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Harry H. Hilton

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THE ELUSIVE AND FICKLE VISCOELASTIC POISSON'S RATIO AND ITS RELATION TO THE ELASTIC-VISCOELASTIC CORRESPONDENCE PRINCIPLE

HARRY H. HILTON

This paper is dedicated to my good friend and colleague Professor Emeritus Georges J. Simitsis.

The conditions for the applicability of the elastic-viscoelastic correspondence principle (analogy) in the presence of any of the five distinct classes of viscoelastic Poisson's ratios (PR) are investigated in detail. It is shown that if Poisson's ratios are time-dependent, no analogy in terms of PRs is possible, except for two of the classes under specifically prescribed highly limited conditions. Separately, the severely restrictive conditions involving time-independent PRs are discussed in detail. Failure to observe all such restrictions leads to ill posed overdeterminate problem formulations. Similarities associated with viscoelastic Timoshenko shear coefficients are also investigated and it is shown that no analogy to equivalent elastic problems can be constructed if these coefficients are time functions. In the final analysis, the PR analogy difficulties can be entirely avoided by characterizing viscoelastic materials in terms of relaxation moduli or creep compliances or creep and relaxation functions without any appeal to PRs.

Introduction

Unlike stress and deformation analyses where approximate solutions are permissible, material characterization must be performed with the highest available degree of precision since thusly defined constitutive relations pervasively impact all subsequent analyses. Consequently, great care must be exercised in modeling material and experimental data and no unnecessary approximations should be introduced[1]. A major case in point is the convenient, but fictitious, introduction of the approximation of time-independent viscoelastic PRs to "simplify" material characterization, but which as will be demonstrated results in overdeterminate ill posed problem formulations and thus leads to unreliable material characterizations as well as stress-strain solutions.

Historically, in elasticity Poisson's ratio [Poisson 1829] has found much justifiable favor in analysis and material characterization. When normal strains can be readily measured in two directions, elastic PRs become a most useful universal cornerstone of elastic property description along with shear, bulk and Young's moduli. Much of the PRs' success is due to their simple concept in elasticity, where constitutive relations between stresses and strains are algebraic, with neither energy dissipation nor time-dependent memory.

Keywords: Bernoulli–Euler beams, correspondence principle, material characterization, Poissonś ratio, Timoshenko shear coefficients, viscoelasticity.

[1] "Nothing is less real than realism. Details are confusing. It is only by selection, by elimination, by emphasis, that we get at the real meaning of things." —Georgia O'Keeffe

While linear elastic materials have been successfully characterized in terms of moduli and Poisson's ratios (PRs) for almost two centuries [Poisson 1829], the transition to viscoelastic PRs is far less simple than the well established equivalence between elastic moduli and viscoelastic relaxation functions/moduli [Hilton 1996; 2001; 2003; Hilton and Yi 1998; Tschoegl 1997; Tschoegl et al. 2002; Lakes and Wineman 2006; Hilton and El Fouly 2007; Shtark et al. 2007]. There are two overriding issues that need to be precisely and properly addressed when using viscoelastic PRs, namely (i) the time and stress dependencies of PRs and (ii) the inapplicability of the elastic-viscoelastic correspondence principle in terms of PRs. In question is the fundamental nature of PRs as a derived quantity in terms of ratios of perpendicular normal strains as opposed to "pure" material properties such as relaxation moduli, creep compliances, relaxation functions, etc. As such viscoelastic PRs are not universal and are specific to loading, deformation and temperature histories for each viscoelastic material.

On the other hand, in viscoelasticity with its time integral constitutive relations PRs become more complex functions dependent on time and stress histories [Hilton 1996; 2001; Hilton and Yi 1998; Tschoegl 1997; Tschoegl et al. 2002, Hilton and El Fouly 2007, Shtark et al. 2007] behaviorally similar to that of the time-dependent viscoelastic shear center [Hilton and Piechocki 1962]. Even in linear viscoelastic theory, PRs are process dependent nonlinear functions of strains and time and non-universal material properties, whereas linear relaxation and creep functions remain invariant with respect to loading histories. The viscoelastic time dependence has been demonstrated analytically [Hilton 1996; 2001; 2003; Hilton and Yi 1998; Tschoegl 1997; Tschoegl et al. 2002; Lakes and Wineman 2006; Hilton and El Fouly 2007] as well as experimentally [Shtark et al. 2007; Lakes 1991]. Auxetic viscoelastic materials, which have negative elastic PRs, have been treated in [Hilton and El Fouly 2007] where it shown that viscoelastic PRs do not follow the negative elastic patterns.

Consequently, time-independent classical PRs require states of stress and strain where each are defined as distinct temporally and spatially separable functions under inertialess conditions with no mixed boundary conditions. Under these conditions the uniquely admissible PR value is one half, the latter condition being restricted solely to incompressible and isotropic viscoelastic materials. Additionally, material characterization in terms of PRs excludes the applicability of any elastic-viscoelastic correspondence principle. The latter analogy can only be derived in terms of relaxation moduli and/or creep compliances and some very limited PR forms. Therefore, material characterization in terms of relaxation moduli (functions) and/or creep compliances, rather than PRs, remains the method of choice.

It must be remembered that isotropic viscoelastic material properties can only be properly determined through experiments involving simultaneous measurements of two-dimensional strains as summarized in [Hilton 2001] with some additional examples described in [Ravi-Chandar 1998; 2000; Lakes et al. 1979; Qvale and Ravi-Chandar 2004; Giovagnoni 1994; Mead and Joannides 1991; Sim and Kim 1990] or through x-ray evaluations [Hoke et al. 2001].

The original separation of variable analogy was formulated in [Alfrey 1944] and [Alfrey 1948] and the more general and inclusive Fourier transform formulation may be found in [Read 1950]. Viscoelasticity theory including the correspondence principle were place on a rational basis in [Lee 1955]. In [Hilton and Russell 1961] and [Hilton and Clements 1964] the analogy was extended to cover temperature dependent viscoelastic material properties, while in [Hilton and Dong 1965] the correspondence principle was derived for anisotropic materials.

In a number of instances [Gottenberg and Christensen 1963; Olesiak 1966; Paulino and Jin 2001a; 2001b; Jin and Paulino 2002; Jin 2006; Ko et al. 2003; Hilton 1964; Freudenthal and Henry 1960; Bieniek et al. 1981; Librescu and Chandiramani 1989b; 1989a; O'Brien et al. 2001; Zhu 2000; Shrotriya 2000; Shrotriya and Sottos 1998; Zhu et al. 2003; Andrianov et al. 2004; di Bernedetto et al. 2007; Noh and Whitcomb 2003; Klasztorny 2004; Bert 1973; Cowper 1966; Hilton 2009; Therriault 2003], time-independent PR assumptions lead to overdetermined ill-posed problems and cause use of the elastic-viscoelastic correspondence principle to become unjustified. In other analyses [Jin and Paulino 2002; Jin 2006; Ko et al. 2003; Hilton 1964; Freudenthal and Henry 1960; Bieniek et al. 1981; Librescu and Chandiramani 1989b; 1989a; O'Brien et al. 2001; Zhu 2000; Shrotriya 2000; Shrotriya and Sottos 1998; Zhu et al. 2003; Andrianov et al. 2004; di Bernedetto et al. 2007; Noh and Whitcomb 2003; Klasztorny 2004; Bert 1973; Cowper 1966; Hilton 2009; Therriault 2003; Nakao et al. 1985; Singh and Abdelnaser 1993; Chen 1995; Hilton and Vail 1993], the elastic-viscoelastic correspondence principle (analogy) has been applied improperly by extending it to viscoelastic time-dependent PRs. In this paper the applicability and predominent inapplicability of this analogy as it relates to Poisson's ratio in elastic and viscoelastic expressions involving bulk, shear and Young's moduli is examined.

Several illustrative examples consider the effects of viscoelastic PRs, namely one-dimensional relaxation loading, simple bending and Timoshenko beams. The Timoshenko beam in particular brings into play an additional parameter, the shear coefficient, which depends on stresses, material properties, loading histories and paths, cross sectional geometry, and boundary and initial conditions. Its characteristics bear some resemblance to those of the PRs and it also does not generally submit to an elastic-viscoelastic analogy, despite a number of publications to the contrary [Therriault 2003; Nakao et al. 1985; Singh and Abdelnaser 1993; Chen 1995], including one by the present author [Hilton and Vail 1993].

Note that all the viscoelastic Timoshenko beam publications [Nakao et al. 1985; Singh and Abdelnaser 1993; Chen 1995; Hilton and Vail 1993] except [Therriault 2003] preceded [Hilton 1996; 2001; 2003; Hilton and Yi 1998; Tschoegl 1997; Tschoegl et al. 2002] where the viscoelastic PR inconsistencies were derived. On the other hand, [Jin and Paulino 2002] were published after [Hilton 1996; 2001; 2003; Hilton and Yi 1998; Tschoegl 1997; Tschoegl et al. 2002].

1. General concepts

The correspondence principle. The elastic-viscoelastic correspondence principle or analogy comes in two flavors, namely (a) separation of variables and (b) integral transforms. The pertinent references are listed in the introduction. Consider a Cartesian coordinate system $x = \{x_1, x_2, x_3\}$ with Einstein's summation notation and where underlined indices indicate no summation.

The separation of variables analogy states that under proper conditions viscoelastic variables are related to equivalent elastic ones by

$$\sigma_{ij}(x,t) \;=\; g(t)\,\sigma_{ij}^{e}(x) \qquad \epsilon_{ij}(x,t) \;=\; h(t)\,\epsilon_{ij}^{e}(x) \tag{1}$$

where the superscripts ^{e} denote equivalent elastic variables or solutions. The severe restrictions associated with these forms are discussed in Section 3. In particular, it is required that the material be incompressible with PRs $\nu^e(x) = \nu(x,t) = \frac{1}{2}$.

