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UNIFORMITY OF STRESSES INSIDE AN ANISOTROPIC ELLIPTICAL INHOMOGENEITY WITH AN IMPERFECT INTERFACE

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By employing the Stroh formalism for two-dimensional anisotropic elasticity, we find that a uniform stress field exists inside an anisotropic elliptical inhomogeneity imperfectly bonded to an infinite anisotropic matrix subject to uniform stresses and strains at infinity. Here, the behavior of the imperfect interface between the inhomogeneity and the matrix is characterized by the linear spring model with vanishing thickness. The degree of imperfections, both normal and in-plane tangential to the interface, are assumed to be equal. A particular form of the interface function that leads to a uniform stress field within the anisotropic elliptical inhomogeneity is identified. Also presented are real form expressions for the stress field inside the inhomogeneity that are shown to be valid for mathematically degenerate (isotropic) material as well. We note that the interpenetration issue that arises from application of the linear spring model to the imperfect interface is not discussed here.

1. Introduction

Eshelby's celebrated results [1957; 1959; 1961] demonstrated that the stress field inside an anisotropic ellipsoidal inhomogeneity is uniform when the infinite matrix is subject to remote uniform stresses. The corresponding two-dimensional anisotropic elliptical inhomogeneity was discussed by Hwu and Ting [1989]. These authors also found that the stress field inside an anisotropic elliptical inhomogeneity is uniform when the infinite matrix is subject to remote uniform stresses. In [Eshelby 1957; 1959; 1961; Hwu and Ting 1989], the inhomogeneity-matrix interface was assumed to be perfect, such that tractions and displacements across the interface are continuous. In recent years, problems involving inhomogeneities with imperfect bonding at the inhomogeneity-matrix interface have attracted great interest [Achenbach and Zhu 1989; Hashin 1991; Gao 1995; Ru and Schiavone 1997; Shen et al. 2001; Antipov and Schiavone 2003]. The behavior of the imperfect interface is commonly simulated by the spring layer model with vanishing thickness. In this model, tractions are continuous but displacements are discontinuous across the interface. More precisely, the jumps in displacement components are proportional (in terms of the 'spring-factor-type' interface functions or interface parameters) to the respective traction components. Hashin [1991] found that the stress field inside a spherical inhomogeneity imperfectly bonded to a three-dimensional matrix is intrinsically *nonuniform* under a remote uniform stress field. The results of [Gao 1995; Shen et al. 2001] also demonstrated that the stress field inside a circular or elliptical inhomogeneity imperfectly bonded to a matrix is nonuniform under a remote uniform stress field.

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More recently, it was found that the stress field inside an elliptical inhomogeneity imperfectly bonded to a matrix can still be uniform under a remote uniform antiplane stress field [Antipov and Schiavone 2003] or a remote uniform in-plane stress field [Wang et al. 2008] for a special form of the interface function. In these investigations, the elastic properties of the inhomogeneity and the matrix were assumed to be *isotropic*. The purpose of the research presented here is to address whether a uniform stress field exists inside a generally anisotropic elliptical inhomogeneity imperfectly bonded to an infinite anisotropic matrix subject to uniform stresses and strains at infinity.

In this research, the Stroh formalism [Stroh 1958; Ting 1996] is employed to investigate the two-dimensional problem associated with an anisotropic elliptical inhomogeneity imperfectly bonded to an infinite matrix under a remote uniform stress field. We find a particular type of interface function that leads to a uniform stress field inside the anisotropic elliptical inhomogeneity. We also present the real-form expressions of the uniform stress field by using the identities established in [Ting 1996].

2. Stroh formalism for two-dimensional anisotropic elasticity

The basic equations for a linear anisotropic elastic material are

$$\sigma_{ij,j}=0, \qquad \varepsilon_{ij}=\tfrac{1}{2}\left(u_{i,j}+u_{j,i}\right), \qquad \sigma_{ij}=C_{ijkl}\varepsilon_{kl}, \tag{1}$$

where u_i are displacement components, σ_{ij} and ε_{ij} are the stresses and strains, and C_{ijkl} are the elastic constants. For a two-dimensional problem, a solution exists of the form

$$\boldsymbol{u}=\begin{bmatrix}u_1 & u_2 & u_3\end{bmatrix}^T=\boldsymbol{a}f\left(x_1+px_2\right), \tag{2}$$

where $\boldsymbol{a}$ is a 3×1 column, p is a complex number, and $f(*)$ is an analytic function. (The justification for this and for many of the assertions in this section can be found in [Ting 1996].)

