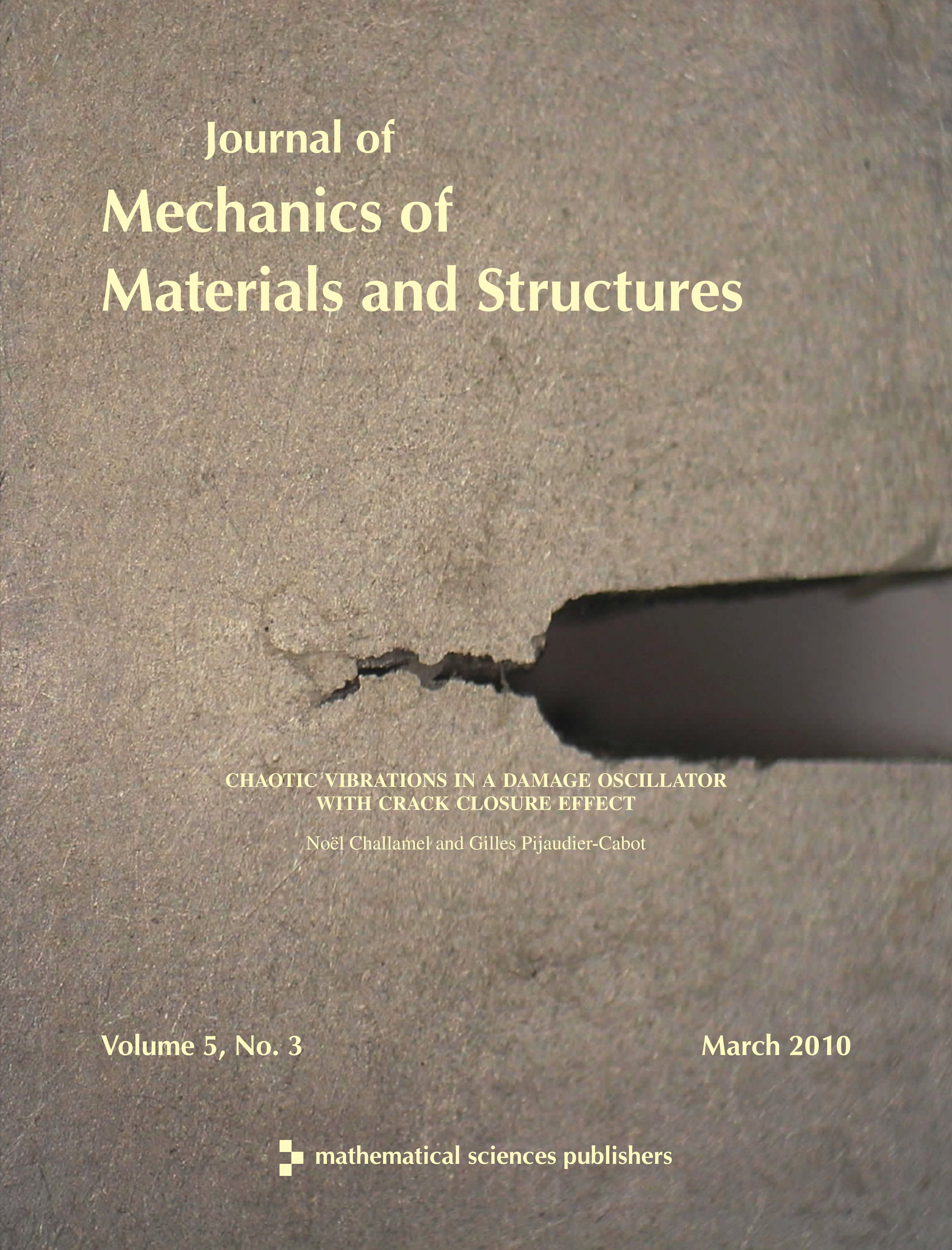

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CHAOTIC VIBRATIONS IN A DAMAGE OSCILLATOR WITH CRACK CLOSURE EFFECT

NOËL CHALLAMEL AND GILLES PIJAUDIER-CABOT

This paper deals with the dynamics of a single-degree-of-freedom unilateral damage oscillator. Using appropriate internal variables, the hysteretic dynamic system can be written as a nonsmooth autonomous system. The free dynamics of such a nonlinear system are simply reduced to periodic motion, eventually attractive trajectories, and divergent motion. The direct Lyapunov method is used to investigate the stability of the free damage system. A critical energy is highlighted that the oscillator can support while remaining stable. The natural frequency of the periodic motion depends on the stationary value of the damage internal variable. The inelastic forced oscillator, however, can exhibit very complex phenomena. When the damage parameter remains stationary, the dynamics are similar to those of an elastic oscillator with nonsymmetric stiffness. The dynamics appear to be controlled by the initial perturbations. Chaotic motions may appear in such a system, specifically for severely damaged oscillators. It is numerically shown that chaos is observed in the vicinity of the divergence zone (the collapse). This closeness of both behaviors — chaos and divergence — is probably related to the perturbation of the homoclinic orbit associated with the critical energy.