For isotropic materials the integral transform analogy one requires that the Fourier transforms (FT) be

$$\text{elastic} \quad \Longrightarrow \quad \begin{cases} \overline{\overline{\sigma}}{}^{e}_{ij}(x,\omega) \;=\; \overline{\overline{\sigma}}{}^{e}_{ij}(x,\omega,G^e,K^e,\overline{\overline{\alpha T}},\overline{\overline{X}},\overline{\overline{U}}) \\ \qquad\qquad\qquad \text{or} \\ \overline{\overline{\sigma}}{}^{e}_{ij}(x,\omega) \;=\; \overline{\overline{\sigma}}{}^{e}_{ij}(x,\omega,\mathcal{G}^e,\nu^e,\overline{\overline{\alpha T}},\overline{\overline{X}},\overline{\overline{U}}) \end{cases} \tag{2}$$

and

$$\text{viscoelastic} \quad \Longrightarrow \quad \begin{cases} \overline{\overline{\sigma}}_{ij}(x,\omega) \;=\; \overline{\overline{\sigma}}{}^{e}_{ij}(x,\omega,\overline{\overline{G}},\overline{\overline{K}},\overline{\overline{q}}\,\overline{\overline{\alpha T}},\overline{\overline{X}},\overline{\overline{U}}) \\ \\ \overline{\overline{\sigma}}_{ij}(x,\omega) \;\neq\; \overline{\overline{\sigma}}{}^{e}_{ij}(x,\omega,\overline{\overline{\mathcal{G}}},\overline{\overline{\nu}},\overline{\overline{q}}\,\overline{\overline{\alpha T}},\overline{\overline{X}},\overline{\overline{U}}) \end{cases} \tag{3}$$

(see Table 2), where $X(x,t)$ and $U(x,t)$ are respectively boundary stresses and displacements. The generic symbols $\mathcal{G}^e$ and $\overline{\overline{\mathcal{G}}}$ refer respectively to G^e, K^e or E^e and $\overline{\overline{G}}$, $\overline{\overline{K}}$ or $\overline{\overline{E}}$. The integral transform analogy or correspondence principle then consists of one to one replacements in elastic FT solutions of elastic *moduli* with corresponding viscoelastic *complex moduli*, i.e.,

$$\overline{\overline{G}} \xrightarrow{\text{for}} G^e, \qquad \overline{\overline{K}} \xrightarrow{\text{for}} K^e, \qquad \overline{\overline{E}} \xrightarrow{\text{for}} E^e, \qquad \underline{\text{but not}}\ \overline{\overline{\nu}} \xrightarrow{\text{for}} \nu^e,\ \text{except when}\ \overline{\overline{\nu}} = \nu^e = \tfrac{1}{2}. \tag{4}$$

The viscoelastic stresses, strains and displacements are the FT inverses of these modified elastic FTs.

Constitutive relations. Isotropic isothermal nonhomogeneous elastic constitutive relations (Hooke's law) at constant temperature are then written as

$$\sigma^e_{\underline{ii}}(x,t) \;=\; \sum_{j=1}^{3} E^e_{\underline{ii}\,jj}(x)\,\epsilon^e_{jj}(x,t), \tag{5}$$

$$\sigma^e_{ij}(x,t) \;=\; 2\,G^e(x)\,\epsilon^e_{ij}(x,t), \qquad i \neq j, \tag{6}$$

and with the classical (original) definition of Poisson's ratio [Poisson 1829] given by

$$\nu^e_{ij}(x,t) \;=\; -\,\frac{\epsilon^e_{\underline{jj}}(x,t)}{\epsilon^e_{\underline{ii}}(x,t)}, \qquad i \neq j, \tag{7}$$

Thus the elastic PR will be time-dependent whenever the strain components are non-separable functions of space and time or distinct time functions regardless whether the elastic moduli E^e_{ijkl} or G^e are time-dependent.

For a case of one-dimensional stress, where

$$\sigma_{11} \;\neq\; 0 \quad \text{and all other } \sigma_{ij} \text{ are } 0, \tag{8}$$

substitution of (7) into (5) in order to eliminate $\epsilon_{22} = \epsilon_{33}$ in favor of ϵ_{11} yields

$$\sigma^e_{11}(x,t) \;=\; \big(E^e_{1111}(x) - 2\,\nu^e_{12}(x,t)\,E^e_{1122}(x)\big)\,\epsilon^e_{11}(x,t) \;=\; E_0(x,t)\,\epsilon^e_{11}(x,t), \tag{9}$$

since $E^e_{1122} = E^e_{1133}$ and where E_0 is the Young's modulus.

Alternately, consider the isotropic constitutive relations in terms of shear and bulk moduli (K^e and G^e):

$$S^e_{ij}(x,t) \;=\; 2\,G^e(x)\,E^e_{ij}(x.t), \qquad \sigma^e(x,t) \;=\; K^e(x)\,\epsilon^e(x,t), \tag{10}$$

where the stress S_{ij} and strain E_{ij} deviators and mean stresses σ and strains ϵ are

$$S_{ij} = \sigma_{ij} - \delta_{ij}\,\sigma, \qquad \sigma = \frac{\sigma_{ii}}{3}, \qquad E_{ij} = \epsilon_{ij} - \delta_{ij}\,\epsilon, \qquad \epsilon = \frac{\epsilon_{ii}}{3}, \tag{11}$$

resulting in

$$\sigma^e_{11}(x,t) = \underbrace{\frac{4\,G^e + K^e}{3}}_{=\,E^e_{1111}}\,\epsilon^e_{11}(x,t) + \underbrace{\frac{K^e - 2\,G^e}{3}}_{=\,E^e_{1122}}\big(\epsilon^e_{22}(x,t) + \epsilon^e_{33}(x,t)\big), \qquad K^e < \infty, \tag{12}$$

and

$$\epsilon^e_{11}(x,t) = \underbrace{\frac{1 + G^e/K^e}{3\,G^e}}_{=\,1/E^e_0\,=\,C^e_0}\,\sigma^e_{11}(x,t), \qquad \epsilon^e_{22}(x,t) = \underbrace{\frac{2\,G^e/K^e - 1}{6\,G^e}}_{=1/E^e_{2211}=C^e_{2211}=-\nu^e_{12}/E^e_0}\,\sigma^e_{11}(x,t), \tag{13}$$

since $\epsilon_{22} = \epsilon_{33}$. Therefore, for any isotropic linear elastic material the PR becomes

$$\nu^e_{12} = \nu^e = \frac{1 - 2\,G^e/K^e}{2\,(1 + G^e/K^e)}, \tag{14}$$

with an upper limit of 0.5 for incompressible materials when $K^e \to \infty$.

The corresponding isotropic nonhomogeneous viscoelastic stress-strain relations at constant temperature are expressable in the form

$$\sigma_{\underline{ii}}(x,t) = \sum_{j=1}^{3}\int_{-\infty}^{t} E_{\underline{ii}jj}(x,t-t')\,\epsilon_{jj}(x,t')\,dt', \tag{15}$$

$$\sigma_{ij}(x,t) = 2\int_{-\infty}^{t} G(x,t-t')\,\epsilon_{ij}(x,t')\,dt', \qquad i \neq j. \tag{16}$$

The initial conditions of any viscoelastic problem are

$$\sigma(x,0) = \sigma^e(x,0) \quad \text{and} \quad \epsilon_{ij}(x,0) = \epsilon^e_{ij}(x,0), \tag{17}$$

with material properties

$$E(x,0) = E^e(x) \qquad G(x,0) = G^e(x) \qquad \nu_{ij}(x,0) = \nu^e_{ij}(x), \tag{18}$$

and where the superscript ^{e} refers to elastic quantities of the corresponding elastic problem (same boundary conditions, geometry, and so on).

In [Hilton 2001] five distinct classes of PR definitions are catalogued:

Class I
[Poisson 1829]
$$\nu_{ij}(x,t) \overset{\text{def}}{=} -\frac{\epsilon_{\underline{jj}}(x,t)}{\epsilon_{\underline{ii}}(x,t)}, \qquad i \neq j; \tag{19}$$

Class II
[Christensen 1982; Pipkin 1972]
$$\nu^{\mathbf{C}}_{ij}(x,t) \overset{\text{def}}{=} -\frac{\epsilon_{\underline{jj}}(x,t)}{\epsilon_{11}(x)}, \qquad j \neq 1; \qquad \epsilon_{11} = \text{const.}; \tag{20}$$

Class III
[Hilton and Yi 1998]
$$\bar{\bar{\nu}}^{\mathbf{A}}_{ij}(x,\omega) \overset{\text{def}}{=} -\frac{\bar{\bar{\epsilon}}_{\underline{jj}}(x,\omega)}{\bar{\bar{\epsilon}}_{\underline{ii}}(x,\omega)}, \qquad i \neq j; \tag{21}$$

Class IV
[Vinogradov and Malkin 1980]

$$\nu_{ij}^{\mathbf{H}}(x,t) \overset{\text{def}}{=} -\frac{\log\bigl(1+\epsilon_{\underline{jj}}(x,t)\bigr)}{\log\bigl(1+\epsilon_{\underline{ii}}(x,t)\bigr)}, \qquad i \neq j; \tag{22}$$

Class V
[Bertilsson et al. 1993]

$$\frac{\partial \nu_{ij}^{\mathbf{V}}(x,t)}{\partial t} \overset{\text{def}}{=} -\frac{\partial \epsilon_{\underline{jj}}(x,t)/\partial t}{\partial \epsilon_{\underline{ii}}(x,t)/\partial t}, \qquad i \neq j. \tag{23}$$

Consider for instance the original classical Class I definition for isothermal viscoelastic materials, resulting in

$$\nu_{ij}(x,t) \;=\; -\frac{\epsilon_{\underline{jj}}(x,t)}{\epsilon_{\underline{ii}}(x,t)} \;=\; -\frac{\int_{-\infty}^{t} C_{\underline{jj}kl}(x,t-t')\,\sigma_{kl}(x,t')\,dt'}{\int_{-\infty}^{t} C_{\underline{ii}mn}(x,t-t')\,\sigma_{mn}(x,t')\,dt'}, \qquad i \neq j, \tag{24}$$

with similar expressions for the other PR classes. It can be readily seen that even in linear viscoelasticity the PRs by any definitions are

(I) nonlinear functions of strains, stresses and their time histories (loading path) and hence process-dependent and not universal material property parameters such as moduli and compliances;

(II) derived or defined quantities and not fundamental ones such as relaxation moduli or creep compliances;

(III) material properties determined from one-dimensional normal loading experiments and PRs may not be exportable to other stress fields, unless proper expressions are used to represent these viscoelastic PRs.

