strains) and anisotropy (in order to consider the mechanics of crystals or of some fiber-reinforced materials). But the three phenomena heralded in the title of this contribution have taken a much greater importance in recent years with the necessary consideration of different types of material inhomogeneity (as in polycrystals, layered composites, and graded materials), of materials newly examined from the mechanical viewpoint (biological materials), and of an interest in new properties (as in ageing). Remarkably enough, all these properties are related to the space-time parametrization of the mechanical (and more generally physical) properties of materials. In a nutshell, most properties were considered independent of this parametrization until very recently, while now we are forced to look into this dependency. This, of course, is a matter of scale of observation (in space and time), but the space and time scales of interest have evolved rendering this matter of foremost interest.

The mathematical apparatus that deals directly with the invariance or lack of invariance with respect to changes of the space-time parametrization is the celebrated theorem of Noether [1918] in field theory. Accordingly, we consider below as a constructive example a simple variational formulation for elastic-like materials in finite strains. This has recently been exploited in a rather new field of research in continuum mechanics, called Eshelbian mechanics or configurational mechanics or mechanics on the material manifold [Noether 1918; Maugin 1993; 1995]. This approach, together with Noether's theorem, yields a canonical thermomechanics of materials where so-called canonical momentum and energy are treated in parallel, corresponding, respectively, to the changes of material coordinates and time. When dissipation is present and a standard variational formulation is no longer feasible, we shall then see how to reformulate a consistent theory. Finally, a glimpse at the future is reported.

The late George Herrmann was much interested in wave propagation in inhomogeneous (layered) structures. He also devoted much of his last research work to material and configurational forces with applications to the strength of materials; see, for example, [Kienzler and Herrmann 2000]. Thus it is natural that the present paper be dedicated to his memory.[1]

2. A sufficiently general variational formulation

We are concerned with simple general features of field theories in a continuum with space-time paramet-rization $\{X, t\}$, where X stands for material coordinates of classical continuum mechanics [Truesdell and Toupin 1960], and t for a timelike scalar variable (Newton's absolute time). We consider first Hamiltonian

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[1]In fact, the material herein was presented as the Introductory Lecture at the *International Symposium on Defect Mechanics in Materials*, ISDMM, Aussois, France, March 2007, also dedicated to G. Herrmann. But the paper was not ready in time for the proceedings of that conference, edited by C. Dascalu and the author, to appear in 2008 as a special issue of the *International Journal of Fracture*.

actions of the type

$$A(\phi; V) = \int_{V \times I} L(\phi^\alpha, \partial_\mu \phi^\alpha; X^\mu) d^4 X, \tag{2-1}$$

where ϕ^α, $\alpha = 1, 2, \ldots, N$, denotes the ordered array of fields, say the independent components of a certain geometric object, and $d^4 X = dV \, dt$. This is a Cartesian-Newtonian notation, with

$$\{\partial_\mu = \partial/\partial X^\mu; \quad \mu = 1, 2, 3, 4\} = \{\partial/\partial X^K, \quad K = 1, 2, 3; \ \partial/\partial X^4 \equiv \partial/\partial t\}, \tag{2-2}$$

since

$$\{X^\mu, \quad \mu = 1, 2, 3, 4\} = \{X = (X^K, \quad K = 1, 2, 3), \quad X^4 = t\}.$$

The summation over dummy indices (Einstein convention) is enforced. We do not pay attention to boundary and initial conditions and impose no external sources. From expression (2-1) two types of equations can be derived: There are those relating to each one of the fields ϕ^α — these are the field equations — and those which express a general conservation law of the system governing all fields simultaneously. The first group is none other than the set of Euler-Lagrange variational equations

$$E_\alpha \equiv \frac{\delta L}{\delta \phi^\alpha} := \frac{\partial L}{\partial \phi^\alpha} - \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial (\dot{\phi}^\alpha)} \right) - \frac{\partial}{\partial X^K} \left(\frac{\partial L}{\partial (\partial \phi^\alpha / \partial X^K)} \right) = 0, \tag{2-3}$$

for each $\alpha = 1, 2, \ldots, N$, at any regular material point X and for any time t.