1. Introduction

Design procedures are becoming more and more oriented towards failure modes and structural ductility control. Modern building codes aim at incorporating basic characteristics that result from the nonlinear analysis of structures. In the case of seismic analysis, knowledge of the basic dynamics of inelastic systems (plastic or damageable systems) is among the main objectives. This is however quite an open issue because these models are nonsmooth dissipative systems and very few results have been established in this particular case [Wiercigroch 2000; Awrejcewicz and Lamarque 2003]. Although concrete structures are inelastic multiple-degree of freedom (DOF) systems, their dynamics are so complex that there is still room for use of a simple approach, based on single-DOF inelastic oscillators, in order to investigate and to illustrate their basic characteristics. The inelastic response of concrete structures is due to several causes: material nonlinearities, geometrical effects, interface, friction and contact problems, and others. Within the framework of a single-DOF inelastic oscillator, the influence of each source of nonlinearity on its dynamic response may be considered separately, its effect being investigated analytically or numerically. Here we consider material nonlinearities only with two consequences: a nonlinear softening response and a damage deactivation effect related to crack closure.

Up to now, most studies have been devoted to plastic oscillators. The pioneer analytical work of Caughey [1960] uses an equivalent asymptotic method to approximate the response of a plastic-kinematic

Keywords: damage oscillator, nonlinear dynamics, seismic design, concrete structures, chaos, unilateral effect, crack closure, breathing crack, bimodular material.

hardening oscillator loaded by a harmonic signal. Periodic motion has been found using numerical simulations of such a plastic oscillator, as in [Savi and Pacheco 1997]; see also [Challamel and Gilles 2007; Challamel et al. 2007; 2008] for the case of perfect plasticity. Limit cycles have been highlighted for the free undamped kinematic-hardening system [Pratap et al. 1994]. The same oscillator forced by a periodic load shows very rich phenomena in dynamics, and sometimes chaotic motion [Pratap and Holmes 1995]. The inelastic response of such a single-DOF oscillator results from a material nonlinear, plastic, response. Coupling with geometrical nonlinearities also leads to chaotic motion [Poddar et al. 1988]. Without being exhaustive, let us mention also that chaos may appear in a Bouc–Wen hysteretic oscillator [Lacarbonara and Vestroni 2003; Awrejcewicz et al. 2008]. It has to be pointed out that these material sources of nonlinearities are mainly investigated from a theoretical point of view, and very few papers have been devoted to the experimental evidence of chaos in these nonsmooth oscillators. [Wiercigroch and Sin 1988] can be cited for extensive experimental investigations on nonsmooth oscillators with piecewise linear restoring forces. [Ing et al. 2008] is also an experimental contribution on an impact oscillator with a one-sided elastic constraint. To the authors' knowledge, no paper has been published on the experimental study of nonlinear dynamics of forced damage oscillators.

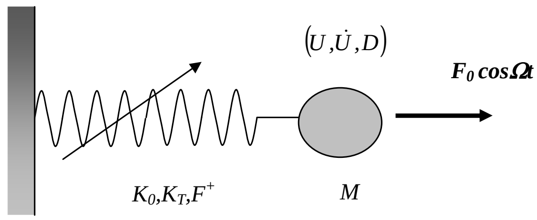
Material damage is a degradation of the elastic stiffness in the oscillator due to microcracking which may become important [Aschheim and Black 1999; Williamson and Hjelmstad 2001]. The free dynamics of a damage oscillator exhibit a stationary periodic motion in a given perturbation domain [Challamel and Pijaudier-Cabot 2004]. The dynamics of a forced damage oscillator (without the crack closure effect) has been studied by DeSimone et al. [2001].

During the loading history, and more specifically for alternated loads, micro and macrocracks may open and close. Crack closure and contact conditions on the crack faces induce a characteristic response called the *unilateral* material response, which depends on the sign of the applied stresses: cracking and damage affect the material response in tension whereas the compression response remains elastic (see [Ortiz 1985; Mazars et al. 1990] or more recently [Challamel et al. 2005] for a discussion of this phenomenon in the framework of damage mechanics). In the case of crack closure, which induces typically some additional discontinuity in the response of the nonlinear single-DOF oscillator, chaotic responses have been found by investigating the solutions of the equations of motion in each subspace and gluing them together [Wiercigroch 2000]. Foong et al. [2003; 2007] compared experiments on a fatigue-testing rig involving crack closure effects to a numerical model of a bending cracked beam which exhibited chaos. Carpinteri and Pugno [2005a; 2005b] investigated the specific role of the breathing crack (the crack closure effect) on the response of a cracked cantilever beam (from an experimental and numerical point of view). They observed the period doubling bifurcation.

The objective of the present study is to investigate the dynamics of a nonlinear damage oscillator with nonsymmetric stiffness due to crack closure. The discontinuity induced by crack closure is taken into account in the model, unlike in the results of [DeSimone et al. 2001; Challamel and Pijaudier-Cabot 2004]. Compared to fracture mechanics, an advantage of continuous damage is that it folds microcracking, crack initiation (localization of damage), and crack propagation into a single framework [Mazars and Pijaudier-Cabot 1996]. As we will see further, the dissipative response of the oscillator can be introduced quite easily into a stability analysis, in addition to the nonsmooth character of the oscillator due to crack closure. We wish to investigate the influence of the degradation of the elastic properties of an oscillator due to (micro) cracking and damage deactivation due to crack closure at the same time within this rather general framework which has become popular in structural analyses of failure (see for example the review by Bažant and Jirásek [2002]). Section 2 presents the basic equations. The free vibrations and the forced vibrations of the oscillator are discussed in the subsequent sections.