Thus equations (1) are satisfied for an arbitrary function $f(*)$ if

$$\left(\boldsymbol{Q}+p\left(\boldsymbol{R}+\boldsymbol{R}^T\right)+p^2\boldsymbol{T}\right)\boldsymbol{a}=\boldsymbol{0}, \tag{3}$$

where the 3×3 real matrix $\boldsymbol{R}$ and the two 3×3 symmetric matrices $\boldsymbol{Q}$ and $\boldsymbol{T}$ are defined by

$$Q_{ik}=C_{i1k1}, \qquad R_{ik}=C_{i1k2}, \qquad T_{ik}=C_{i2k2}. \tag{4}$$

For a stable material with positive-definite energy density, the six roots of (3) form three distinct conjugate pairs with nonzero imaginary parts. If p_i (where $i=1,2,3$) are the three distinct roots with positive imaginary parts, and $\boldsymbol{a}_i$ are the associated eigenvectors, then the general solution is given by

$$\boldsymbol{u}=\begin{bmatrix}u_1 & u_2 & u_3\end{bmatrix}^T=\boldsymbol{A}\boldsymbol{f}(z)+\bar{\boldsymbol{A}}\,\overline{\boldsymbol{f}(z)}, \qquad \boldsymbol{\Phi}=\begin{bmatrix}\Phi_1 & \Phi_2 & \Phi_3\end{bmatrix}^T=\boldsymbol{A}\boldsymbol{f}(z)+\bar{\boldsymbol{B}}\,\overline{\boldsymbol{f}(z)}, \tag{5}$$

where

$$\begin{gathered}
\boldsymbol{b}_i=\left(\boldsymbol{R}^T+p_i\boldsymbol{T}\right)\boldsymbol{a}_i=-p_i^{-1}\left(\boldsymbol{Q}+p_i\boldsymbol{R}\right)\boldsymbol{a}_i, \quad (i=1,2,3),\\
\boldsymbol{A}=\begin{bmatrix}\boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3\end{bmatrix}, \qquad \boldsymbol{B}=\begin{bmatrix}\boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3\end{bmatrix},\\
\boldsymbol{f}(z)=\begin{bmatrix}f_1(z_1) & f_2(z_2) & f_3(z_3)\end{bmatrix}^T,\\
z_i=x_1+p_ix_2, \qquad \operatorname{Im}\{p_i\}>0, \qquad (i=1,2,3).
\end{gathered} \tag{6}$$

Thus, the stresses are given by

$$\sigma_{i1} = -\Phi_{i,2}, \qquad \sigma_{i2} = \Phi_{i,1}, \qquad (i = 1, 2, 3). \tag{7}$$

The second-order eigenvalue problem (3) can also be recast into the standard first-order eigenvalue problem,

$$\boldsymbol{N} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = p \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}, \tag{8}$$

where

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}_1 & \boldsymbol{N}_2 \\ \boldsymbol{N}_3 & \boldsymbol{N}_1^T \end{bmatrix} = \begin{bmatrix} -\boldsymbol{T}^{-1}\boldsymbol{R}^T & \boldsymbol{T}^{-1} \\ \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^T - \boldsymbol{Q} & -\boldsymbol{R}\boldsymbol{T}^{-1} \end{bmatrix}. \tag{9}$$

Due to the fact that the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy the normalized orthogonal relationship

$$\begin{bmatrix} \boldsymbol{B}^T & \boldsymbol{A}^T \\ \bar{\boldsymbol{B}}^T & \bar{\boldsymbol{A}}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \bar{\boldsymbol{A}} \\ \boldsymbol{B} & \bar{\boldsymbol{B}} \end{bmatrix} = \boldsymbol{I}, \tag{10}$$

the real matrices $\boldsymbol{S}$, $\boldsymbol{H}$, and $\boldsymbol{L}$ can be introduced,

$$\boldsymbol{S} = i\big(2\boldsymbol{A}\boldsymbol{B}^T - \boldsymbol{I}\big), \qquad \boldsymbol{H} = 2i\boldsymbol{A}\boldsymbol{A}^T, \qquad \boldsymbol{L} = -2i\boldsymbol{B}\boldsymbol{B}^T, \tag{11}$$

Furthermore, $\boldsymbol{H}$ and $\boldsymbol{L}$ are symmetric, while $\boldsymbol{S}\boldsymbol{H}$, $\boldsymbol{L}\boldsymbol{S}$, $\boldsymbol{H}^{-1}\boldsymbol{S}$, and $\boldsymbol{S}\boldsymbol{L}^{-1}$ are antisymmetric. The development presented here uses the identities

$$\begin{aligned} 2\boldsymbol{A}\langle p_\alpha\rangle\boldsymbol{A}^T &= \boldsymbol{N}_2 - i\big(\boldsymbol{N}_1\boldsymbol{H} + \boldsymbol{N}_2\boldsymbol{S}^T\big), \\ 2\boldsymbol{A}\langle p_\alpha\rangle\boldsymbol{B}^T &= \boldsymbol{N}_1 + i\big(\boldsymbol{N}_2\boldsymbol{L} - \boldsymbol{N}_1\boldsymbol{S}\big), \\ 2\boldsymbol{B}\langle p_\alpha\rangle\boldsymbol{B}^T &= \boldsymbol{N}_3 + i\big(\boldsymbol{N}_1^T\boldsymbol{L} - \boldsymbol{N}_3\boldsymbol{S}\big), \end{aligned} \tag{12}$$

where $\langle * \rangle$ is a 3×3 diagonal matrix in which each component is varied according to the index α.