The second group of equations are the result of the variation of the parametrization, and these results, on account of the former group, express the invariance or lack of invariance of the whole system under changes of this parametrization. The resulting equations are nothing but the expression of the celebrated Noether's theorem [1918], which states that to any symmetry of the system there corresponds the conservation (or lack of strict conservation) of a current [Soper 1976]. Here, with respect to time and space-like parametrization we have the following two equations, a scalar conservation law in the explicit form

$$\frac{\partial H}{\partial t}\bigg|_X - \nabla_R \cdot \boldsymbol{Q} = h, \tag{2-4}$$

and a co-vectorial conservation (material) law

$$\frac{\partial \boldsymbol{P}}{\partial t}\bigg|_X - \operatorname{div}_R \boldsymbol{b} = \boldsymbol{f}^{\text{inh}}, \tag{2-5}$$

where we defined the following canonical quantities:

- Energy (Hamiltonian density)

$$H := \sum_\alpha \dot{\phi}^\alpha (\partial L / \partial \dot{\phi}^\alpha) - L, \qquad \dot{\phi}^\alpha \equiv \partial \phi^\alpha / \partial t; \tag{2-6}$$

- Energy flux vector

$$\boldsymbol{Q} = \left\{ Q^K := -\sum_\alpha \dot{\phi}^\alpha \frac{\partial L}{\partial (\partial_K \phi^\alpha)} \right\}; \tag{2-7}$$

- Canonical (material) momentum

$$\boldsymbol{P} = \left\{ P_K := -\sum_\alpha \frac{\partial \phi^\alpha}{\partial X^K} \frac{\partial L}{\partial (\partial \phi^\alpha / \partial t)} \right\}; \tag{2-8}$$

- Canonical stress tensor

$$
\boldsymbol{b} = \left\{ b^{K}_{\cdot L} := -\left(L\delta^{K}_{\cdot L} - \sum_{\alpha} \frac{\partial \phi^{\alpha}}{\partial X^{L}} \frac{\partial L}{\partial(\partial\phi^{\alpha}/\partial X^{K})} \right) \right\},
\tag{2-9}
$$

with source terms

$$
h := -\left. \frac{\partial L}{\partial t} \right|_{\mathrm{expl}}, \qquad \boldsymbol{f}^{\mathrm{inh}} := \left. \frac{\partial L}{\partial \boldsymbol{X}} \right|_{\mathrm{expl}},
\tag{2-10}
$$

where the notation expl means that the derivatives are taken at fixed fields ϕ^{α}.

Definitions of various systems. Physical systems in which h does not vanish are said to be rheonomic (after Boltzmann cited by [Lanczos 1962, p.32]). Systems in which the material force $\boldsymbol{f}^{\mathrm{inh}}$ does not vanish are said to be materially inhomogeneous. This is best illustrated by the case of hyperelasticity.

Other symmetries, and thus other consequences of Noether's theorem can be applied — such as for rotations [Maugin 1993, Section 5.2–5.5] — but these are not discussed further here as they will not be exploited. More is to be found on Noether's theorem and its relationship to the theory of Lie groups in [Olver 1986]. The above is quite sufficient for our purpose.

The case of hyperelasticity. This corresponds to a trivial and isomorphic application of the above given formulae with a special choice of the fields ϕ^{α}. They are the physical components (components in physical space at time t) of the placement of the material particle $\boldsymbol{X}$; that is

$$
\boldsymbol{x} = \bar{\boldsymbol{x}}(\boldsymbol{X}, t).
\tag{2-11}
$$

The physical velocity of $\boldsymbol{X}$ and the deformation gradient are defined by

$$
\boldsymbol{v} := \left. \frac{\partial \bar{\boldsymbol{x}}}{\partial t} \right|_{X}, \qquad \boldsymbol{F} := \left. \frac{\partial \bar{\boldsymbol{x}}}{\partial \boldsymbol{X}} \right|_{t} \equiv \nabla_{R} \bar{\boldsymbol{x}}.
\tag{2-12}
$$