2. The general system

Consider the simple system shown on the right. A mass M is attached to a damage spring. The inelastic system is loaded by an external harmonic force. This oscillator is characterized by the displacement U, the velocity $\dot{U}$, and an additional internal variable characterizing the inelastic damage process. This damage variable, classically denoted by D, encodes the effect of microcracking in the spring of the oscillator in tension. It varies between 0 (the initial virgin state) and 1 (at failure). The damage law is schematized in Figure 1. Concrete, like many other geomaterials, has a nonsymmetric response in traction and compression. In tension, linear softening is assumed. This law depends on three parameters: the initial stiffness K_0, the tangent stiffness K_T in the postpeak regime, which rules the damage evolution, and the maximum force F^+. In the case of a softening process such as is considered in this paper, the tangent stiffness is negative ($K_T \leq 0$). It is assumed that concrete remains elastic in compression. Load levels where some nonlinear response is observed in compression are very high compared to tension and will not be considered for simplicity. The material response is unilateral, with a discontinuous stiffness upon the change of sign of the stress.



The material parameters of the model may be easily expressed in terms of characteristic displacements: U_Y is the maximum displacement of the initial elastic domain, and U_f is the displacement at failure, defined as

$$U_Y = \frac{F^+}{K_0}, \qquad \frac{U_f}{U_Y} = 1 - \frac{K_0}{K_T}. \tag{1}$$

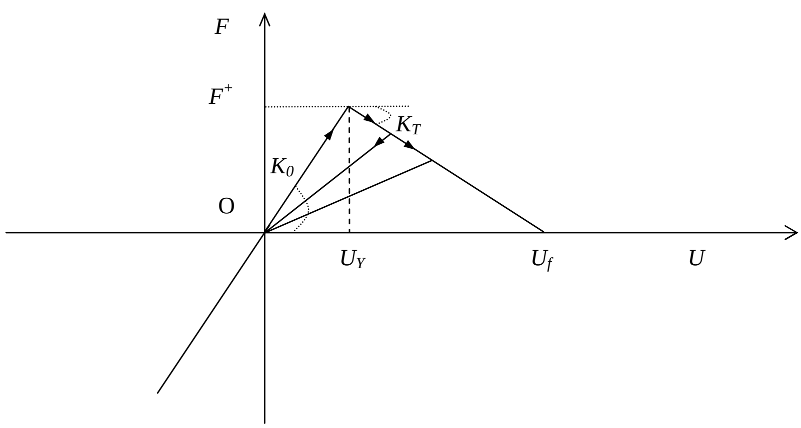


Figure 1. Damage incremental law for the inelastic spring.

In the meantime, the softening damage process can be characterized by one of the two dimensionless parameters u_f or κ^2:

$$u_f = \frac{U_f}{U_Y}, \quad \kappa^2 = -\frac{K_T}{K_0} \quad \Rightarrow \quad u_f = 1 + \frac{1}{\kappa^2}. \tag{2}$$

The damage variable D can directly be expressed in terms of a memory variable V, defined as

$$V(t) = \max_t U(t), \tag{3}$$

and the relation between D and V is given by

$$D = \left\langle 1 + \frac{K_T}{K_0}\left\langle -1 - \frac{K_0 - K_T}{K_T}\frac{U_Y}{V}\right\rangle\right\rangle, \qquad \text{where } \langle x \rangle = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{4}$$

It is easy to verify that the rate of damage is necessarily positive:

$$\dot{D} \geq 0, \tag{5}$$

and the classical thermodynamic inequality prescribing the positiveness of the damage dissipation rate of such scalar damage model is recovered.

Three dynamical states can be distinguished. These three states correspond to a reversible state (or elastic state) in the tension domain $\hat{E}^+$, a reversible state in the compression domain $\hat{E}^-$, and an irreversible state $\hat{D}$ (necessarily in the tension domain) associated with the evolution of damage:

$$\begin{cases} \hat{E}^+ \text{ state:} & M\ddot{U} + K_0\big(1 - D(V)\big)U = F_0 \cos \Omega t, & \dot{D} = 0; \\ \hat{E}^- \text{ state:} & M\ddot{U} + K_0 U = F_0 \cos \Omega t, & \dot{D} = 0; \\ \hat{D} \text{ state:} & M\ddot{U} + \langle K_T\big(U - U_f\big)\rangle = F_0 \cos \Omega t, & \dot{V} = \dot{U}. \end{cases} \tag{6}$$

Each state is defined from a partition of the phase space:

$$\begin{cases} \hat{E}^+ : \big(U > 0 \text{ or } (U = 0 \text{ and } \dot{U} \geq 0)\big) \text{ and } \big((\dot{U} \leq 0) \text{ or } (\dot{U} \geq 0 \text{ and } U < V) \text{ or } (V < U_Y)\big); \\ \hat{E}^- : \big(U < 0 \text{ or } (U = 0 \text{ and } \dot{U} \leq 0)\big); \\ \hat{D} : (\dot{U} > 0) \text{ and } (U = V) \text{ and } (V \geq U_Y). \end{cases} \tag{7}$$

One can recognize in this set Kuhn–Tucker like conditions defining the growth of damage. In the $\hat{E}^+$ state (loading or unloading within the elastic domain), damage does not grow because the displacement is less than the history variable, or, if it is equal, unloading is being performed. In the $\hat{E}^-$ state, crack closure occurs and the original stiffness of the oscillator is recovered. In the $\hat{D}$ state, damage grows and the displacement U is equal to the history variable V at all times; the elastic domain grows at the same time. Note that the dissipative nature induces an additional regime compared to the usual nonsmooth oscillator.