Let $\boldsymbol{t}$ be the surface traction on a boundary Γ. If s is the arc-length measured along Γ so that, when facing the direction of increasing s, the material is on the right-hand side, it can be shown that [Stroh 1958; Ting 1996]

$$\boldsymbol{t} = \frac{d\boldsymbol{\Phi}}{ds}. \tag{13}$$

To simplify the analysis, we set $x = x_1$, $y = x_2$, and $z = x_3$.

3. Uniform stress field inside the anisotropic elliptical inhomogeneity

Consider an infinite domain in R^2 that contains a single internal anisotropic elastic inhomogeneity with elastic properties that are different from those of the surrounding anisotropic matrix, as shown in Figure 1. The elastic constants are $C_{ijkl}^{(1)}$ for the inhomogeneity and $C_{ijkl}^{(2)}$ for the surrounding matrix. The inhomogeneity occupies the elliptical region

$$S_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1,$$

and the matrix domain is given by

$$S_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \ge 1.$$

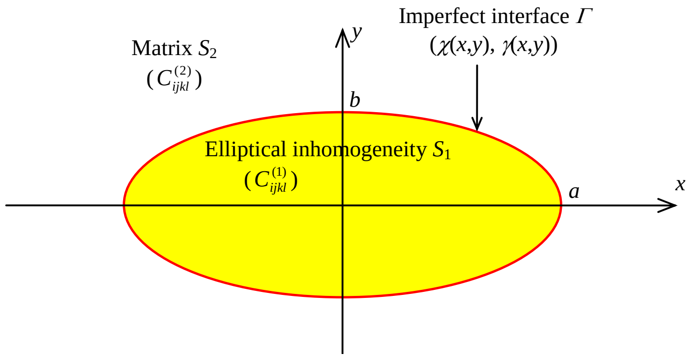

Figure 1. An anisotropic elliptical inhomogeneity imperfectly bonded to an infinite matrix.

The ellipse Γ, whose semimajor and semiminor axes are a and b, respectively, denotes the interface between the inhomogeneity and matrix. In the following discussion, the subscripts or superscripts 1 and 2 refer to the regions S_1 and S_2, respectively. At infinity, the matrix is subject to remote uniform stresses

$$\boldsymbol{t}_2^{\infty} = [\, \sigma_{12}^{\infty} \;\sigma_{22}^{\infty} \;\sigma_{32}^{\infty} \,]^T,$$

and remote uniform strains

$$\boldsymbol{e}_1^{\infty} = [\, \varepsilon_{11}^{\infty} \;\varepsilon_{12}^{\infty} \;2\varepsilon_{31}^{\infty} \,]^T.$$

Without losing generality, the rigid body rotation at infinity is assumed to be zero. The inhomogeneity is imperfectly bonded to the matrix. The boundary conditions on the imperfect interface Γ are given by

$$\begin{aligned} \sigma_{rr}^{(1)} &= \sigma_{rr}^{(2)} = \chi(x, y)\big(u_r^{(2)} - u_r^{(1)}\big), \\ \sigma_{r\theta}^{(1)} &= \sigma_{r\theta}^{(2)} = \chi(x, y)\big(u_\theta^{(2)} - u_\theta^{(1)}\big), \\ \sigma_{r3}^{(1)} &= \sigma_{r3}^{(2)} = \gamma(x, y)\big(u_3^{(2)} - u_3^{(1)}\big), \end{aligned} \tag{14}$$

where $\chi(x, y)$ and $\gamma(x, y)$ are nonnegative imperfect interface functions whose values depend on the coordinates x and y of Γ; $\sigma_{rr}^{(k)}$, $\sigma_{r\theta}^{(k)}$, and $\sigma_{r3}^{(k)}$ $(k = 1, 2)$ denote the traction components along the normal, in-plane tangential, and antiplane directions of the interface, respectively; and $u_r^{(k)}$, $u_\theta^{(k)}$, and $u_3^{(k)}$ $(k = 1, 2)$ denote the displacement components along the normal, in-plane tangential, and antiplane directions of the interface, respectively. Equation (14) demonstrates that the same degree of imperfection is realized in both the normal and in-plane tangential directions. A perfectly bonded interface is achieved for $\chi \to \infty$, $\gamma \to \infty$, while a traction-free surface is achieved for $\chi \to 0$, $\gamma \to 0$. Note that a jump in the normal displacement may give rise to interpenetration of the inhomogeneity and the matrix in some regions of the interface. This issue will not be discussed here.