Elimination of ϵ_{22} from (15) now results in

$$\sigma_{11}(x,t) \;=\; \int_{-\infty}^{t} \bigl(E_{1111}(x,t-t') - 2\,\nu_{12}(x,t')\,E_{1122}(x,t-t')\bigr)\,\epsilon_{11}(x,t')\,dt'. \tag{25}$$

This isotropic constitutive relation form can be achieved in temporal space only through the use of the Class I PR definition of (19), *i.e.*, *Poisson's original definition* [1829], *since the strain substitutions must be based on the actual instantaneous strains. Indeed, this viscoelastic protocol is identical to what is employed in the theory of elasticity when Hooke's law is extended to three dimensions and is the proper approach for formulating general relations between dynamic moduli.*

Taking Fourier transforms (FT) of (9) and (25) yields respectively

$$\text{elastic} \implies \bar{\bar{\sigma}}^{e}_{11}(x,\omega) \;=\; \bar{\bar{E}}^{e}_{1111}(x,\omega)\,\bar{\bar{\epsilon}}^{e}_{11}(x,\omega) \;-\; 2\,\bar{\bar{E}}^{e}_{1122}(x,\omega)\,\overline{\overline{\nu^{e}_{12}\,\epsilon^{e}_{11}}}(x,\omega), \tag{26}$$

$$\text{viscoelastic} \implies \bar{\bar{\sigma}}_{11}(x,\omega) \;=\; \bar{\bar{E}}_{1111}(x,\omega)\,\bar{\bar{\epsilon}}_{11}(x,\omega) \;-\; 2\,\bar{\bar{E}}_{1122}(x,\omega)\,\overline{\overline{\nu_{12}\,\epsilon_{11}}}(x,\omega). \tag{27}$$

It can be readily seen that (26) and (27) are not in the proper form for the correspondence principle to be applicable, since they contain the transforms of the Class I PR and strain as opposed to the necessary product of the transforms, i.e. $\overline{\overline{\nu_{12}\,\epsilon_{11}}} \neq \bar{\bar{\nu}}_{12}\,\bar{\bar{\epsilon}}_{11}$. This inequality can be removed if and only if either the PR or the strain or both are time-independent or if and only if the strains are separable functions as described in (1). Time independent strains are the degenerate case of relations (1). Additional examples are analyzed in detail in the next section.

The relationship between relaxation moduli $G(t)$, compliances $C(t)$ and relaxation and creep functions $\Phi(t)$ and $\Psi(t)$ in the Fourier transform space is

$$\bar{\bar{C}}(x,\omega) = \frac{1}{\bar{\bar{G}}(x,\omega)} = \imath\,\omega\,\bar{\bar{\Psi}}(x,\omega) = \frac{1}{\imath\,\omega\,\bar{\bar{\Phi}}(x,\omega)}; \tag{28}$$

see [Christensen 1982; Hilton 1964]. The Laplace transform can be obtained from the FT as

$$\mathrm{LT}\{f(x,t)\} = \bar{f}(x,p) = \bar{\bar{f}}(x,\omega)\big|_{\imath\,\omega=1/p} \tag{29}$$

These proper definitions then lead to shear viscoelastic constitutive relations

$$\epsilon_S(x,t) = \int_{-\infty}^{t} C(t-t')\,\sigma_S(x,t')\,dt' = \int_{-\infty}^{t} \Psi(t-t')\,\frac{\partial\sigma_S(x,t')}{\partial t'}\,dt' = \int_{-\infty}^{t} \frac{\partial\Psi(t-t')}{\partial t'}\,\sigma_S(x,t')\,dt' \tag{30}$$

and

$$\sigma_S(x,t) = \int_{-\infty}^{t} G(t-t')\,\epsilon_S(x,t')\,dt' = \int_{-\infty}^{t} \Phi(t-t')\,\frac{\partial\epsilon_S(x,t')}{\partial t'}\,dt' = \int_{-\infty}^{t} \frac{\partial\Phi(t-t')}{\partial t'}\,\epsilon_S(x,t')\,dt'. \tag{31}$$

With similar expressions for the normal stresses and strains given by

$$\begin{aligned}\epsilon_{\underline{ii}}(x,t) &= \sum_{k=1}^{3}\int_{-\infty}^{t} C^N_{\underline{ii}kk}(x,t-t')\,\sigma_{kk}(x,t')\,dt' \\ &= \sum_{k=1}^{3}\int_{-\infty}^{t} \Psi^N_{\underline{ii}kk}(x,t-t')\,\frac{\partial\sigma_{kk}(x,t')}{\partial t'}\,dt' = \sum_{k=1}^{3}\int_{-\infty}^{t} \frac{\partial\Psi^N_{\underline{ii}kk}(x,t-t')}{\partial t'}\sigma_{kk}(x,t')\,dt'\end{aligned} \tag{32}$$

and

$$\begin{aligned}\sigma_{\underline{ii}}(x,t) &= \sum_{k=1}^{3}\int_{-\infty}^{t} E_{\underline{ii}kk}(x,t-t')\,\epsilon_{kk}(x,t')\,dt' \\ &= \sum_{k=1}^{3}\int_{-\infty}^{t} \Phi^N_{\underline{ii}kk}(x,t-t')\,\frac{\partial\epsilon_{kk}(x,t')}{\partial t'}\,dt' = \sum_{k=1}^{3}\int_{-\infty}^{t} \frac{\partial\Phi^N_{\underline{ii}kk}(x,t-t')}{\partial t'}\epsilon_{kk}(x,t')\,dt'.\end{aligned} \tag{33}$$

The shear constitutive equations are

$$\sigma_{ij}(x,t) = 2\int_{-\infty}^{t} G(x,t-t')\,\epsilon_{ij}(x,t')\,dt', \qquad i \neq j. \tag{34}$$

Application of Fourier transforms leads to

$$\bar{\bar{\sigma}}_{\underline{ii}}(x,\omega) = \sum_{k=1}^{3} \bar{\bar{E}}_{\underline{ii}kk}(x,\omega)\,\bar{\bar{\epsilon}}_{kk}(x,\omega), \tag{35}$$

$$\bar{\bar{\sigma}}_{ij}(x,\omega) = 2\,\bar{\bar{G}}(x,\omega)\,\bar{\bar{\epsilon}}_{ij}(x,\omega), \qquad i \neq j, \tag{36}$$

which leads to the proper elastic-viscoelastic correspondence principle in terms of relaxation moduli.

2. The elastic-viscoelastic correspondence principle or analogy

Class I Poisson ratios: original definition. The elastic-viscoelastic analogy cannot be expressed in terms of PRs with classical definitions of (7) and (19) when these elastic or viscoelastic PRs are functions of time, since

$$\begin{aligned}\bar{\bar{\epsilon}}_{22}(x,\omega) &= \overline{\overline{\nu_{12}\,\epsilon_{11}}}(x,\omega) = \int_{-\infty}^{\infty} \nu_{12}(x,t)\,\epsilon_{11}(x,t)\ \exp(-\imath\,\omega\,t)\,dt \\ &\neq \bar{\bar{\nu}}_{12}(x,\omega)\,\bar{\bar{\epsilon}}_{11}(x,\omega) = \int_{-\infty}^{\infty} \nu_{12}(x,t)\ \exp(-\imath\,\omega\,t)\,dt \int_{-\infty}^{\infty} \epsilon_{11}(x,t)\ \exp(-\imath\,\omega\,t)\,dt; \qquad (37)\end{aligned}$$

the inequality arises because the quantity on the first line of (37) is the transform of the ν and ϵ_{11} product, while the one on the second line is a product of their individual transforms. Either elastic and viscoelastic PR will be time independent if and only if all the strains are time-independent or separable functions of space and time with identical time functions [Hilton and Yi 1998; Hilton 2001].

Similarly, the elastic PR relation (14) is not receptive to the application of the elastic-viscoelastic correspondence principle, since, by virtue of (27),

$$\bar{\bar{\nu}}_{12}(x,\omega) \neq \frac{1-2\,\bar{\bar{G}}(x,\omega)/\bar{\bar{K}}(x,\omega)}{2\left(1+\bar{\bar{G}}(x,\omega)/\bar{\bar{K}}(x,\omega)\right)} \qquad (38)$$

except for incompressible materials when $K(x,t) \to \infty$ and $\nu_{12} \to 0.5$.