Equations (6) and (7) describe a piecewise linear oscillator (see for instance [Shaw and Holmes 1983] for some fundamental properties of piecewise linear oscillators). The dimensionless phase variables are introduced as

$$(u, \dot{u}, v) = \left(\frac{U}{U_Y}, \frac{\dot{U}}{U_Y}, \frac{V}{U_Y}\right), \quad v = \max_t u(t). \tag{8}$$

New temporal derivatives are written directly with respect to the dimensionless time parameter

$$\tau = \frac{t}{t^*} \quad \text{with } t^* = \sqrt{\frac{M}{K_0}}, \tag{9}$$

where t^* is a time constant of the dynamical system. With dimensionless variables, the system of equations of the oscillator becomes

$$\left\{\begin{array}{lrl} \hat{E}^+: & \ddot{u} + (1 - D(\upsilon))u = f_0 \cos\omega\tau, & \dot{D} = 0, \\ \hat{E}^-: & \ddot{u} + u = f_0 \cos\omega\tau, & \dot{D} = 0, \\ \hat{D}: & \ddot{u} + \left\langle \dfrac{u - u_f}{1 - u_f} \right\rangle = f_0 \cos\omega\tau, & \dot{\upsilon} = \dot{u}, \end{array}\right\} \quad \text{with } f_0 = \frac{F_0}{F^+}, \ \omega = \Omega t^*. \tag{10}$$

The damage function depends on the new dimensionless memory variable υ:

$$D = \left\langle 1 + \frac{1}{1 - u_f} \left\langle -1 + \frac{u_f}{\upsilon} \right\rangle \right\rangle, \tag{11}$$

and the three states are now governed by

$$\left\{\begin{array}{ll} \hat{E}^+: & \big(u > 0 \text{ or } (u = 0 \text{ and } \dot{u} \geq 0)\big) \text{ and } \big((\dot{u} \leq 0) \text{ or } (\dot{u} \geq 0 \text{ and } u < \upsilon) \text{ or } (\upsilon < 1)\big); \\ \hat{E}^-: & \big(u < 0 \text{ or } (u = 0 \text{ and } \dot{u} \leq 0)\big); \\ \hat{D}: & (\dot{u} > 0) \text{ and } (u = \upsilon) \text{ and } (\upsilon \geq 1). \end{array}\right. \tag{12}$$

For $f_0 = 0$ (free vibrations), the system is autonomous with a three-dimensional phase space associated with the variables $(u, \dot{u}, \upsilon)$. The periodically forced oscillator ($f_0 \neq 0$) can be studied using an extended four-dimensional phase space with coordinates $(u, \dot{u}, \upsilon, \tau)$.

Local solutions of (10) are known explicitly for each state. The solution of the $\hat{E}^+$ state, based on the initial conditions

$$\big(u(\tau_i), \dot{u}(\tau_i), \upsilon(\tau_i)\big) = (u_i, \dot{u}_i, \upsilon_i), \tag{13}$$

is written as

$$\left\{\begin{array}{l} u(\tau) = A\cos\omega_i(\tau - \tau_i) + B\sin\omega_i(\tau - \tau_i) + \dfrac{f_0}{\omega_i^2 - \omega^2}\cos\omega\tau, \\ \dot{u}(\tau) = -\omega_i A\sin\omega_i(\tau - \tau_i) + \omega_i B\cos\omega_i(\tau - \tau_i) - \dfrac{f_0 \cdot \omega}{\omega_i^2 - \omega^2}\sin\omega\tau, \\ \upsilon(\tau) = \upsilon_i, \end{array}\right. \tag{14}$$

where

$$\begin{aligned} \omega_i &= \sqrt{1 - D(\upsilon_i)}, \\ A &= u_i - \frac{f_0}{\omega_i^2 - \omega^2}\cos\omega\tau_i, \\ B &= \frac{\dot{u}_i}{\omega_i} + \frac{f_0\omega}{\omega_i(\omega_i^2 - \omega^2)}\sin\omega\tau_i. \end{aligned}$$

The solution of the $\hat{E}^-$ state, based on the initial conditions (13), is

$$\begin{cases} u(\tau) = A\cos(\tau-\tau_i) + B\sin(\tau-\tau_i) + \dfrac{f_0}{1-\omega^2}\cos\omega\tau, \\ \dot{u}(\tau) = -A\sin(\tau-\tau_i) + B\cos(\tau-\tau_i) - \dfrac{f_0\cdot\omega}{1-\omega^2}\sin\omega\tau, \\ \upsilon(\tau) = \upsilon_0, \end{cases} \tag{15}$$

where

$$A = u_i - \frac{f_0}{1-\omega^2}\cos\omega\tau_i \quad B = \dot{u}_i + \frac{f_0\omega}{1-\omega^2}\sin\omega\tau_i.$$

The solution of the $\hat{D}$ state, based on the initial conditions (13), is

$$\begin{cases} u(\tau) = Ae^{-\kappa(\tau-\tau_i)} + Be^{\kappa(\tau-\tau_i)} - \dfrac{f_0}{\kappa^2+\omega^2}\cos\omega\tau + 1 + \dfrac{1}{\kappa^2}, \\ \dot{u}(\tau) = -A\kappa e^{-\kappa(\tau-\tau_i)} + B\kappa e^{\kappa(\tau-\tau_i)} + \dfrac{f_0\omega}{\kappa^2+\omega^2}\sin\omega\tau, \\ \upsilon(\tau) = u(\tau), \end{cases} \tag{16}$$

where

$$A = \frac{f_0}{2\kappa(\kappa^2+\omega^2)}(\kappa\cos\omega\tau_i + \omega\sin\omega\tau_i) - \frac{1}{2}\Big(1+\frac{1}{\kappa^2}\Big) - \frac{1}{2\kappa}\dot{u}_i + \frac{1}{2}u_i,$$
$$B = \frac{f_0}{2\kappa(\kappa^2+\omega^2)}(\kappa\cos\omega\tau_i - \omega\sin\omega\tau_i) - \frac{1}{2}\Big(1+\frac{1}{\kappa^2}\Big) + \frac{1}{2\kappa}\dot{u}_i + \frac{1}{2}u_i.$$