Combining (13) and (14), we can equivalently write the boundary conditions on Γ as

$$\boldsymbol{\Phi}_1 = \boldsymbol{\Phi}_2, \qquad -\frac{d\boldsymbol{\Phi}_1}{ds} = \boldsymbol{t} = \boldsymbol{\Omega}(x, y)\big(\boldsymbol{u}_2 - \boldsymbol{u}_1\big), \tag{15}$$

where the increasing s is in the counterclockwise direction of the interface, and

$$\mathbf{\Omega}(x, y) = \begin{bmatrix} \chi(x, y) & 0 & 0 \\ 0 & \chi(x, y) & 0 \\ 0 & 0 & \gamma(x, y) \end{bmatrix}. \tag{16}$$

In order to address the boundary value problem, we first consider the mapping functions

$$z_\alpha = x + p_\alpha y = m_\alpha(\zeta_\alpha) = \tfrac{1}{2}(a - ip_\alpha b)\zeta_\alpha + \tfrac{1}{2}(a + ip_\alpha b)\zeta_\alpha^{-1} \qquad (\alpha = 1, 2, 3), \tag{17}$$

which map the elliptical region with a cut in the z_α-plane onto the annulus

$$\sqrt{\left|\frac{a + ip_\alpha b}{a - ip_\alpha b}\right|} \le |\zeta_\alpha| \le 1,$$

in the ζ_α-plane. p_α $(\alpha = 1, 2, 3)$ are the three Stroh eigenvalues pertaining to the inhomogeneity.

Second, we consider the mapping functions

$$z_\alpha^* = x + p_\alpha^* y = m_\alpha^*(\zeta_\alpha^*) = \tfrac{1}{2}(a - ip_\alpha^* b)\zeta_\alpha^* + \tfrac{1}{2}(a + ip_\alpha^* b)\zeta_\alpha^{*-1} \qquad (\alpha = 1, 2, 3), \tag{18}$$

which map the outside of an elliptical region in the z_α^*-plane onto the outside of unit circle, $|\zeta_\alpha^*| \ge 1$, in the ζ_α^*-plane. p_α^*, $(\alpha = 1, 2, 3)$ are the three Stroh eigenvalues pertaining to the matrix.

Third, we consider the mapping function [Muskhelishvili 1953]

$$z = x + iy = m(\zeta) = \tfrac{1}{2}(a + b)\zeta + \tfrac{1}{2}(a - b)\zeta^{-1}, \tag{19}$$

which maps the region $S_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \ge 1$ onto $|\zeta| \ge 1$ in the ζ-plane.

Given that $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_1^* = \zeta_2^* = \zeta_3^* = \zeta$ on Γ, we can replace ζ_α and ζ_α^* by ζ. The variable ζ can be easily substituted with ζ_α or ζ_α^* at the end of the analysis.

Extending the developments outlined in [Antipov and Schiavone 2003] and [Wang et al. 2008], the two interface functions $\chi(x, y)$ and $\gamma(x, y)$ are chosen to be

$$\chi(x, y) = \frac{1}{\lambda_1 |m'(\zeta)|}, \qquad \gamma(x, y) = \frac{1}{\lambda_2 |m'(\zeta)|} \qquad (|\zeta| = 1), \tag{20}$$

where λ_1 and λ_2 are two nonnegative constants, and $|m'(\zeta)|$ is explicitly given by

$$|m'(\zeta)| = \tfrac{1}{2}|(a + b) - (a - b)\zeta^{-2}| = b\sqrt{1 + b^* \sin^2\theta}, \qquad \zeta = e^{i\theta} \text{ on } \Gamma, \tag{21}$$

with $b^* = (a^2 - b^2)/b^2$.

In this case, $\mathbf{\Omega}(x, y)$ in (16) can also be expressed as

$$\mathbf{\Omega}(x, y) = \frac{\mathbf{\Lambda}^{-1}}{|m'(\zeta)|}, \qquad \text{where } \mathbf{\Lambda} = \operatorname{diag}[\lambda_1\ \lambda_1\ \lambda_2]. \tag{22}$$

The boundary conditions in (15) on the imperfect interface can also be expressed in terms of the analytic function vectors $\boldsymbol{f}_1(\zeta)$ and $\boldsymbol{f}_2(\zeta)$, for $|\zeta| = 1$, as

$$\begin{gathered} \boldsymbol{B}_2 \boldsymbol{f}_2^-(\zeta) + \bar{\boldsymbol{B}}_2 \bar{\boldsymbol{f}}_2^+(1/\zeta) = \boldsymbol{B}_1 \boldsymbol{f}_1^+(\zeta) + \bar{\boldsymbol{B}}_1 \bar{\boldsymbol{f}}_1^-(1/\zeta), \\ \boldsymbol{A}_2 \boldsymbol{f}_2^-(\zeta) + \bar{\boldsymbol{A}}_2 \bar{\boldsymbol{f}}_2^+(1/\zeta) - \boldsymbol{A}_1 \boldsymbol{f}_1^+(\zeta) - \bar{\boldsymbol{A}}_1 \bar{\boldsymbol{f}}_1^-(1/\zeta) = -i\mathbf{\Lambda}\big(\zeta \boldsymbol{B}_1 \boldsymbol{f}_1'^+(\zeta) - \zeta^{-1} \bar{\boldsymbol{B}}_1 \bar{\boldsymbol{f}}_1'^-(1/\zeta)\big). \end{gathered} \tag{23}$$

To ensure that the stress field inside the elliptical inhomogeneity is uniform, assume that

$$f_1(\zeta) = \tfrac{1}{2}\langle (a - i p_\alpha b)\zeta + (a + i p_\alpha b)\zeta^{-1}\rangle \boldsymbol{k}, \tag{24}$$

where $\boldsymbol{k}$ is a 3×1 vector to be determined.