If classical (Class I) Poisson ratios are introduced into the isotropic constitutive relations, then from (19) one obtains their FT as the transform of the products and not the product of the transforms as is required for the correspondence principle [Hilton 1996; 2001; Hilton and Yi 1998]. This can be readily seen by substituting (27) into (36), resulting in

$$\bar{\bar{\sigma}}_{\underline{ii}}(x,\omega) = \bar{\bar{E}}_{\underline{iiii}}(x,\omega)\,\bar{\bar{\epsilon}}_{\underline{ii}}(x,\omega) - 2\,\bar{\bar{E}}_{\underline{iikk}}(x,\omega)\,\underbrace{\overline{\overline{\nu_{\underline{ik}}\epsilon_{\underline{ii}}}}(x,\omega)}_{=-\bar{\bar{\epsilon}}_{\underline{kk}}(x,\omega)}, \qquad i \neq k. \qquad (39)$$

This is not the proper form of the elastic-viscoelastic correspondence principle and the analogy, therefore, fails to materialize. Upon inversion one obtains

$$\sigma_{\underline{ii}}(x,t) = \int_{-\infty}^{t} \Big(E_{\underline{iiii}}(x,t-t')\,\epsilon_{\underline{ii}}(x,t') - 2\,E_{\underline{iikk}}(x,t-t')\,\underbrace{\nu_{\underline{ik}}(x,t')\,\epsilon_{\underline{ii}}(x,t')}_{=-\epsilon_{\underline{kk}}(x,t')} \Big)\,dt', \quad i \neq k. \quad (40)$$

Consequently, the conventional isotropic elastic material property relations

$$G^e = \frac{E^e}{2(1+\nu^e)} \quad \text{and} \quad K^e = \frac{E^e}{1-2\,\nu^e}, \qquad (41)$$

involving the Young's (E^e), shear (G^e) and bulk (K^e) moduli together with PRs, have no counterpart in viscoelasticity except, when PRs are time-independent, because of the inability to arrive at corresponding Laplace or Fourier transforms of ν^e and ν. Therefore

$$\bar{\bar{G}} \neq \frac{\bar{\bar{E}}}{2\left(1+\bar{\bar{\nu}}\right)} \quad \text{and} \quad \bar{\bar{K}} \neq \frac{\bar{\bar{E}}}{1-2\,\bar{\bar{\nu}}} \quad \text{and} \quad \bar{\bar{\nu}} \neq \frac{1-2\,\bar{\bar{G}}/\bar{\bar{K}}}{2\left(1+\bar{\bar{G}}/\bar{\bar{K}}\right)}, \qquad (42)$$

due to (37). Unfortunately, these inequalities prevent conversion by the correspondence principle of the extensive elastic formulas developed in [Hahn 1980] and further amplified in [Whitney and McCullough 1990]. However, relations involving only moduli, such as

$$E^e \;=\; \frac{3\,G^e}{1+G^e/K^e}, \tag{43}$$

possess an equivalent viscoelastic integral transform expression of the type

$$\bar{\bar{E}}(x,\omega) \;=\; \frac{3\,\bar{\bar{G}}(x,\omega)}{1+\bar{\bar{G}}(x,\omega)/\bar{\bar{K}}(x,\omega)}. \tag{44}$$

Hence, the integral transform elastic-viscoelastic analogy cannot involve Poisson's ratios, except when the viscoelastic PRs are time-independent, with all the attendant severe restrictions outlined above and developed in detail in [Hilton 1996; 2001; Hilton and Yi 1998].

Class II Poisson ratios: one strain component, time-independent. Class II is a special degenerate case of Class I with a time-independent loaded direction strain $\epsilon_{11}(x)$. Taking the FT of (20) leads to

$$\bar{\bar{\nu}}^{\mathbf{C}}_{1j}(x,\omega)\,\epsilon_{11}(x) \;=\; \bar{\bar{\epsilon}}_{\underline{jj}}(x,\omega), \qquad j\neq 1, \tag{45}$$

with corresponding constitutive FT relations

$$\bar{\bar{\sigma}}_{\underline{ii}}(x,\omega) \;=\; \big(\bar{\bar{E}}_{\underline{ii}11}(x,\omega) \;-\; 2\,\bar{\bar{E}}_{\underline{iikk}}(x,\omega)\,\bar{\bar{\nu}}^{\mathbf{C}}_{1\underline{k}}(x,\omega)\big)\,\epsilon_{11}(x), \qquad k\neq i,\ k\neq 1. \tag{46}$$

which inverts to

$$\sigma_{\underline{ii}}(x,t) \;=\; E_{\underline{ii}11}(x,t)\,\epsilon_{11}(x) \;-\; 2\int_{-\infty}^{t} E_{\underline{iikk}}(x,t-t')\;\underbrace{\nu^{\mathbf{C}}_{1\underline{k}}(x,t')\,\epsilon_{11}(x)}_{=\,-\,\epsilon_{\underline{kk}}(x,t')}\;dt', \qquad k\neq i,\ k\neq 1. \tag{47}$$

This indicates that for this special case, the elastic-viscoelastic analogy is applicable in the FT space even though the PR is time-dependent, but one of the normal strains, $\epsilon_{11}(x)$, must be time-independent. *However*, (46) *cannot be generalized to and are inapplicable for time-dependent strains* $\epsilon_{11}(x,t)$, *which have to be treated as Class I PRs.* (See (25) and (27).)

Class III Poisson ratios: alternate definition based on Fourier transforms. The alternate or transform Poisson ratio [Hilton and Yi 1998] defined by (21) will change the FT of (39) to

$$\bar{\bar{\sigma}}_{\underline{ii}}(x,\omega) \;=\; \big(\bar{\bar{E}}_{\underline{iiii}}(x,\omega) \;-\; 2\;\underbrace{\bar{\bar{E}}_{\underline{i}ikk}(x,\omega)\,\bar{\bar{\nu}}^{\mathbf{A}}_{\underline{i}k}(x,\omega)\big)}_{=\,\bar{\bar{E}}^{\mathbf{A}}_{\underline{i}ikk}(x,\omega)}\;\bar{\bar{\epsilon}}_{\underline{ii}}(x,\omega), \qquad k\neq i, \tag{48}$$

with an inverse relation

$$\sigma_{\underline{ii}}(x,t) \;=\; \int_{-\infty}^{t}\big(E_{\underline{iiii}}(x,t-t') \;-\; 2\,E^{\mathbf{A}}_{\underline{iikk}}(x,t-t')\big)\,\epsilon_{\underline{ii}}(x,t')\,dt', \tag{49}$$

where

$$E^{\mathbf{A}}_{\underline{iikk}}(x,t) \;=\; \int_{-\infty}^{t} E_{\underline{iikk}}(x,t-t')\,\nu^{\mathbf{A}}_{\underline{ik}}(x,t')\,dt' \;=\; \int_{-\infty}^{t} E_{\underline{iikk}}(x,t)\,\nu^{\mathbf{A}}_{\underline{ik}}(x,t-t')\,dt'. \tag{50}$$

The form (49) restores a format for the correspondence principle in terms of a pseudo relaxation modulus $E^{\mathbf{A}}_{\underline{iikk}}$. It must be remembered, however, that neither $\nu^{\mathbf{A}}_{ij}$ nor $\overline{\overline{\nu}}^{\mathbf{A}}_{ij}$ is a physical quantity.

These inherent difficulties associated with viscoelastic PRs stem from the fact that unlike moduli, compliances, relaxation and creep functions, etc., PRs are "derived" rather than fundamental material properties, as seen from (19) and (20)–(23) and discussed in detail in [Hilton 2001] All but one of these five do not accommodate the elastic-viscoelastic correspondence principle. However, the alternate PR definition based on Fourier transforms [Hilton and Yi 1998], namely

$$\overline{\overline{\nu}}^{\mathbf{A}}_{ij}(x,\omega) = -\frac{\overline{\overline{\epsilon}}_{\underline{jj}}(x,\omega)}{\overline{\overline{\epsilon}}_{\underline{ii}}(x,\omega)} \quad \text{and} \quad \epsilon_{\underline{jj}}(x,t) = -\int_{-\infty}^{t} \nu^{\mathbf{A}}_{\underline{ij}}(x,t-t')\,\epsilon_{\underline{ii}}(x,t')\,dt', \qquad i\neq j, \tag{51}$$

lends itself to an elastic-viscoelastic analogy in terms of $\overline{\overline{\nu}}^{\mathbf{A}}_{ij}$, but this alternate or transform PR has no physical counter part nor relation to the the classical PR as given by (7) and (37). Furthermore, the Class III PR bears no relation to its classical Class I counterpart

$$\nu^{\mathbf{A}}_{ij}(x,t) = -\int_{-\infty}^{\infty} \frac{\overline{\overline{\epsilon}}_{\underline{jj}}(x,\omega)}{\overline{\overline{\epsilon}}_{\underline{ii}}(x,\omega)} \exp(\imath\,\omega t)\;d\omega \neq \nu_{ij}(x,t), \qquad i\neq j. \tag{52}$$

The viscoelastic PR situation is further aggravated since under many conditions classical (19) and alternate PRs (51) become stress as a well as time-dependent for linear viscoelastic materials [Hilton and Yi 1998; Hilton 2001]. Therefore, unlike relaxation moduli and creep compliances which in the linear case are stress independent, viscoelastic PRs in any form are not global material properties which can be used interchangeably among different loading conditions without re-computation to fit each specific set of conditions and time histories.

Class IV Poisson ratios: Hencky definition. The Hencky definition of (22) does not lend itself to any form of the elastic-viscoelastic analogy because of its inherent presence of the logarithmic terms.

Class V Poisson ratios: strain velocity ratios. For this PR class one can use the relaxation form of (33) and substitute the PR from (23) to yield

$$\sigma_{\underline{ii}}(x,t) = \int_{-\infty}^{t} \left(\Phi^{N}_{\underline{iiii}}(x,t-t') - 2\,\Phi^{N}_{\underline{iikk}}(x,t-t')\,\frac{\partial \nu^{\mathbf{V}}_{\underline{ik}}(x,t')}{\partial t'} \right) \frac{\partial \epsilon_{\underline{ii}}(x,t')}{\partial t'}\,dt' \qquad k\neq i \tag{53}$$

There are visible similarities between Class I and V definitions and, hence, it is not surprising that the velocity based PR suffers from the same limitations as the Class I representation since

$$\overline{\overline{\frac{\partial \nu^{\mathbf{V}}_{\underline{ik}}}{\partial t}\,\frac{\partial \epsilon_{\underline{ii}}}{\partial t}}}(x,\omega) = \int_{-\infty}^{\infty} \frac{\partial \nu^{\mathbf{V}}_{\underline{ik}}(x,t)}{\partial t}\,\frac{\partial \epsilon_{\underline{ii}}(x,t)}{\partial t} \exp(-\imath\,\omega t)\;dt = \overline{\overline{\frac{\partial \epsilon_{\underline{kk}}}{\partial t}}}(x,\omega) \qquad k\neq 1 \tag{54}$$

This leads to constitutive relations in the FT domain

$$\overline{\overline{\sigma}}_{\underline{ii}}(x,\omega) = \imath\,\omega\,\overline{\overline{\Phi}}^{N}_{\underline{iiii}}(x,\omega)\,\overline{\overline{\epsilon}}_{\underline{ii}}(x,\omega) - 2\,\overline{\overline{\Phi}}^{N}_{\underline{iikk}}(x,\omega)\,\overline{\overline{\frac{\partial \nu^{\mathbf{V}}_{\underline{ik}}}{\partial t}\,\frac{\partial \epsilon_{\underline{ii}}}{\partial t}}}(x,\omega), \qquad k\neq i, \tag{55}$$

and, therefore, is not suited for any form of the elastic-viscoelastic analogy.