The limit case of the oscillator completely broken ($D = 1$) yields the differential equation $\ddot{u} = f_0\cos\omega\tau$ and the solution:

$$u(\tau) = A\tau + B - \frac{f_0}{\omega^2}\cos\omega\tau, \quad \dot{u}(\tau) = A + \frac{f_0}{\omega}\sin\omega\tau, \tag{17}$$

where

$$A = \dot{u}_i - \frac{f_0}{\omega}\sin\omega\tau_i, \quad B = u_i + \frac{f_0}{\omega^2}\cos\omega\tau_i - \Big(\dot{u}_i - \frac{f_0}{\omega}\sin\omega\tau_i\Big)\tau_i.$$

The times of flight in each region (each state) cannot be found in closed form in the general case and piecing together these known solutions is not directly possible directly. Before that, the time which characterizes the transition between each state is computed from a Newton–Raphson procedure. Note that this solution is considerably more accurate than the usual numerical solutions of ordinary differential equations, the only approximations being made at the boundary of each state.

3. Free vibrations: $f_0 = 0$

3.1. ***Existence of a stability domain.*** The following generic perturbation is considered:

$$\tau_0 = 0 : (u_0, \dot{u}_0, \upsilon_0) = (0, \dot{u}_0, 0), \quad \text{with } \dot{u}_0 \leq 0. \tag{18}$$

If $\dot{u}_0 \geq -1$, elastic behavior prevails and the trajectory is a circle in the phase space restricted to $D = 0$. On the opposite case, if $\dot{u}_0 < -1$, the motion is also composed of a damage inelastic phase. The time τ_1

necessary to initiate this damage phase is computed from

$$u(\tau_1) = 1, \quad \dot{u}(\tau_1) = \sqrt{\dot{u}_0^2 - 1}. \tag{19}$$

During this damage phase, the solution $u(\tau)$ is expressed by (16). Three types of dynamic responses can be distinguished from the sign of the constant B. A critical speed is then introduced:

$$\dot{u}_c = \sqrt{u_f}. \tag{20}$$

The size of the perturbation governs the stability of the origin point:

$$\begin{cases} |\dot{u}_0| > \dot{u}_c & \Rightarrow \quad \lim_{\tau \to \infty} u(\tau) = \infty, \\ |\dot{u}_0| = \dot{u}_c & \Rightarrow \quad \lim_{\tau \to \infty} u(\tau) = 0, \\ |\dot{u}_0| < \dot{u}_c & \Rightarrow \quad \text{stationary periodic regime.} \end{cases} \tag{21}$$

In the last case, another elastic phase is initiated and the motion is periodic. Free dynamics of such an inelastic system can be reduced to the periodic regime (waiting for a certain time), and the attractive or divergent trajectories (see also [Challamel and Pijaudier-Cabot 2004]). For the broken oscillator, the autonomous system at failure is characterized by

$$\ddot{u} = 0 \quad \Rightarrow \quad \begin{cases} u(\tau) = \dot{u}_i(\tau - \tau_i) + u_f, \\ \dot{u}(\tau) = \dot{u}_i. \end{cases} \tag{22}$$

The phase portrait is a horizontal line parallel to the u-axis.

The same kind of classification would be observed for the free vibrations of the plastic softening oscillator [Challamel and Pijaudier-Cabot 2006]. These three cases are distinguished by the value of the initial speed $\dot{u}_0$ with respect to the critical speed $\dot{u}_c$. The different types of dynamics are plotted on Figure 2. For the simulations, parameters are chosen as $u_f = 3$ with the following initial conditions:

$$\begin{aligned} &u_0 = v_0 = 0, \qquad \dot{u}_0 = -0.5; \\ &u_0 = v_0 = 0, \qquad \dot{u}_0 = -1; \\ &u_0 = v_0 = 0, \qquad \dot{u}_0 = -1.5; \\ &u_0 = v_0 = 0, \qquad \dot{u}_0 = -\sqrt{3} = -\dot{u}_c; \\ &u_0 = v_0 = 0, \qquad \dot{u}_0 = -2. \end{aligned} \tag{23}$$

For sufficiently large perturbations, the motion diverges. For sufficiently small perturbations, the motion is described by a circular (in the compression domain) or an elliptic (in the tension domain) periodic trajectory after a damage phase. The intermediate trajectory, represented on Figure 2, is an attractive trajectory. It asymptotically converges towards a fixed point. This attractive trajectory (homoclinic orbit) is structurally unstable. It is in fact the limit of the domain of perturbations generating bounded evolutions and also the limit of the domain associated with stability of the origin (in the sense of Lyapunov). This domain is defined by:

$$u_0^2 + \dot{u}_0^2 \le u_f, \quad u_0 \le v_0 \le 1. \tag{24}$$

For seismic design applications, (24) can be interpreted as a critical energy (induced by seismic solicitation for instance) that the oscillator can support in order to remain stable. For a higher seismic energy