Substituting (24) into (23), we obtain, for $|\zeta| = 1$,

$$\begin{aligned} \boldsymbol{B}_2 \boldsymbol{f}_2^-(\zeta) + \overline{\boldsymbol{B}}_2 \bar{\boldsymbol{f}}_2^+(1/\zeta) &= \tfrac{1}{2}\zeta\big(\boldsymbol{B}_1\langle a - i p_\alpha b\rangle \boldsymbol{k} + \overline{\boldsymbol{B}}_1\langle a - i\bar{p}_\alpha b\rangle \bar{\boldsymbol{k}}\big) \\ &\quad + \tfrac{1}{2}\zeta^{-1}\big(\boldsymbol{B}_1\langle a + i p_\alpha b\rangle \boldsymbol{k} + \overline{\boldsymbol{B}}_1\langle a + i\bar{p}_\alpha b\rangle \bar{\boldsymbol{k}}\big), \\ \boldsymbol{A}_2 \boldsymbol{f}_2^-(\zeta) + \overline{\boldsymbol{A}}_2 \bar{\boldsymbol{f}}_2^+(1/\zeta) &= \tfrac{1}{2}\zeta\big((\boldsymbol{A}_1 - i\boldsymbol{\Lambda}\boldsymbol{B}_1)\langle a - i p_\alpha b\rangle \boldsymbol{k} + (\overline{\boldsymbol{A}}_1 - i\boldsymbol{\Lambda}\overline{\boldsymbol{B}}_1)\langle a - i\bar{p}_\alpha b\rangle \bar{\boldsymbol{k}}\big) \\ &\quad + \tfrac{1}{2}\zeta^{-1}\big((\boldsymbol{A}_1 + i\boldsymbol{\Lambda}\boldsymbol{B}_1)\langle a + i p_\alpha b\rangle \boldsymbol{k} + (\overline{\boldsymbol{A}}_1 + i\boldsymbol{\Lambda}\overline{\boldsymbol{B}}_1)\langle a + i\bar{p}_\alpha b\rangle \bar{\boldsymbol{k}}\big). \end{aligned} \tag{25}$$

Given that $\boldsymbol{f}_2(\zeta) \cong \tfrac{1}{2}\zeta\langle a - i p_\alpha^* b\rangle\big(\boldsymbol{B}_2^T \boldsymbol{e}_1^\infty + \boldsymbol{A}_2^T \boldsymbol{t}_2^\infty\big)$ as $\zeta \to \infty$, it follows from $(25)_1$ that, for $|\zeta| \geq 1$,

$$\begin{aligned} \boldsymbol{f}_2(\zeta) = \tfrac{1}{2}\zeta^{-1}\Big(&\boldsymbol{B}_2^{-1}\boldsymbol{B}_1\langle a + i p_\alpha b\rangle \boldsymbol{k} + \boldsymbol{B}_2^{-1}\overline{\boldsymbol{B}}_1\langle a + i\bar{p}_\alpha b\rangle \bar{\boldsymbol{k}} \\ &- \boldsymbol{B}_2^{-1}\overline{\boldsymbol{B}}_2\langle a + i\bar{p}_\alpha^* b\rangle\big(\overline{\boldsymbol{B}}_2^T \boldsymbol{e}_1^\infty + \overline{\boldsymbol{A}}_2^T \boldsymbol{t}_2^\infty\big)\Big) + \tfrac{1}{2}\zeta\langle a - i p_\alpha^* b\rangle\big(\boldsymbol{B}_2^T \boldsymbol{e}_1^\infty + \boldsymbol{A}_2^T \boldsymbol{t}_2^\infty\big), \end{aligned} \tag{26}$$

Similarly, it follows from $(25)_2$ that, for $|\zeta| \geq 1$,

$$\begin{aligned} \boldsymbol{f}_2(\zeta) = \tfrac{1}{2}\zeta^{-1}\Big(&\boldsymbol{A}_2^{-1}\big(\boldsymbol{A}_1 + i\boldsymbol{\Lambda}\boldsymbol{B}_1\big)\langle a + i p_\alpha b\rangle \boldsymbol{k} + \boldsymbol{A}_2^{-1}\big(\overline{\boldsymbol{A}}_1 + i\boldsymbol{\Lambda}\overline{\boldsymbol{B}}_1\big)\langle a + i\bar{p}_\alpha b\rangle \bar{\boldsymbol{k}} \\ &- \boldsymbol{A}_2^{-1}\overline{\boldsymbol{A}}_2\langle a + i\bar{p}_\alpha^* b\rangle\big(\overline{\boldsymbol{B}}_2^T \boldsymbol{e}_1^\infty + \overline{\boldsymbol{A}}_2^T \boldsymbol{t}_2^\infty\big)\Big) + \tfrac{1}{2}\zeta\langle a - i p_\alpha^* b\rangle\big(\boldsymbol{B}_2^T \boldsymbol{e}_1^\infty + \boldsymbol{A}_2^T \boldsymbol{t}_2^\infty\big). \end{aligned} \tag{27}$$