For a first-order gradient theory of elasticity, the Lagrangian density L per unit volume in the reference configuration of the continuous body reads

$$
L = \bar{L}(\boldsymbol{v}, \boldsymbol{F}; \boldsymbol{X}, t) = K(\boldsymbol{v}; \boldsymbol{X}, t) - W(\boldsymbol{F}; \boldsymbol{X}, t),
\tag{2-13}
$$

where W is the elasticity potential and K is the kinetic energy given in a general manner by

$$
K = \tfrac{1}{2}\rho_{0}(\boldsymbol{X}, t)\boldsymbol{v}^{2}.
\tag{2-14}
$$

Accordingly, Equations (2-3) through (2-5) render the balance of linear physical momentum (three components of the field $\boldsymbol{x}$ in physical space) as

$$
\left. \frac{\partial \boldsymbol{p}}{\partial t} \right|_{X} - \operatorname{div}_{R} \boldsymbol{T} = \boldsymbol{0},
\tag{2-15}
$$

and the balance of energy (a scalar equation), and the balance of material momentum (a covectorial equation on the material manifold) in the form of Equations (2-4) and (2-5), with the following expressions

for the quantities involved:

$$\boldsymbol{p} := \rho_0 \boldsymbol{v}, \qquad\qquad \boldsymbol{T} := \frac{\partial W}{\partial \boldsymbol{F}}, \qquad\qquad (2\text{-}16)$$

$$H = K + W, \qquad\qquad \boldsymbol{Q} = \boldsymbol{T} \cdot \boldsymbol{v}, \qquad\qquad (2\text{-}17)$$

$$\boldsymbol{P} := -\rho_0 \boldsymbol{v} \cdot \boldsymbol{F}, \qquad\qquad \boldsymbol{b} = -(L\mathbf{1}_R + \boldsymbol{T} \cdot \boldsymbol{F}), \qquad\qquad (2\text{-}18)$$

$$h := -(K/\rho_0)\frac{\partial \rho_0}{\partial t}\Big|_X + \frac{\partial W}{\partial t}\Big|_{\text{expl}}, \qquad\qquad (2\text{-}19)$$

$$\boldsymbol{f}^{\text{inh}} := (K/\rho_0)\frac{\partial \rho_0}{\partial \boldsymbol{X}}\Big|_t - \frac{\partial W}{\partial \boldsymbol{X}}\Big|_{\text{expl}}. \qquad\qquad (2\text{-}20)$$

These are, respectively, the physical linear momentum $\boldsymbol{p}$, the first Piola-Kirchhoff stress $\boldsymbol{T}$, the Hamiltonian (energy) density H, the energy flux or Poynting vector $\boldsymbol{Q}$, the material momentum $\boldsymbol{P}$, the Eshelby material stress $\boldsymbol{b}$ (see [Maugin and Trimarco 1992], who gave this name to what Eshelby himself used to refer to as the energy-momentum tensor or Maxwell stress), the energy source h in a rheonomic system, and the material inhomogeneity force $\boldsymbol{f}^{\text{inh}}$ [Maugin 1993; 1995]. Rheonomic systems are rarely studied but in such systems energy is not conserved. (2-4) becomes a strict conservation law (no source), only when the system considered is scleronomic (that is, with no explicit time dependence of the Lagrangan density, according to the classification of Boltzmann and Lanczos). The possible explicit time dependency comes through the matter density in the kinetic energy and through the elasticity potential. This is seldom envisioned. But the possibility that the reference material density be an explicit function of time is nonetheless the basic working hypothesis in the theory of material growth (as in biological tissues [Epstein and Maugin 2000]), according to which

$$\frac{\partial \rho_0}{\partial t}\Big|_X \neq 0. \qquad\qquad (2\text{-}21)$$

The selection of the word "growth" and the sign of the right-hand side in (2-21) are arbitrary since the inverse phenomenon, resorption, with the opposite sign is not forbidden.

The possibility that the elasticity potential W might depend explicitly on time,

$$\frac{\partial W}{\partial t}\Big|_{\text{expl}} \neq 0, \qquad\qquad (2\text{-}22)$$

would refer to the phenomenon of ageing (evolution in time of material elasticity coefficients), a domain of research that remains practically unexplored probably due to the difficulty to measure experimentally the time evolution of interest. Note that the selection of the word ageing indicates an arrow of time since the opposite phenomenon, rejuvenation, is certainly forbidden by the second law of thermodynamics.