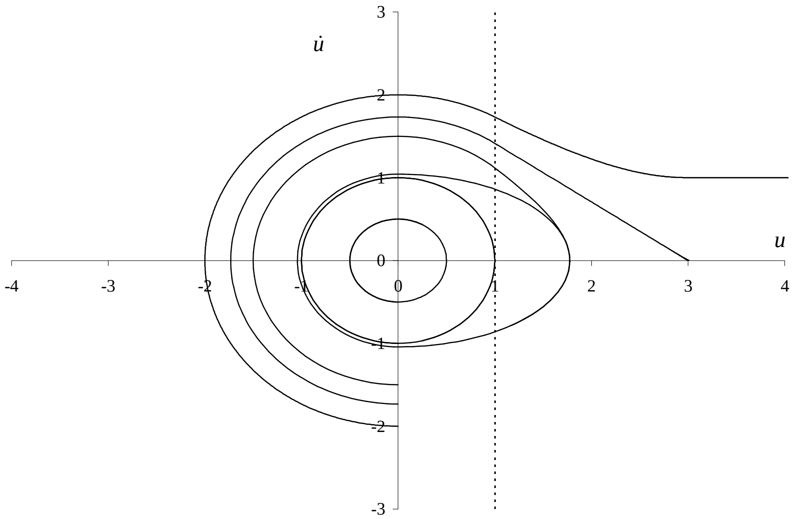


Figure 2. Dynamics of the free damage system.

level, a divergent evolution is present, leading to structural collapse. This critical energy is nothing but the energy dissipated by completely damaging the spring and is equal to the area under the monotonic force extension diagram (see [Challamel and Pijaudier-Cabot 2006] for the plastic softening oscillator). Moreover, when the motion is periodic, the damage reaches a stationary value denoted by $\overline{D}$. The global pulsation of this periodic motion can be obtained in closed-form solution from

$$\overline{\omega} = \frac{2\sqrt{1-\overline{D}}}{1+\sqrt{1-\overline{D}}}. \tag{25}$$

A similar relationship can be found in [Ryue and White 2007] where the damage parameter is replaced by the relative crack depth.

3.2. ***The direct Lyapunov method.*** The critical energy related to the stability domain can also be determined by means of a stability analysis based on the direct Lyapunov method. We follow the reasoning of [Kounadis 1996] for a smooth softening elastic system. We stress that the direct Lyapunov method was initially developed by Lyapunov for smooth systems; see for instance [la Salle and Lefschetz 1961]. The extension of such methodology to nonsmooth systems is a recent topic since the pioneer work of Filippov [1960; 1988]; for examples see [Shevitz and Paden 1994; Wu and Sepehri 2001; Bourgeot and Brogliato 2005; Leine 2006]. An application of this method to plastic systems can be found in [Challamel and Gilles 2007]. For the nonsmooth damage system, the Lyapunov function can be chosen as

$$V(u, \dot{u}, D) = \tfrac{1}{2}u^2 - \tfrac{1}{2}D\langle u\rangle^2 + \tfrac{1}{2}\dot{u}^2. \tag{26}$$

In this energy function, the damage variable is coupled to the positive part of the displacement (this is similar to the unilateral continuum damage model of [Challamel et al. 2005] at the material scale). $V(u, \dot{u}, D)$ is not a positive definite function over the complete space. The function V is vanishing for the trivial state $(u, \dot{u}, D) = (0, 0, 0)$, but also at failure when the oscillator is fully damaged: $(u, \dot{u}, D) = (u \geq 0, 0, 1)$.

However, if it is assumed that the oscillator is not fully damaged ($D < 1$), it can be rigorously proven that $V(u, \dot{u}, D)$ is effectively a definite positive function.

The direct Lyapunov method is based on the calculation of the time derivative of V. The time derivative of V involves the growth of damage, that is, the dissipative nature of the response of the oscillator in addition to the nonsmooth character captured with the positive part of the displacement introduced in the function. The time derivative of the Lyapunov function does not exist in the classical sense at the intersection of the elastic and the damage states (it only exists almost everywhere). It can be convenient to present the dynamic system with the nonsmooth functions:

$$\begin{cases} \ddot{u} + u - D(\upsilon)\langle u \rangle = 0, \\ \dot{\upsilon} = 2h(u-\upsilon)\dot{u}, \end{cases} \qquad h(x) = \begin{cases} 1 & \text{if } x > 0, \\ \frac{1}{2} & \text{if } x = 0, \\ 0 & \text{if } x < 0, \end{cases} \tag{27}$$

where h is a step function. The nonsmooth character of such a system is no longer ambiguous with this unified presentation. In particular, the damage rate is discontinuous at the elastic-damage interface. Additionally, the displacement rates can be discontinuous at the origin if the damage is nonzero.