The obtained expressions for $\boldsymbol{f}_2(\zeta)$ must be compatible with each other (or must be equal), and the following set of linear algebraic equations can be obtained

$$\begin{aligned} &\big(\boldsymbol{B}_1\langle a{+}i p_\alpha b\rangle - \boldsymbol{B}_2\boldsymbol{A}_2^{-1}\big(\boldsymbol{A}_1{+}i\boldsymbol{\Lambda}\boldsymbol{B}_1\big)\langle a{+}i p_\alpha b\rangle\big)\boldsymbol{k} + \big(\overline{\boldsymbol{B}}_1\langle a{+}i\bar{p}_\alpha b\rangle - \boldsymbol{B}_2\boldsymbol{A}_2^{-1}\big(\overline{\boldsymbol{A}}_1{+}i\boldsymbol{\Lambda}\overline{\boldsymbol{B}}_1\big)\langle a{+}i\bar{p}_\alpha b\rangle\big)\bar{\boldsymbol{k}} \\ &\qquad = \big(\overline{\boldsymbol{B}}_2 - \boldsymbol{B}_2\boldsymbol{A}_2^{-1}\overline{\boldsymbol{A}}_2\big)\langle a + i\bar{p}_\alpha^* b\rangle\big(\overline{\boldsymbol{B}}_2^T \boldsymbol{e}_1^\infty + \overline{\boldsymbol{A}}_2^T \boldsymbol{t}_2^\infty\big). \end{aligned} \tag{28}$$

Consequently, the unknown vector $\boldsymbol{k}$ can be uniquely determined,

$$\boldsymbol{k} = \boldsymbol{B}_1^{-1}\big(\boldsymbol{E}_2^{-1}\boldsymbol{E}_1 - \overline{\boldsymbol{E}}_1^{-1}\overline{\boldsymbol{E}}_2\big)^{-1}\big(\boldsymbol{E}_2^{-1}\boldsymbol{g} - \overline{\boldsymbol{E}}_1^{-1}\bar{\boldsymbol{g}}\big), \tag{29}$$

where the 3×3 matrices $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, and the 3×1 vector $\boldsymbol{g}$ are given by

$$\begin{aligned} \boldsymbol{E}_1 &= a\boldsymbol{I} - a\boldsymbol{M}_2\big(\boldsymbol{M}_1^{-1} - \boldsymbol{\Lambda}\big) + b\big(\boldsymbol{I} + \boldsymbol{M}_2\boldsymbol{\Lambda}\big)\big(\boldsymbol{N}_3\boldsymbol{L}_1^{-1} + i\big(\boldsymbol{N}_1^T - \boldsymbol{N}_3\boldsymbol{S}_1\boldsymbol{L}_1^{-1}\big)\big) \\ &\qquad + b\boldsymbol{M}_2\big(\big(\boldsymbol{N}_2 - \boldsymbol{N}_1\boldsymbol{S}_1\boldsymbol{L}_1^{-1}\big) - i\boldsymbol{N}_1\boldsymbol{L}_1^{-1}\big), \\ \boldsymbol{E}_2 &= a\boldsymbol{I} + a\boldsymbol{M}_2\big(\overline{\boldsymbol{M}}_1^{-1} + \boldsymbol{\Lambda}\big) - b\big(\boldsymbol{I} + \boldsymbol{M}_2\boldsymbol{\Lambda}\big)\big(\boldsymbol{N}_3\boldsymbol{L}_1^{-1} + i\big(\boldsymbol{N}_3\boldsymbol{S}_1\boldsymbol{L}_1^{-1} - \boldsymbol{N}_1^T\big)\big) \\ &\qquad + b\boldsymbol{M}_2\big(\big(\boldsymbol{N}_2 - \boldsymbol{N}_1\boldsymbol{S}_1\boldsymbol{L}_1^{-1}\big) + i\boldsymbol{N}_1\boldsymbol{L}_1^{-1}\big), \\ \boldsymbol{g} &= \big(\boldsymbol{H}_2^{-1}\big(b\boldsymbol{N}_1^* + a\boldsymbol{S}_2\big) + i\boldsymbol{H}_2^{-1}\big(b\boldsymbol{N}_1^*\boldsymbol{S}_2 - b\boldsymbol{N}_2^*\boldsymbol{L}_2 - a\boldsymbol{I}\big)\big)\boldsymbol{e}_1^\infty \\ &\qquad + \big(\big(a\boldsymbol{I} + b\boldsymbol{H}_2^{-1}\boldsymbol{N}_2^*\big) + ib\boldsymbol{H}_2^{-1}\big(\boldsymbol{N}_1^*\boldsymbol{H}_2 + \boldsymbol{N}_2^*\boldsymbol{S}_2^T\big)\big)\boldsymbol{t}_2^\infty, \end{aligned} \tag{30}$$

with

$$\boldsymbol{M}_k = -i\boldsymbol{B}_k\boldsymbol{A}_k^{-1} = \boldsymbol{H}_k^{-1}(\boldsymbol{I}+i\boldsymbol{S}_k) = \boldsymbol{L}_k(\boldsymbol{I}-i\boldsymbol{S}_k)^{-1} \quad (k=1,2). \tag{31}$$