3. The seldom time-independent viscoelastic Poisson ratio

In elasticity time-independent strains can be achieved only under time-independent stresses and displacements regardless of boundary conditions. In viscoelasticity time free strains are attainable under considerably more restrictive conditions. (15) can be inverted in order to express strains in terms stress as

$$\epsilon_{\underline{ii}}(x,t) \;=\; \sum_{j=1}^{3} \int_{-\infty}^{t} C_{\underline{ii}jj}(x, t-t')\, \sigma_{jj}(x,t')\, dt', \tag{56}$$

with compliances defined by (28).

As has been pointed out in [Hilton and Yi 1998; Hilton 2001] and as can be seen from (19), the viscoelastic PRs are time-independent if and only if the corresponding viscoelastic solution is separable into products of temporal and spatial parts, such that

$$E_{ijkl}(x,t) \;=\; F(t)\, E^{*}_{ijkl}(x), \qquad C_{ijkl}(x,t) \;=\; F_c(t)\, C^{*}_{ijkl}(x), \tag{57}$$

$$\epsilon_{ij}(x,t) \;=\; h(t)\, \epsilon^{e}_{ij}(x), \qquad \sigma_{ij}(x,t) \;=\; g(t)\, \sigma^{e}_{ij}(x), \tag{58}$$

with

$$g(t) \;=\; \int_{-\infty}^{t} F(t-t')\, h(t')\, dt' \quad \text{or} \quad h(t) \;=\; \int_{-\infty}^{t} F_c(t-t')\, g(t')\, dt' \tag{59}$$

depending on whether $g(t)$ or $h(t)$ is defined *a priori* on the boundary. It must be emphasized that the requirement that the E_{ijkl} and C_{ijkl} all have the same time functions has serious implications. In isotropic viscoelasticity it means that the shear and bulk relaxation moduli all must have identical time functions, which is not the case in real materials. It is not uncommon to witness bulk moduli with relaxation times three to six orders of magnitude larger than those of shear moduli. Therefore, the requirements on the $F(t)$ and $F_c(t)$ functions of (57) are unrealistic means to simply achieve the desired time-independent PRs.

These severe restrictions necessary for the existence of separable variable solutions are discussed in [Hilton 1996; 1964; Alfrey 1944; 1948; Christensen 1982]. Each and every one of the following conditions must be enforced for separation of variable formulations to exist:

- Elastic and viscoelastic materials must be isotropic, homogeneous and incompressible with $\nu^{e}(t) = \nu(t) = \frac{1}{2}$.
- No dynamic effects and no body forces can be included.
- No moving boundaries, i.e., no penetration or ablation problems, and boundary surface $\Gamma = \Gamma(x)$ only.
- No mixed boundary conditions; only separable stress or separable displacement BCs may be prescribed, i.e.,

  $$\sigma_{ij}(x,t) \;=\; g(t)\, \sigma^{*}_{ij}(x) \;=\; g(t)\, n_i(x)\, X^{*}_{j}(x) \quad \text{on } \Gamma(x) \tag{60}$$

  or

  $$u_i(x,t) \;=\; h(t)\, U^{*}_{i}(x) \quad \text{on } \Gamma(x). \tag{61}$$

- No thermal expansions, i.e. $\alpha T = 0$, except for special cases of stress free boundaries [Hilton and Russell 1961].

- Only separable functions for material properties are permissible (relaxation moduli, compliances, etc.; see (57)).
- Viscoelastic materials must be isotropic [Hilton 1996].
- Relaxation moduli in all directions must have the same separable time function as defined by (57), but $K(x,t) \to \infty$. An exception occurs when $E(t)$, $G(t)$ and $K(t)$ all obey the same time functions:

$$\frac{E(t)}{E_0} = \frac{G(t)}{G_0} = \frac{K(t)}{K_0} = \mathscr{F}(t), \tag{62}$$

and then $-1 \le \nu_0 \le 0.5$. The equal relaxation time function concept was was first introduced in [Tsien 1950] and its implications and limitations are discussed in detail in [Hilton 1996]. A time-independent PR other than 0.5 must satisfy the conditions

$$\text{the special case} \quad \Longrightarrow \quad \frac{\bar{\bar{\bar{E}}}}{\bar{\bar{\bar{G}}}} = \frac{E_0}{G_0} = 2\,(1+\nu_0) = \frac{3}{1+G_0/K_0} \tag{63}$$

Having $E(t) \sim G(t)$ is physically achievable, but bulk relaxation moduli generally have relaxation times 3 to 5 orders of magnitude larger than those for $E(t)$ [Hilton 1996; 2001; Hilton and Yi 1998; [Qvale and Ravi-Chandar 2004]]. (See Figure 1.)

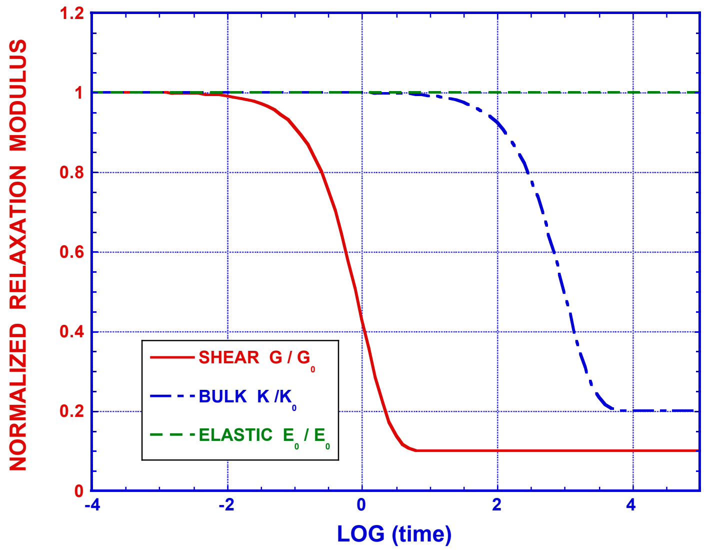

Figure 1. Elastic and viscoelastic relaxation moduli.

One can next ask whether it is possible to obtain a time-independent Class II PR. An examination of (45) indicates that this can only occur if all normal strains are time-independent, i.e., $\epsilon_{\underline{jj}} = \epsilon_{\underline{jj}}(x)$.

Consider an isothermal isotropic material with a special 1-D loading in the x_1 direction, such that

$$\epsilon_{11}(x) = \int_{-\infty}^{t} \overbrace{C_{1111}(x, t-t')}^{=\,C(x,t-t')} \, \sigma_{11}(x, t')\, dt' \tag{64}$$

and

$$\sigma_{11}(x,t) \;=\; \int_{-\infty}^{t} \overbrace{E_{1111}(x,t-t')}^{=\,E(x,t-t')}\,\epsilon_{11}(x)\,dt', \tag{65}$$

indicating that the one-dimensional relaxation stress $\sigma_{11}(x,t)$ necessary to maintain a time-independent strain $\epsilon_{11}(x)$ in the loaded direction must be time-dependent. Similarly, the strains in the other two normal directions are

$$\epsilon_{22}(x,t) \;=\; \epsilon_{33}(x,t) \;=\; \int_{-\infty}^{t}\left(C_{2211}(x,t-t')\underbrace{\int_{-\infty}^{t'} E_{1111}(x,t-s)\,\epsilon_{11}(x)\,ds}_{=\,\sigma_{11}(x,t')}\right)dt'. \tag{66}$$

Consequently, these two strains cannot be time-independent in this one-dimensional configuration and a time-independent PR is impossible under this loading. On the other hand, a special three-dimensional loading with all $\sigma_{\underline{ii}}(x,t)$ necessary to maintain time-independent strains can be imposed. Such a special stress field is material dependent and in a sense is specifically contrived to produce the desired time-independent strains leading to time independent Class II PRs.

4. Error analysis

Consider realistic simulations of a one-dimensional experiment consisting of a prismatic isotopic viscoelastic bar as described above where $\sigma_{11} \neq 0$ and all other $\sigma_{ij} = 0$. One generally measures $\epsilon_{11}(t)$ and $\sigma_{11}(t)$ and determines $C(t)$ or $E(t)$ from (64) or (65). This can be accomplished in either the time or FT or LT spaces by a least square fit of the the coefficients E_n and by using the approximation [Schapery 1962]

$$\tau_n \;=\; 10^n \tag{67}$$

such that

$$\text{FT} \implies \frac{\bar{\bar{\sigma}}_{11}(\omega)}{\bar{\bar{\epsilon}}_{11}(\omega)} \;=\; \bar{\bar{E}}(\omega) \;=\; \frac{E_\infty}{\imath\,\omega} + \sum_{n=1}^{N}\frac{E_n}{\imath\,\omega + 1/\tau_n}; \qquad \text{LT} \implies \bar{E}(p) \;=\; \bar{\bar{E}}(\omega)\big|_{\imath\,\omega = p}. \tag{68}$$

If one does not assume values of relaxation times as indicated in (67), then nonlinear algebraic solvers can be used to determine sets of E_n and τ_n from the experimental data. The number of terms N is selected to meet a prescribed accuracy of fit.

The experimental difficulties arise from attempts to simultaneously measure normal strains in the other directions, i.e., $\epsilon_{22}(t)$. Instead, a number of authors have assumed time-independent PRs $\nu_{AS} =$ constant $\neq 0.5$, obtaining approximate shear and bulk relaxation moduli from

$$\bar{\bar{E}} \;=\; \frac{3\,\bar{\bar{G}}}{1+\bar{\bar{G}}/\bar{\bar{K}}} \;=\; \frac{1}{\bar{\bar{C}}} \qquad \text{and} \qquad \bar{\bar{G}}_{AS} \;\approx\; \frac{\bar{\bar{E}}}{2\,(1+\nu_{AS})}, \tag{69}$$

and thereby creating an ill posed overdetermined problem, resulting in nonuniversal shear and bulk relaxation moduli G_{AS} and K_{AS}. The correct protocol for one-dimensional experiments is formulated in [Shtark et al. 2007].