Together with the two conditions

$$\frac{\partial \rho_O}{\partial \boldsymbol{X}}\Big|_t \neq 0, \qquad \frac{\partial W}{\partial \boldsymbol{X}}\Big|_{\text{expl}} \neq 0, \qquad\qquad (2\text{-}23)$$

relating to the two forms of material inhomogeneities, inertial ones — these appear only while dealing with problems of dynamics — and (here) elastic ones, Equations (2-21) and (2-22) define the "non" properties in which we are interested. However, in contrast to conditions (2-23), which bear no relationship

to irreversible thermodynamics, conditions (2-21) and (2-22) are related to it since they result in a non-conservation of energy. As a result, the nonspecified right-hand sides in these conditions are subjected the second law while those in Equations (2-23) are data in a specific problem.

Of course, the vast majority of elastic materials considered until now correspond to scleronomic systems. However, the frequent dependency of both ρ_0 and W on the material point X in materially inhomogeneous solids is an evidence, and its manifestation as a source term in (2-5) provides the basis for the theory of all types of material forces in solids. As a conclusion of this point, we note that hyperelasticity clearly is a paradigmatic theory of fields.

Now, the dynamic elasticity encapsulated in a Lagrangian such as (2-13) may be fully nonlinear as regards deformation processes, but it does not contain any characteristic length, except for inhomogeneous materials where the material spatial variation of the material properties (density, elasticity coefficients) may provide such a length. Homogeneous elasticity exhibiting a characteristic intrinsic length of necessity involves gradients of higher order than the first, e.g, the second material gradient of the placement or, just the same, the material gradient of the deformation gradient. That is, we may have to consider a generalization of (2-13) of the type

$$L = \bar{L}(v, F, \nabla_R F; X, t) = K(v; X, t) - W(F, \nabla_R F; X, t), \tag{2-24}$$

where the inertial contribution is left unchanged. In this case more lengthy representations than (2-7) and (2-9) have to be used.

3. Dissipation: elementary approach

The variational approach in Section 2, paradigmatic as it is, is not equipped to deal with true dissipative processes. However a simple argument shows the way for remedying this. We would have energy conservation if the right-hand side of (2-4) would vanish. Without introducing any thermal processes, this could be accomplished by adding from the outside a term that makes this right-hand side vanish. This will be a dissipation rate Φ. Thus we would have

$$\Phi = -\left(-\frac{\partial L}{\partial t}\bigg|_{\text{expl}}\right) = \Phi_{\text{growth}} + \Phi_{\text{age}}, \tag{3-1}$$

with (2-19)

$$\Phi_{\text{growth}} = K \Pi_0/\rho_0, \qquad \Phi_{\text{age}} = -\frac{\partial W}{\partial t}\bigg|_{\text{expl}}, \tag{3-2}$$

where Π_0 is the right-hand side of (2-21). This right-hand side will typically be of the form

$$\Pi_0 = R_0 + \nabla_R \cdot M \tag{3-3}$$

(see [Epstein and Maugin 2000]), where R_0 is an external supply — of, say, nutrients, via some biophysical process — and M is an influx of mass. Since K and ρ_0 are obviously positive, so must be Π_0 in order to guarantee a positive Φ_{growth}. As to Φ_{age} a simplistic case exemplifies the situation. We may assume that the explicit time dependence of W is only through material functions (with certain tensorial properties, a set of scalars being the simplest case) $\alpha(t)$. Then using a notation intentionally akin to that of the thermodynamics of irreversible processes with internal variables [Maugin 1999], we will set and

have

$$A := -\frac{\partial W}{\partial \alpha}, \qquad \Phi_{\text{age}} = A\dot{\alpha}, \tag{3-4}$$

where the appropriate inner product must be used in the last expression which is classically written as a bilinear form, the product of a thermodynamical force, A, and a time rate. The total expression must be non-negative according to the second law of thermodynamics. Consider a one-dimensional homogeneous elasticity in small strains ε such that $W(\varepsilon; t) = \frac{1}{2}\mu(t)\varepsilon^2$. Here α is a positive shear modulus, which may decrease exponentially with time: $\mu(t) = \mu_0[1 - \exp((t - t_0)/\tau)]$, with μ_0 and τ positive. Then $\Phi_{\text{age}} = \frac{1}{2}\mu(t_0)\varepsilon^2/\tau$, as shown by an elementary calculation. Of course, Φ_{age} (otherwise always positive) goes to zero for finite elastic energies as the characteristic ageing time τ increases indefinitely. This is but a rather naïve description of the phenomenon of creep. We shall return to this in further works.