In (30), $\boldsymbol{N}_1$, $\boldsymbol{N}_2$, $\boldsymbol{N}_3$ are the real matrices defined in (9) for the inhomogeneity, and $\boldsymbol{N}_1^*$, $\boldsymbol{N}_2^*$, $\boldsymbol{N}_3^*$ are those for the surrounding matrix. Note that in the course of this derivation, we have utilized the identities (11) and (12).

The uniform stress field within the elliptical inhomogeneity can now be given by

$$\begin{bmatrix}\sigma_{12}\\ \sigma_{22}\\ \sigma_{32}\end{bmatrix} = 2\,\mathrm{Re}\{\boldsymbol{B}_1\boldsymbol{k}\} = 2\,\mathrm{Re}\{(\boldsymbol{E}_2^{-1}\boldsymbol{E}_1-\bar{\boldsymbol{E}}_1^{-1}\bar{\boldsymbol{E}}_2)^{-1}(\boldsymbol{E}_2^{-1}\boldsymbol{g}-\bar{\boldsymbol{E}}_1^{-1}\bar{\boldsymbol{g}})\}, \tag{32}$$

$$\begin{aligned}\begin{bmatrix}\sigma_{11}\\ \sigma_{21}\\ \sigma_{31}\end{bmatrix} &= -2\,\mathrm{Re}\{\boldsymbol{B}_1\langle p_\alpha\rangle\boldsymbol{k}\}\\ &= 2\,\mathrm{Re}\{(\boldsymbol{N}_3\boldsymbol{S}_1\boldsymbol{L}_1^{-1}-\boldsymbol{N}_1^T+i\boldsymbol{N}_3\boldsymbol{L}_1^{-1})(\boldsymbol{E}_2^{-1}\boldsymbol{E}_1-\bar{\boldsymbol{E}}_1^{-1}\bar{\boldsymbol{E}}_2)^{-1}(\boldsymbol{E}_2^{-1}\boldsymbol{g}-\bar{\boldsymbol{E}}_1^{-1}\bar{\boldsymbol{g}})\}.\end{aligned} \tag{33}$$

We note that $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{g}$, defined in (30), are expressed in terms of the real Barnett–Lothe tensors $\boldsymbol{H}_k$, $\boldsymbol{L}_k$, $\boldsymbol{S}_k$ $(k=1,2)$, whose explicit expressions are given in [Dongye and Ting 1989; Ting 1997], and (ii) the three 3×3 real matrices $\boldsymbol{N}_i$ $(i=1,2,3)$ for the inhomogeneity, and the three 3×3 real matrices $\boldsymbol{N}_i^*$ $(i=1,2,3)$ for the matrix. In other words, the expressions of $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{g}$ do not contain the Stroh eigenvalues or eigenvectors, thus the expressions of stresses (and also strains and rigid-body rotation) within the elliptical inhomogeneity are also valid for (mathematically) degenerate materials, such as an isotropic material. Note from (29) that the internal stress field depends on the imperfection of the interface characterized by the two nonnegative interface constants λ_1 and λ_2. Finally, the full field expression of $\boldsymbol{f}_2(\zeta)$ can be easily obtained,

$$\begin{aligned}\boldsymbol{f}_2(\zeta) = \tfrac{1}{2}\langle\zeta_\alpha^{*-1}\rangle\big(\boldsymbol{B}_2^{-1}\boldsymbol{B}_1\langle a+ip_\alpha b\rangle\boldsymbol{k}+\boldsymbol{B}_2^{-1}\bar{\boldsymbol{B}}_1\langle a+i\bar{p}_\alpha b\rangle\bar{\boldsymbol{k}}-\boldsymbol{B}_2^{-1}\bar{\boldsymbol{B}}_2\langle a+i\bar{p}_\alpha^* b\rangle(\bar{\boldsymbol{B}}_2^T\boldsymbol{e}_1^\infty+\bar{\boldsymbol{A}}_2^T\boldsymbol{t}_2^\infty)\big)\\ +\tfrac{1}{2}\langle\zeta_\alpha^*\rangle\langle a-ip_\alpha^* b\rangle(\boldsymbol{B}_2^T\boldsymbol{e}_1^\infty+\boldsymbol{A}_2^T\boldsymbol{t}_2^\infty),\end{aligned} \tag{34}$$

for $|\zeta_\alpha^*|\geq 1$, which clearly indicates that the remote uniform stresses and strains are disturbed by the imperfectly bonded elliptical inhomogeneity.