The application of the direct Lyapunov method is much simplified when one considers the internal variable υ instead of D. It is possible to show that

$$\frac{dV(u, \dot{u}, \upsilon)}{d\tau} \in \dot{\tilde{V}}, \quad \text{where } \dot{\tilde{V}} = \left(\frac{\partial V}{\partial u}, \frac{\partial V}{\partial \dot{u}}, \frac{\partial V}{\partial \upsilon}\right) \begin{pmatrix} \dot{u} \\ -u + D(\upsilon)\langle u \rangle \\ 2K[h(u-\upsilon)]\dot{u} \end{pmatrix}. \tag{28}$$

K is called Filippov's set. It can be calculated for the Heaviside function:

$$K[h(x)] = H(x), \quad \text{with } H(x) = \begin{cases} 1 & \text{if } x > 0, \\ [0, 1] & \text{if } x = 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{29}$$

$\dot{\tilde{V}}$ can then be simplified as:

$$\dot{\tilde{V}} = \frac{\partial V}{\partial D}\frac{\partial D}{\partial \upsilon} 2\dot{u} H(u-\upsilon) = -\frac{1}{2}\langle u \rangle^2 \frac{\partial D}{\partial \upsilon} 2\dot{u} H(u-\upsilon). \tag{30}$$

The final result is obtained:

$$\begin{cases} \hat{E}^+, \hat{E}^- : & \dot{\tilde{V}} = 0, \\ \hat{D} : & \dot{\tilde{V}} = -\frac{u^2}{2}\dot{D} \leq 0. \end{cases} \tag{31}$$

V is a positive and definite function. Each element of $\dot{\tilde{V}}$ is negative or zero — as the damage is necessarily an increasing function of the time, see (5). Then, the origin (0, 0, 0) is stable in the sense of Lyapunov for sufficiently small perturbations (in fact all perturbations leading to $D < 1$). The boundary of this stability domain is exactly the stability domain exhibited by Equation (24). It is worth mentioning that the particular case of the fully damaged oscillator leads to:

$$D = 1 \quad \Rightarrow \quad V(u, \dot{u}) = \tfrac{1}{2}u^2 + \tfrac{1}{2}\dot{u}^2 \quad \Rightarrow \quad \dot{V} = u\dot{u} \geq 0 \quad \text{since } \ddot{u} = 0. \tag{32}$$

In this case, one cannot apply rigorously the instability theorem of the direct Lyapunov method, but (32) gives us some information about the instability of the fully damaged oscillator.

4. Forced vibrations: general case

Numerical simulations show that two types of responses may be observed, namely the shakedown response (damage shakedown means that $\dot{D} = 0$ after a critical time), and the collapse characterized by a divergent evolution (in such a case, failure is reached and $D = 1$). The theoretical analysis consists in treating the bounded dynamics (in the case of damage shakedown) as an equivalent elastic oscillator after a critical time (as in [Poddar et al. 1988], for instance). The extended four-dimensional phase space with coordinates $(u, \dot{u}, v, \tau)$ can be reduced to a three-dimensional phase space with coordinates $(u, \dot{u}, \tau)$. The new oscillator is an elastic oscillator with different stiffnesses in tension and compression. The results of [Shaw and Holmes 1983; Thompson et al. 1983; Mahfouz and Badrakhan 1990] can be used for the dynamics of the oscillator studied in the three-dimensional phase space.

We shall employ the Poincaré section method to investigate the response of the forced inelastic oscillator. In the case of elastic evolutions, damage remains constant ($D = \overline{D}$) and the motion can be studied using the three-dimensional phase space with coordinates $(u, \dot{u}, \tau)$. The vector field defined by (6) is easily seen to be $2\pi/\omega$ periodic in τ. The Poincaré section is useful to investigate properties of the dynamical system: the phase space is sliced by the map

$$\frac{(u, \dot{u}, \tau)}{\tau} \equiv \tau_0\left[\frac{2\pi}{\omega}\right]. \tag{33}$$

Numerical simulations have been conducted with the following realistic parameters:

$$u_f = 3, \quad f_0 = 0.05, \quad \omega = 0.2, \quad u_0 = 0, \quad \dot{u}_0 = 0. \tag{34}$$

The value of v_0 (equivalently, the value of the initial damage D_0) was varied in order to investigate the damage effect. Periodic, quasiperiodic, chaotic, and divergent behaviors are observed, distributed as Figure 3.

For the virgin material ($v_0 \leq 1$ or $D_0 = 0$), with the parameters chosen, the motion is periodic (Figure 4, left) and the Poincaré map of the harmonic motion (Figure 4, right) is a single point $(0, 0)$. With the chosen initial conditions ($u_0 = 0$; $\dot{u}_0 = 0$), the damage parameter remained stable ($\overline{D} = D_0 = 0$). Quasiperiodic motion (Figure 5) has also been found for a system with moderate damage ($v_0 > 1$ or $D_0 \neq 0$). The quasiperiodic nature of the motion is checked in the Poincaré map on Figure 5, right. The damage value has not varied during the simulation ($v_0 = 1.25$, $\overline{D} = D_0 = 0.30$): the constant stiffness ratio in tension and compression is equal to $1 - \overline{D}$. The parameter f_0 is not so significant if only elastic

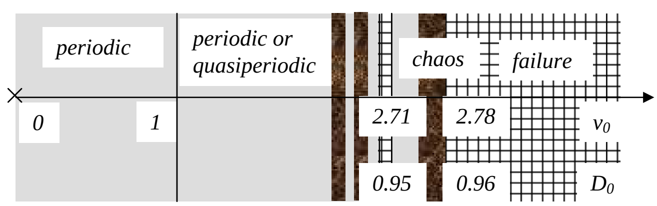


Figure 3. Bifurcation diagram for $f_0 = 0.05$; $\omega = 0.2$; $u_f = 3$; $u_0 = 0$; $\dot{u}_0 = 0$.