An error analysis will be undertaken next to evaluate the effects of this PR assumption as part of a computational simulation. Consider a state of one-dimensional stress where σ_{11} and ϵ_{11} produce creep compliances with $C_0 < C_\infty$ and of the forms

$$C(t) \;=\; C_\infty \;-\; (C_\infty \;-\; C_0)\;\exp\Bigl(-\frac{t}{\tau_c}\Bigr), \tag{70}$$

$$C_{2211}(t) \;=\; -\,C_{2211\infty} \;+\; (C_{2211\infty} \;-\; C_{22110})\exp\Bigl(-\frac{t}{\tau_{2211}}\Bigr). \tag{71}$$

This is equivalent to determining any other two moduli such as $E(t)$, $G(t)$ and the bulk relaxation modulus $K(t)$. Note that as discussed in a previous section $\tau_K > \tau_G$, it follows from (69) and (72) that $\tau_C \neq \tau_{2211}$.

The exact strains for this illustrative example are obtained from the constitutive relations as

$$\bar{\bar{\epsilon}}_{11} \;=\; \frac{\bar{\bar{\sigma}}_{11}}{\bar{\bar{E}}} \;=\; \frac{1+\bar{\bar{G}}/\bar{\bar{K}}}{3\,\bar{\bar{G}}}\;\bar{\bar{\sigma}}_{11} \;=\; \bar{\bar{C}}\;\bar{\bar{\sigma}}_{11}, \tag{72}$$

$$\bar{\bar{\epsilon}}_{22} \;=\; \frac{2\,\bar{\bar{G}}/\bar{\bar{K}}-1}{2\,\bar{\bar{E}}\;\Bigl(1+\bar{\bar{G}}/\bar{\bar{K}}\Bigr)}\;\bar{\bar{\sigma}}_{11} \;=\; \frac{2\,\bar{\bar{G}}/\bar{\bar{K}}-1}{6\,\bar{\bar{G}}}\;\bar{\bar{\sigma}}_{11} \;=\; \bar{\bar{C}}_{2211}\;\bar{\bar{\sigma}}_{11}, \tag{73}$$

and the shear and bulk moduli can be determined from (69) and (72) to be

$$\bar{\bar{G}} \;=\; \frac{1}{2\,(\bar{\bar{C}}-\bar{\bar{C}}_{2211})} \qquad \text{and} \qquad \bar{\bar{K}} \;=\; \frac{1}{\bar{\bar{C}}+2\,\bar{\bar{C}}_{2211}}, \tag{74}$$

with $C(t) \geq 0$ and $C_{2211}(t) \leq 0$. Note that for $\bar{\bar{K}}(\omega) \to \infty$ the compliance $\bar{\bar{C}}_{2211}(\omega)$ tends to $-\bar{\bar{C}}(\omega)/2$. As seen from (72) the simulation can also be formulated in terms of G and K instead of the Cs above, but the latter approach renders the moduli/compliance relations considerably more involved.

The previously discussed exception of time-independent PRs with $\nu \neq .5$ is evident from (69) and (72) when $G(t) \sim K(t)$ and $\bar{\bar{G}}/\bar{\bar{K}} \to G_0/K_0$. Then

$$\nu_{12}(t) \;=\; -\,\frac{\epsilon_{22}(t)}{\epsilon_{11}(t)} \;\to\; -\,\frac{2\,G_0/K_0-1}{2\;(G_0/K_0+1)}. \tag{75}$$

On the other hand, it is quite evident from (72) that when $G(t)$ and $K(t)$ respond with different time functions, the isotropic compliances $C(t)$ and $C_{2211}(t)$ obey another set of two distinct time functions.

One can now compare exact $\bar{\bar{G}}(\omega)$ with approximate $\bar{\bar{G}}_{AS}(\omega)$ and obtain the error resulting from the introduction of ν_{AS}

$$\bar{\bar{G}}_{err} \;=\; \frac{\bar{\bar{G}}-\bar{\bar{G}}_{AS}}{\bar{\bar{G}}}, \tag{76}$$

where the variables without subscripts AS are exact quantities. Similarly, the error between approximate strains ϵ_{22AS} and correct strains is determined by

$$\epsilon_{22AS}(t) \;=\; -\,\nu_{AS}\;\epsilon_{11}(t) \qquad \text{or} \qquad \bar{\bar{\epsilon}}_{22AS}(\omega) \;=\; -\,\nu_{AS}\;\bar{\bar{\epsilon}}_{11}(\omega), \tag{77}$$

and for a one-dimensional loading from (72) one obtains

$$\bar{\bar{\epsilon}}_{22}(\omega) \;=\; \frac{\bar{\bar{C}}_{2211}(\omega)}{\bar{\bar{C}}(\omega)}\;\bar{\bar{\epsilon}}_{11}(\omega) \qquad \text{with} \qquad \bar{\bar{\epsilon}}_{err}(\omega) \;=\; \frac{\bar{\bar{\epsilon}}_{22}(\omega)\;-\;\bar{\bar{\epsilon}}_{22AS}(\omega)}{\bar{\bar{\epsilon}}_{22}(\omega)}. \tag{78}$$

Typical compliance values and the corresponding viscoelastic PR are displayed in Figure 2.

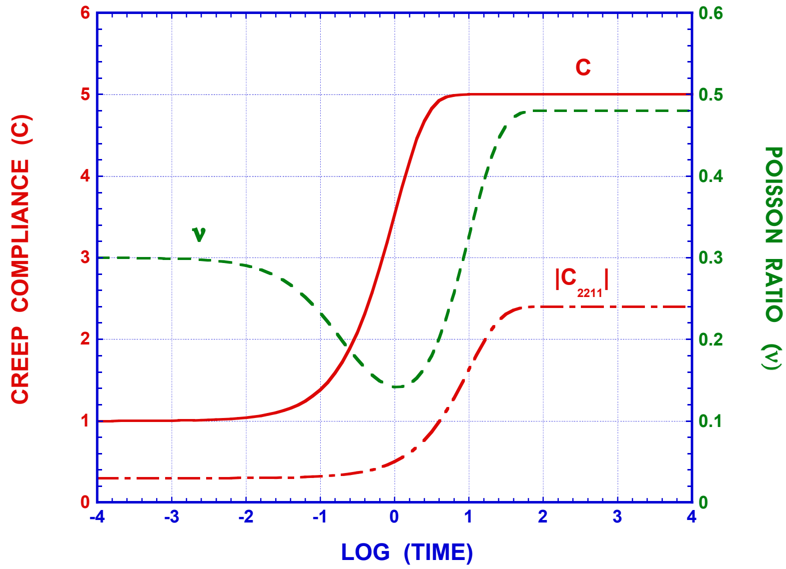

Figure 2. Compliances and PRs.

Note the pattern of initially decreasing from 0.3 and then rising PRs to a long time value of 0.48 for this configuration. Consequently, lower and upper limit estimations based on time-independent initial ν_{AS} and maximum values of 0.5 as was reported in [Therriault 2003] are erroneous and misleading, because they disregard the time history as exemplified by the the constitutive relation convolution time integrals. Furthermore, such arbitrarily assumed time-independent PRs ν_{AS} do not even lead to upper and lower bounds which could replace and bracket the experimentally unrecorded relaxation moduli, strains, etc. This fact is further amplified by next examining the experimentally unmeasured shear relaxation moduli and strains in the direction normal to the one-dimensional loading.

Figure 3 depicts the per cent error between the exact LT shear modulus of (74) and the one based on assumed values of the PR ν_{AS} of (69). For this configuration, the estimates for shear moduli based on constant PR values of .3 and .5 lead to maximum errors in shear moduli of 43% and 56% respectively. Errors of such magnitude render the constant PR approach totally unsatisfactory and unacceptable for shear relaxation modulus determination from uniaxial experimental data with only single directional stress and strain measurements.

The errors between the LT of the unmeasured and exact strains ϵ_{22} for $0 \leq p \leq \infty$ or conversely for $\infty \geq t \geq 0$ are shown in Figure 4 based on (72).

It is patent from these graphs that the arbitrary selection of constant Poisson ratios — in the present examples PR values between 0.3 and 0.5 — produces errors in predicted unmeasured strains ϵ_{22} varying from 130% to 270% from the exact values. These errors are so excessive as to make the constant PR approach meaningless. These conclusions should come as no surprise, since earlier (and different) time-independent PR error analyses in [Hilton and Yi 1998] and [Hilton 2001] showed similar undesirable results.

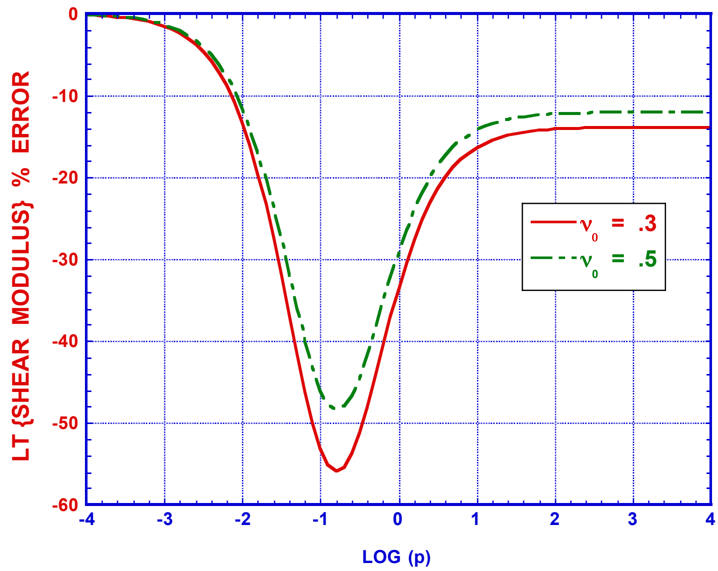

Figure 3. Percent shear modulus error.