4. Conclusions

In this investigation, we found that a uniform stress field exists inside an anisotropic elliptical inhomogeneity imperfectly bonded to an infinite matrix provided that: (i) the same degree of imperfection on the interface is realized in both the normal and the in-plane tangential directions; and (ii) the interface functions satisfy equation (20). Condition (i) has been adopted in previous studies of isotropic materials [Wang et al. 2005; Wang et al. 2008], while the circumferential inhomogeneity of the imperfect interface reflects a realistic scenario of inhomogeneous interface damage in which the extent of bonding varies along the interface [Ru and Schiavone 1997]. The interface functions given by equation (20), which lead to a uniform stress field inside the inhomogeneity, depend only on the shape of the ellipse Γ, that is, the semimajor and semiminor axes a and b are independent of the material properties of both inhomogeneity and matrix. For a circular inhomogeneity $a=b$, it follows from (20) and (21) that $\chi(x,y)=1/(\lambda_1 a)$

and $\gamma(x, y) = 1/(\lambda_2 a)$ are constant. In other words, the stress field inside an anisotropic circular inhomogeneity will be uniform when the interface is homogeneously imperfect and when the same degree of imperfection on the circular interface is realized in both the normal and the in-plane tangential directions. The interpenetration issue due to the introduction of the imperfect interface is not discussed here. Finally, our work poses an interesting question: Can the stress field inside an imperfectly bonded anisotropic ellipsoidal inhomogeneity still remain uniform when the infinite anisotropic matrix is subject to remote uniform stresses?

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References

[Achenbach and Zhu 1989] J. D. Achenbach and H. Zhu, "Effect of interfacial zone on mechanical behavior and failure of fiber-reinforced composites", *J. Mech. Phys. Solids* **37**:3 (1989), 381–393.

[Antipov and Schiavone 2003] Y. A. Antipov and P. Schiavone, "On the uniformity of stresses inside an inhomogeneity of arbitrary shape", *IMA J. Appl. Math.* **68**:3 (2003), 299–311.

[Dongye and Ting 1989] C. Dongye and T. C. T. Ting, "Explicit expressions of Barnett–Lothe tensors and their associated tensors for orthotropic materials", *Quart. Appl. Math.* **47** (1989), 723–734.

[Eshelby 1957] J. D. Eshelby, "The determination of the elastic field of an ellipsoidal inclusion, and related problems", *Proc. R. Soc. Lond. A* **241**:1226 (1957), 376–396.

[Eshelby 1959] J. D. Eshelby, "The elastic field outside an ellipsoidal inclusion", *Proc. R. Soc. Lond. A* **252**:1271 (1959), 561–569.

[Eshelby 1961] J. D. Eshelby, "Elastic inclusion and inhomogeneities", pp. 222–246 in *Progress in solid mechanics*, vol. 2, edited by I. N. Sneddon and R. Hill, North-Holland, Amsterdam, 1961.

[Gao 1995] J. Gao, "A circular inclusion with imperfect interface: Eshelby's tensor and related problems", *J. Appl. Mech.* (*ASME*) **62**:4 (1995), 860–866.

[Hashin 1991] Z. Hashin, "The spherical inclusion with imperfect interface", *J. Appl. Mech.* (*ASME*) **58**:2 (1991), 444–449.

[Hwu and Ting 1989] C. Hwu and T. C. T. Ting, "Two-dimensional problems of the anisotropic elastic solid with an elliptic inclusion", *Q. J. Mech. Appl. Math.* **42**:4 (1989), 553–572.

[Muskhelishvili 1953] N. I. Muskhelishvili, *Some basic problems of the mathematical theory of elasticity*, Noordhoff, Groningen, 1953.

[Ru and Schiavone 1997] C. Q. Ru and P. Schiavone, "A circular inclusion with circumferentially inhomogeneous interface in antiplane shear", *Proc. R. Soc. Lond. A* **453**:1967 (1997), 2551–2572.

[Shen et al. 2001] H. Shen, P. Schiavone, C. Q. Ru, and A. Mioduchowski, "Stress analysis of an elliptic inclusion with imperfect interface in plane elasticity", *J. Elasticity* **62**:1 (2001), 25–46.

[Stroh 1958] A. N. Stroh, "Dislocations and cracks in anisotropic elasticity", *Philos. Mag.* **3**:30 (1958), 625–646.

[Ting 1996] T. C. T. Ting, *Anisotropic elasticity: theory and applications*, Oxford University Press, New York, 1996.

[Ting 1997] T. C. T. Ting, "New explicit expression of the Barnett–Lothe tensors for anisotropic linear elastic materials", *J. Elasticity* **47**:1 (1997), 23–50.

[Wang et al. 2005] X. Wang, J. Q. Zhang, and X. M. Guo, "Two circular inclusions with inhomogeneously imperfect interfaces in plane elasticity", *Int. J. Solids Struct.* **42**:9–10 (2005), 2601–2623.

[Wang et al. 2008] X. Wang, E. Pan, and L. J. Sudak, "Uniform stresses inside an elliptical inhomogeneity with an imperfect interface in plane elasticity", *J. Appl. Mech.* (*ASME*) **75**:5 (2008), 054501.

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