4. Full irreversible thermodynamics

In that general case, like in the rest of continuum mechanics, we must start with a statement of global balance laws for a continuous simply connected material region B of the body, bounded by a sufficiently regular boundary ∂B equipped with unit outward normal N. We can thus state (omitting the balance of angular momentum):

- Balance of mass

$$\frac{d}{dt}\int_B \rho_0 \, dV = \int_B R_0 \, dV + \int_{\partial B} N \cdot M \, dA; \tag{4-1}$$

- Balance of linear (physical) momentum (in the absence of body forces)

$$\frac{d}{dt}\int_B p \, dV = \int_{\partial B} N \cdot T \, dV; \tag{4-2}$$

- First law of thermodynamics (E is the internal energy per unit reference volume; Q is the material heat flux; h_0 is a possible body heat source)

$$\frac{d}{dt}\int_B (K + E) \, dV = \int_B (h_0 - \tfrac{1}{2}v^2 R_0) \, dV + \int_{\partial B} (N \cdot T \cdot v - N \cdot Q - (\tfrac{1}{2}v^2)N \cdot M) dA; \tag{4-3}$$

- Second law of thermodynamics (S is the entropy per unit reference volume and θ is the non-negative thermodynamical temperature)

$$\frac{d}{dt}\int_B S \, dV \geq \int_B \theta^{-1}(h_0 - \tfrac{1}{2}v^2 R_0) \, dV - \int_{\partial B} \theta^{-1} N \cdot (Q + \tfrac{1}{2}v^2 M) dA. \tag{4-4}$$

In making these statements we have (i) implicitly assumed the necessity to include sources and fluxes of energy and entropy via energy mass source and transfer, and (ii) consistently written the sources of energy and entropy with the integrating factor θ^{-1}.

On account of the usual argument of continuity of the involved quantities, we immediately have the local form of Equations (4-1) through (4-4) at any regular material point X in B:

- Equation governing the mass density

$$\left.\frac{\partial \rho_0}{\partial t}\right|_X = R_0 + \nabla_R \cdot M; \tag{4-5}$$

- Balance of linear (physical) momentum

$$\left.\frac{\partial \boldsymbol{p}}{\partial t}\right|_X - \operatorname{div}_R \boldsymbol{T} = \boldsymbol{0}; \tag{4-6}$$

- First law of thermodynamics

$$\left.\frac{\partial}{\partial t}(K + E)\right|_X - \nabla_R \cdot (\boldsymbol{T} \cdot \boldsymbol{v} - \boldsymbol{Q} - \tfrac{1}{2}v^2\boldsymbol{M}) = h_0 - \tfrac{1}{2}v^2 R_0; \tag{4-7}$$

- Second law of thermodynamics

$$\left.\theta\frac{\partial S}{\partial t}\right|_X \geq h_0 - \tfrac{1}{2}v^2 R_0 - \nabla_R \cdot (\boldsymbol{Q} + \tfrac{1}{2}v^2\boldsymbol{M}) + \boldsymbol{S} \cdot \nabla_R \theta, \tag{4-8}$$

where

$$\boldsymbol{S} := \theta^{-1}(\boldsymbol{Q} + \tfrac{1}{2}v^2\boldsymbol{M}). \tag{4-9}$$