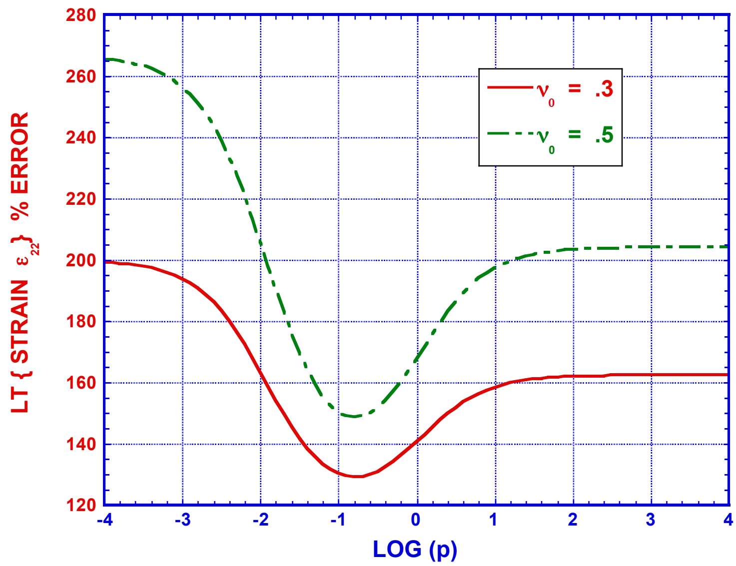

Figure 4. Percent Laplace transform transverse strain errors.

Furthermore, it can be readily seen from (77) and (69) that calculations based solely on the erroneous time-independent PR ν_{as} with values between 0.3 and 0.5 produce unmeasured strains ϵ_{22AS} which differ by 66.7% and corresponding changes in shear moduli G_{AS} of 76.9%. These errors are smaller than the true errors described in the preceding paragraph, but they are much too large to be acceptable in their own right. The generally accepted standard in deviations of elastic moduli is ±3%. Although no firm equivalent standards has been established for viscoelastic relaxation moduli, material property characterization protocols based on arbitrarily defined time-independent PRs which yield different relaxation time histories and maximum errors ranging from 43% to 67% must definitively be rejected as indefensible. (Note that the corresponding maximum strain errors are in excess of 250%.)

Different viscoelastic materials and other temperature conditions would change the specific numerical results, but would not alter the general large discrepancies between exact viscoelastic compliances, strains, PRs, etc. and those based on assumed time-independent values ν_{AS}. While the comparison were conducted in the LT space, the transforms can be inverted analytically or in the presence of complicated transforms by fast Fourier transform (FFT) protocols [van Loan 1992]. In the present simulations the LT results were not inverted into the time plane in order to avoid any additional possible errors resulting from the approximate IFFT.

In summary, the arguments advanced in [O'Brien et al. 2001; Zhu 2000; Shrotriya 2000; Shrotriya and Sottos 1998; Zhu et al. 2003; Andrianov et al. 2004; di Bernedetto et al. 2007; Noh and Whitcomb 2003] to mention a few, that analyses based on time-independent PRs are reasonable approximations to exact solutions — particularly for material characterizations — are disproved by the present simple simulations of exact conditions and their comparison with assumed time-independent PR responses.

5. Some illustrative examples

One-dimensional relaxation loading. The foregoing analysis has direct implications in a number of "simple" problems. Consider a prismatic viscoelastic bar subjected to a one-dimensional loading in the x_1-direction with $\epsilon_{11}(x)$ only and a relaxation stress $\sigma_{11}(x, t)$ with all other $\sigma_{ij} = 0$. Clearly from (66) the other two normal strains are time-dependent and so is the classical PR, as well as the other PRs of Classes II through V.

Euler–Bernoulli viscoelastic beams. Another case in point is that of a prismatic, isotropic and isothermal Euler-Bernoulli viscoelastic beam of length L, moment of inertia I, height $2c$ and $h(x_2) < c$ the beam thickness or width with static loads $q(x_1)$ and statically determinate boundary conditions (Figure 5).

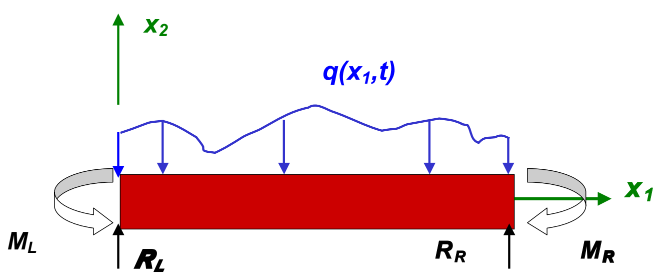

Figure 5. Viscoelastic beam.

Here the self-equilibrating bending and shear stresses are time-independent, but the strains and deflections are not since

$$\int_{-\infty}^{t} E(t-t')\, I\, \frac{\partial^4 w(x_1, t')}{\partial x_1^4}\, dt' = q(x_1), \qquad 0 \le x_1 \le L, \tag{79}$$

or, taking the FT,

$$I\, \frac{\partial^4 \overline{\overline{w}}(x_1, \omega)}{\partial x_1^4} = \overline{\overline{C}}(\omega)\, \frac{q(x_1)}{\imath\, \omega}, \qquad 0 \le x_1 \le L, \tag{80}$$

which upon inversion leads to

$$I\,\frac{\partial^4 w(x_1,t)}{\partial x_1^4} \;=\; q(x_1)\int_{-\infty}^{t} C(t-t')\,dt' \;=\; h(t)\,q(x_1), \qquad 0\le x_1\le L, \tag{81}$$

with

$$\sigma_{11}(x_1,x_2) \;=\; \frac{M(x_1)\,x_2}{I} \quad\text{and}\quad \sigma_{12}(x_1,x_2) \;=\; \frac{1}{b(x_2)}\int_{x_2}^{c}\frac{\partial\sigma_{11}(x_1,x_2')}{\partial x_1}\,b(x_2')\,dx_2' \tag{82}$$

for $0\le x_1\le L$ and $-c\le x_2\le c$.

In this special one-dimensional case, the strains are also separable functions by virtue of the constitutive relations (56) and (81), resulting in a time-independent PR with the required value of 0.5 and the mandatory incompressible material ($K\to\infty$). However, for an anisotropic beam made of say composite materials, no such separation of variables solution is admissible [Hilton 1996] and the corresponding PR for such beams must be time-dependent.

Furthermore, if the applied loads are time-dependent then (79) changes to

$$m\,\frac{\partial^2 w(x_1,t)}{\partial t^2} \;+\; \int_{-\infty}^{t} E(t-t')\,I\,\frac{\partial^4 w(x_1,t')}{\partial x_1^4}\,dt' \;=\; q(x_1,t), \qquad 0\le x_1\le L, \tag{83}$$

and its solution is no longer separable even if the inertia term is neglected, unless the load is limited to the special expression $q(x_1,t)=g(t)\,f(x_1)$. In general the load can be represented by a Fourier series whose summands are of this form:

$$q(x_1,t) \;=\; \sum_{n=1}^{\infty} g_n(t)\;f_n(x_1), \qquad 0\le x_1\le L, \tag{84}$$

and the deflection $w(x_1,t)$ will also be a sum of separable functions

$$w(x,t) \;=\; \sum_{n=1}^{\infty} h_n(t)\;W_n(x), \tag{85}$$

where each of the the functions $W_n(x)$ individually satisfy all boundary conditions for all $n\ge 1$.

In this case the PRs will be time-dependent regardless of whether or not the inertia term is included. Table 1 summarizes these effects.

Load	Inertia	E	Deflection	PR
$q(x_1)$	Yes or No	$F(t)\,E^*_{ijkl}(x_1)$	$h(t)\;w^*(x_1)$	$\nu(x_1)=0.5$
$q(x_1)$	Yes or No	$E_{ijkl}(x_1,t)$	$w(x_1,t)$	$\nu(x_1,t)$
$g(t)\;q^*(x_1)$	No	$F(t)\,E^*_{ijkl}(x_1)$	$h(t)\;w^*(x_1)$	$\nu(x_1)=0.5$
$g(t)\;q^*(x_1)$	Yes	$F(t)\,E^*_{ijkl}(x_1)$	$w(x_1,t)$	$\nu(x_1,t)$
$g(t)\;q^*(x_1)$	No	$E_{ijkl}(x_1,t)$	$w(x_1,t)$	$\nu(x_1,t)$
$q(x_1,t)$	Yes or No	$F(t)\,E^*_{ijkl}(x_1)$	$w(x_1,t)$	$\nu(x_1,t)$
$q(x_1,t)$	Yes or No	$E_{ijkl}(x_1,t)$	$w(x_1,t)$	$\nu(x_1,t)$

Table 1. Euler–Bernoulli bending effects on class I PR.

Viscoelastic Timoshenko beams. Although the definition of the elastic Timoshenko shear coefficient is somewhat arbitrary, in that it is based on equalities of strain energies [Bert 1973] or deformations [Cowper 1966] between exact and approximate solutions to mention a few examples, the concept leads to relatively simple expressions depending only on beam cross sectional geometry and its elastic material properties. However, under either definitions the shear coefficient is dependent on the elastic PR, thus making it impossible to construct an elastic-viscoelastic analogy for this problem [Hilton 2009]. Unfortunately, a number of authors [Therriault 2003; Nakao et al. 1985; Singh and Abdelnaser 1993; Chen 1995] including the present one [Hilton and Vail 1993], have misinterpreted the possibility of the K_{SC} analogy and used it in inappropriate and incorrect settings. Space limitations do not allow to present the correct solution for the viscoelastic Timoshenko beam here; for a complete treatment see [Hilton 2009].

6. Concluding remarks

Poisson ratios are defined quantities and not fundamental material properties such as relaxation moduli and creep compliances which can be derived from first principles through the latter's dependence on thermodynamic derivatives. In linear viscoelasticity PRs are functions of time and stresses as well as their time histories and, therefore, are not universal universal property parameters such as moduli and compliances. It is, therefore, best to formulate viscoelastic analyses in terms of relaxation moduli or creep compliances without involving Poisson ratios.

The following points emerge from the above analyses:

(1) The fundamental problem with viscoelastic Poisson's ratios is not so much the diversity of their definitions, i.e. five classes, as it is with their proper use in constructing constitutive relations and correspondence principles involving PRs.

(2) In general, viscoelastic Poisson ratios can be time-independent if and only if displacements, strains and stresses as well as relaxation moduli and creep compliances are all separable unequal functions in time and space, and then PRs are limited to a single value of 0.5 for incompressible materials.

(3) A specific exception to the above exists if bulk, shear and Young's relaxation moduli obey identical time functions and stresses, displacements and moduli are separable spatial and temporal functions, then PRs are time-independent and in the elastic range $-1 \leq \nu \leq 0.5$. However, such a phenomenon where changes in shape and in volume exhibit the same time response remain unobserved in nature.

(4) Linear viscoelastic PRs are not limited to the elastic value range and may exceed it considerably in either direction, because of their dependence on stresses and stress time histories [Shtark et al. 2007; Lakes 1991].

(5) An assumption of time-independent viscoelastic Poisson ratios $\neq 0.5$ and without enforcement of the above enumerated conditions is not an admissible approximation, because ill posed overdeterminate problems result.

(6) The conventional elastic-viscoelastic analogy does not apply to expressions involving elastic or viscoelastic PRs based on the classical Poisson and other definitions (Classes I, IV and V), as seen in Table 2.

(7) Additionally, when the correspondence principle is inapplicable then no relations exist between complex PRs and complex moduli.

(8) Class II PRs based on one time-independent normal strain are always time-dependent, unless constant volume deformations are maintained.

(9) An elastic-viscoelastic correspondence principle based on the alternate Fourier transform PR definition (Class III) may be constructed, but these PRs have no physical counterparts.

(10) Viscoelastic PRs are derived material properties and unlike relaxation moduli are neither universal nor path-independent of loading conditions, since they depend on stress (loading) conditions and relaxation/creep properties as well their time histories.

(11) In the time space, it is possible to formulate viscoelastic constitutive relations in terms of PRs which bear resemblances to their elastic counterparts (Table 3). However, their forms do not lend themselves to the elastic-viscoelastic correspondence principle, except under very restrictive conditions; see Table 2.

(12) Simulation study results displayed in Figures 2, 3, and 4 clearly demonstrate that even for a simple time independent loading shear relaxation modulus, PRs and strains based on time-independent PRs are no measure of the exact values of these variables as the former lead to excessively large errors (ranging from 130% to 270% for the strain error in the examples considered), and constitute extremely poor approximations. Furthermore, any such arbitrarily assumed time-independent PRs ν_{AS} values do not lead to upper and lower bounds which could replace and bracket the experimentally unrecorded relaxation moduli, strains, etc.

(13) The time dependence of viscoelastic PRs makes them unsuitable to be characterized from experimental data and measurements in two normal directions must be employed. Alternately, simultaneous loadings, such as tractions and twisting for instance, may be employed on the same specimen.

Class	Name	Viscoelastic Poisson's Ratio $i \neq j$	Eq.	Analogy	Eq.
I	Classical	$\nu_{ij}(x,t) \stackrel{\text{def}}{=} -\dfrac{\epsilon_{\underline{jj}}(x,t)}{\epsilon_{\underline{ii}}(x,t)}$	(19)	NO	(39)
II	Constant strain	$\nu_{ij}^{\mathbf{C}}(x,t) \stackrel{\text{def}}{=} -\dfrac{\epsilon_{\underline{jj}}(x,t)}{\epsilon_{\underline{ii}}(x)}$	(20)	YES, but limited to $\epsilon_{11}(x)$ only	(47)
III	Transform	$\bar{\bar{\nu}}_{ij}^{\mathbf{A}}(x,\omega) \stackrel{\text{def}}{=} -\dfrac{\bar{\bar{\epsilon}}_{\underline{jj}}(x,\omega)}{\bar{\bar{\epsilon}}_{\underline{ii}}(x,\omega)}$	(21)	YES, but $\bar{\bar{\nu}}_{ij}^{\mathbf{A}}$ has no physical meaning	(48)
IV	Hencky	$\nu_{ij}^{\mathbf{H}}(x,t) \stackrel{\text{def}}{=} -\dfrac{\log\big(1+\epsilon_{\underline{jj}}(x,t)\big)}{\log\big(1+\epsilon_{\underline{ii}}(x,t)\big)}$	(22)	NO	–
V	Velocity	$\dfrac{\partial\nu_{ij}^{\mathbf{V}}(x,t)}{\partial t} \stackrel{\text{def}}{=} -\dfrac{\partial\epsilon_{\underline{jj}}(x,t)/\partial t}{\partial\epsilon_{\underline{ii}}(x,t)/\partial t}$	(23)	NO	(54)

Table 2. Poisson ratio elastic-viscoelastic correspondence principle (analogy). See equations (19)–(23) for the bibliographical references for each class.

Class	Constitutive relations
I	$\bar{\bar{\sigma}}_{\underline{ii}}(x,\omega) = \bar{\bar{E}}_{\underline{iiii}}(x,\omega)\,\bar{\bar{\epsilon}}_{\underline{ii}}(x,\omega) - 2\,\bar{\bar{E}}_{\underline{iikk}}(x,\omega)\,\overbrace{\overline{\overline{\nu_{\underline{ik}}\epsilon_{\underline{ii}}}}(x,\omega)}^{=-\bar{\bar{\epsilon}}_{\underline{kk}}(x,\omega)}, \quad i \neq k$
I	$\sigma_{\underline{ii}}(x,t) = \int_{-\infty}^{t} \big(E_{\underline{iiii}}(x,t-t')\,\epsilon_{\underline{ii}}(x,t') - 2\,E_{\underline{iikk}}(x,t-t')\,\underbrace{\nu_{\underline{ik}}(x,t')\,\epsilon_{\underline{ii}}(x,t')}_{=-\epsilon_{\underline{kk}}(x,t')}\big)\,dt', \quad i \neq k$
II	$\bar{\bar{\sigma}}_{\underline{ii}}(x,\omega) = \bar{\bar{E}}_{\underline{ii}11}(x,\omega)\,\epsilon_{11}(x) - 2\,\bar{\bar{E}}_{\underline{iikk}}(x,\omega)\,\overbrace{\bar{\bar{\nu}}^{\mathbf{C}}_{1\underline{k}}(x,\omega)\,\epsilon_{11}(x)}^{=-\bar{\bar{\epsilon}}_{\underline{kk}}(x,\omega)}, \quad k \neq i,\ k \neq 1$
II	$\sigma_{\underline{ii}}(x,t) = E_{\underline{ii}11}(x,t)\,\epsilon_{11}(x) - 2\int_{-\infty}^{t} E_{\underline{iikk}}(x,t-t')\,\underbrace{\nu^{\mathbf{C}}_{1\underline{k}}(x,t')\,\epsilon_{11}(x)}_{=-\epsilon_{\underline{kk}}(x,t')}\,dt', \quad k \neq i,\ k \neq 1$
III	$\bar{\bar{\sigma}}_{\underline{ii}}(x,\omega) = \big(\bar{\bar{E}}_{\underline{iiii}}(x,\omega) - 2\,\overbrace{\bar{\bar{E}}_{\underline{iikk}}(x,\omega)\,\bar{\bar{\nu}}^{\mathbf{A}}_{\underline{ik}}(x,\omega)}^{=\bar{\bar{E}}^{\mathbf{A}}_{\underline{iikk}}(x,\omega)}\big)\,\bar{\bar{\epsilon}}_{\underline{ii}}(x,\omega), \quad k \neq i$
III	$\sigma_{\underline{ii}}(x,t) = \int_{-\infty}^{t} \big(E_{\underline{iiii}}(x,t-t') - 2\,E^{\mathbf{A}}_{\underline{iikk}}(x,t-t')\big)\,\epsilon_{\underline{ii}}(x,t')\,dt', \quad k \neq i$
V	$\bar{\bar{\sigma}}_{\underline{ii}}(x,\omega) = \imath\,\omega\,\bar{\bar{\Phi}}^{N}_{\underline{iiii}}(x,\omega)\,\bar{\bar{\epsilon}}_{\underline{ii}}(x,\omega) - 2\,\bar{\bar{\Phi}}^{N}_{\underline{iikk}}(x,\omega)\,\overline{\overline{\frac{\partial \nu^{\mathbf{V}}_{\underline{ik}}}{\partial t}\,\frac{\partial \epsilon_{\underline{ii}}}{\partial t}}}(x,\omega), \quad k \neq i$
V	$\sigma_{\underline{ii}}(x,t) = \int_{-\infty}^{t} \left(\Phi^{N}_{\underline{ii}11}(x,t-t') - 2\,\Phi^{N}_{\underline{iikk}}(x,t-t')\,\frac{\partial \nu^{\mathbf{V}}_{\underline{ik}}(x,t')}{\partial t'}\right)\frac{\partial \epsilon_{\underline{ii}}(x,t')}{\partial t'}\,dt', \quad k \neq i$

Table 3. Linear isotropic constitutive relations with Poisson's ratios.

(14) In the final analysis, relaxation moduli, compliances, and creep and relaxation functions should be the characterizations of choice since they do not suffer the severe limitations of PRs, such as — even for linear materials — dependence on stress, strain and displacement time histories. Furthermore, they properly allow use of the elastic-viscoelastic correspondence principle without additional constraints.

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HARRY H. HILTON: h-hilton@illinois.edu
Aerospace Engineering Department (AE) and Technology Research, Education and Commercialization Center (TRECC), National Center for Supercomputing Applications (NCSA), University of Illinois at Urbana–Champaign, 104 South Wright Street, MC-236, Urbana, IL 61801-2935, United States