From (4-6) we obtain the kinetic-energy theorem by scalar product by $\boldsymbol{v}$. Accounting for the fact that a simple calculation allows one to show that

$$\left(\frac{\partial}{\partial t}\rho_0\boldsymbol{v}\right) \cdot \boldsymbol{v} = \left.\frac{\partial K}{\partial t}\right|_X + \left.\tfrac{1}{2}v^2\frac{\partial \rho_0}{\partial t}\right|_X, \tag{4-10}$$

we obtain

$$\left.\frac{\partial K}{\partial t}\right|_X + \left.\tfrac{1}{2}v^2\frac{\partial \rho_0}{\partial t}\right|_X - \nabla_R \cdot (\boldsymbol{T} \cdot \boldsymbol{v}) + \boldsymbol{T}{:}(\nabla_R\boldsymbol{v})^T = 0. \tag{4-11}$$

Substituting into (4-7) we have the internal energy equation

$$\left.\frac{\partial E}{\partial t}\right|_X + \nabla_R \cdot \boldsymbol{Q} - \boldsymbol{T}{:}\dot{\boldsymbol{F}} + \boldsymbol{M} \cdot \nabla_R(\tfrac{1}{2}v^2) = h_0, \tag{4-12}$$

since

$$(\nabla_R\boldsymbol{v})^T = \dot{\boldsymbol{F}} := \left.\frac{\partial}{\partial t}(\boldsymbol{F}(\boldsymbol{X}, t))\right|_X. \tag{4-13}$$

We now introduce the Helmholtz free energy W by

$$W = E - S\theta. \tag{4-14}$$

and note that since E, W, and S are constitutive functions, if the physical system is rheonomic, we must isolate the implicit partial time derivatives. Thus we shall write as an example

$$\left.\frac{\partial W}{\partial t}\right|_X = \left.\frac{\partial W}{\partial t}\right|_{\text{impl}} + \left.\frac{\partial W}{\partial t}\right|_{\text{expl}}, \tag{4-15}$$

with a similar decomposition for E and S. On taking the partial time derivative of (4-14) we have thus

$$\left.\frac{\partial E}{\partial t}\right|_X = \left.\frac{\partial W}{\partial t}\right|_{\text{impl}} + \left.\frac{\partial W}{\partial t}\right|_{\text{expl}} - \left.S\frac{\partial \theta}{\partial t}\right|_X - \left.\theta\frac{\partial S}{\partial t}\right|_X. \tag{4-16}$$

Combining (4-8), (4-1) and (4-16) we arrive at an expression of the second law of thermodynamics known as the Clausius-Duhem inequality, here in the remarkable form

$$-\left(\left.\frac{\partial W}{\partial t}\right|_{\text{impl}} + \left.S\frac{\partial \theta}{\partial t}\right|_X\right) + \boldsymbol{T}{:}\dot{\boldsymbol{F}} + \Phi_{\text{age}} + \Phi_{\text{growth}} - \boldsymbol{S} \cdot \nabla_R \theta \geq 0, \tag{4-17}$$

where the dissipation rates due to ageing and growth are defined just as in Equations (3-2)–(3-3). Note that neither anisotropy nor material inhomogeneity play any role in the above derivation. Herein above no constitutive equations have been assumed, even for the Piola-Kirchhoff stress $\boldsymbol{T}$. The example closest to the consideration of Sections 2 and 3 is that dealing with a materially inhomogeneous, rheonomic thermoelastic conductor of heat for which the free energy W may be considered a function

$$W = \bar{W}(\boldsymbol{F}, \theta; \boldsymbol{X}, t). \tag{4-18}$$

The usual argument to exploit (4-17) is not altered by the presence of the last two arguments in (4-18). Accordingly, we have the constitutive equations

$$\boldsymbol{T} = \frac{\partial \bar{W}}{\partial \boldsymbol{F}}, \qquad S = -\frac{\partial \bar{W}}{\partial \theta}, \tag{4-19}$$

while there remains the dissipation inequality

$$\Phi_{\text{age}} + \Phi_{\text{growth}} - \boldsymbol{S} \cdot \nabla_R \theta \geq 0. \tag{4-20}$$

It remains to establish the expression of the balance of material momentum in this case to compare to the developments of Section 2.

5. Balance of material momentum

In the presence of dissipation we know that this can be obtained only by parodying the Noether's identity of variational field theory [Maugin 2006a]. This is mainly achieved by taking the inner product of (4-6) by $\boldsymbol{F}$ to the right. We can try to go as far as possible without specifying the functional dependence of the energy density [Maugin 2006b], or work out the special case (4-18) for which we already know that thermal effects introduce a pseudoforce of material inhomogeneity as a contribution in the right-hand side of the canonical (4-5) [Epstein and Maugin 1995]. What about growth and ageing [Chudnovsky and Preston 1996]? Do they create such material pseudoforces? Although we could follow the general approach of [Maugin 2006b], we simply consider the case of the known form (4-18). Since the operations involved in this manipulation concern only space derivatives (material gradients) and no time derivatives, we can state that ageing will not modify the balance of material momentum, and we could state more generally that nothing would a priori be modified in quasi-statics. But inertia (and hence matter density) is involved in the full dynamics so that this is the only point to be examined more carefully. As a matter of fact, following a now-traditional computation we clearly have

$$\left(\frac{\partial}{\partial t}\rho_0 \boldsymbol{v}\right) \cdot \boldsymbol{F} = -\frac{\partial \boldsymbol{P}}{\partial t}\bigg|_X - \nabla_R \left(\rho_0 \frac{v^2}{2}\right) + \left(\frac{v^2}{2}\right) \nabla_R \rho_0, \quad \boldsymbol{P} := -\rho_0 \boldsymbol{v} \cdot \boldsymbol{F}. \tag{5-1}$$

This shows that there is no alteration brought to the classical derivation of the equation of balance of material momentum from the balance of physical momentum due to growth processes since the first term in the right-hand side of (5-1) is the classical term. Accordingly, in the case where (4-18) holds true, we would obtain in that manner the same formal expression as in classical (scleronomic) materially inhomogeneous thermoelasticity

$$\frac{\partial \boldsymbol{P}}{\partial t}\bigg|_X - \operatorname{div}_R \boldsymbol{b} = \boldsymbol{f}^{\text{inh}} + \boldsymbol{f}^{\text{th}}, \tag{5-2}$$

wherein

$$\boldsymbol{b} = -(L^{\text{th}}\mathbf{1}_R + \mathbf{TF}), \qquad \boldsymbol{f}^{\text{inh}} := \left.\frac{\partial L^{\text{th}}}{\partial \boldsymbol{X}}\right|_{\text{expl}}, \qquad \boldsymbol{f}^{\text{th}} := S\nabla_R\theta, \qquad L^{\text{th}} = K - \bar{W}(\boldsymbol{F}, \theta; \boldsymbol{X}, t).$$

So much for a short exploration of the works on ageing to be expanded later on.

6. A glimpse at the future of adaptive structures

So far, only natural objects have been envisaged in our considerations. But in these times of man-designed new materials and more particularly smart materials and adaptive structures, we may think of designing structures made of materials that are artificially rendered materially inhomogeneous and rheonomic and this at a scale that may suit certain exploitation at the microscopic and nanoscopic levels. Imagine, for instance that we have a material of which the material property such as elasticity may vary in time according to a certain time scale while, the mass distribution may vary in space in the way that we like. Because of the good choice of material coordinates and Newtonian time as space-time coordinates, an equation of motion such as (2-15), expressed componentwise as

$$\frac{\partial}{\partial t}\rho_0(\boldsymbol{X})v_i\Big|_X - \frac{\partial}{\partial X^K}T_{.i}^K\Big|_t = 0, \tag{6-1}$$

will yield an equation where the space-dependent density comes out of the time derivative in the first term and the purely time-dependent material property will come out of the material divergence in the second term. In the case of linear elasticity in small strains this will give in one-dimension a wave equation of the type

$$u_{tt} - c^2(x, t)u_{xx} = 0, \tag{6-2}$$

with a space-time dependent acoustic velocity. This can be realized in several ways in an essentially one-dimensional structure such as a rod with more or less smoothly varying cross section and local elasticity controlled in time by an external field such as an electromagnetic one. An equation such as (6-2) is mathematically linear (in the elastic displacement u), but the dynamical overall behavior of the resulting structure may be akin to some nonlinear structure as regards certain signals carried by the structure. The latter becomes thus a dynamical structure in the sense of [Lurie 2007], and from here we can dream of several interesting possibilities that we let the reader discover by himself. An allied problem appears in the form of homogenization in both space and time.

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