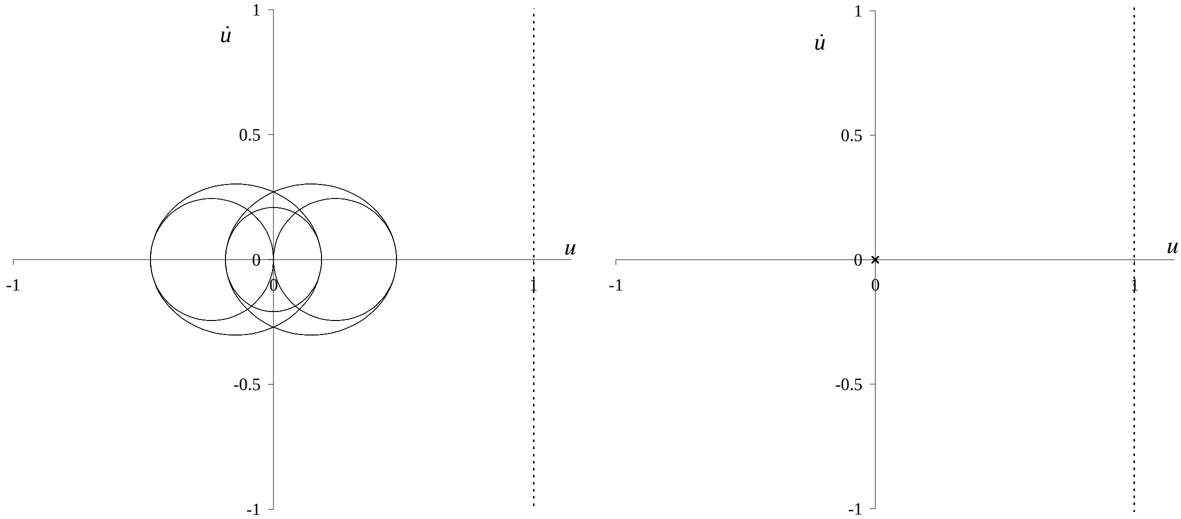


Figure 4. Periodic motion: phase portrait (left) and Poincaré map (right) for $f_0 = 0.25$, $\omega = 0.2$, $u_f = 3$, $\upsilon_0 = 0$ ($D_0 = 0$).

states prevail during the evolution ($\dot{D} = 0$). In such a case, all phase portraits are geometrically similar, transformed from a reference case (with adapted initial conditions). For size reasons, $f_0 = 0.25$ has been adopted for simulations of Figures 4 and 5 but the case $f_0 = 0.05$ can be directly deduced from these graphics (as in the bifurcation diagram of Figure 3). With the initial conditions chosen, failure can be reached for $\upsilon_0 \in [2.62, 2.66]$ or $\upsilon_0 > 2.78$ (except for the marginal value of $\upsilon_0 = 2.82$). It strongly depends on initial conditions. In the simulation plotted in Figure 6 ($\upsilon_0 = 2.65$ or $D_0 = 0.934$), damage shakedown does not occur and failure is reached after several cycles ($\overline{D} = 1$). For $\upsilon_0 = 2.71$ ($D_0 = \overline{D} = 0.946$),

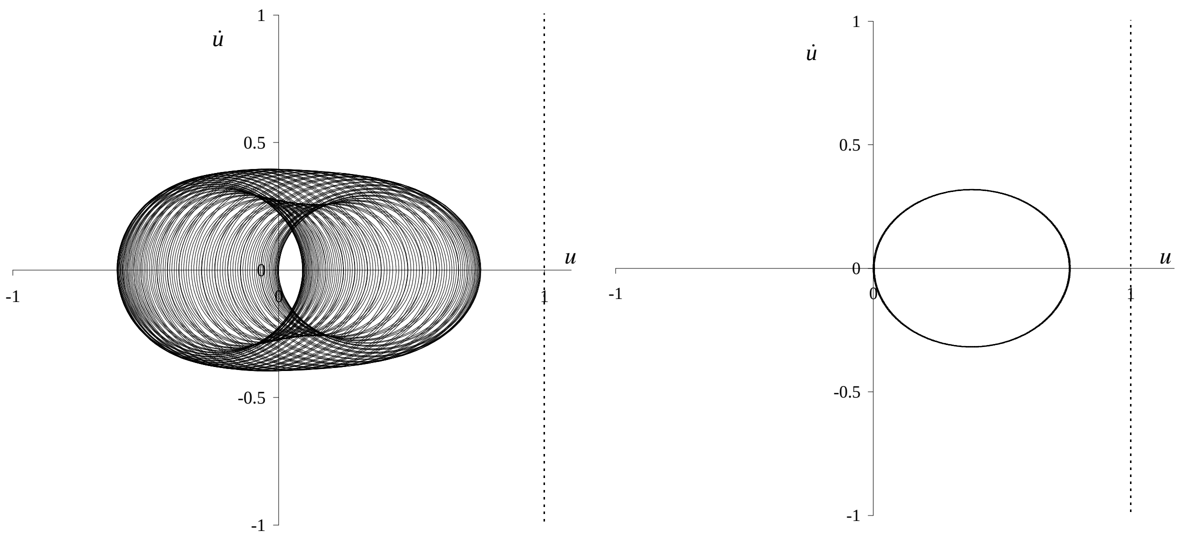


Figure 5. Quasiperiodic motion: phase portrait (left) and Poincaré map (right) for $f_0 = 0.25$, $\omega = 0.2$, $u_f = 3$, $\upsilon_0 = 1.25$ ($D_0 = 0.3$).

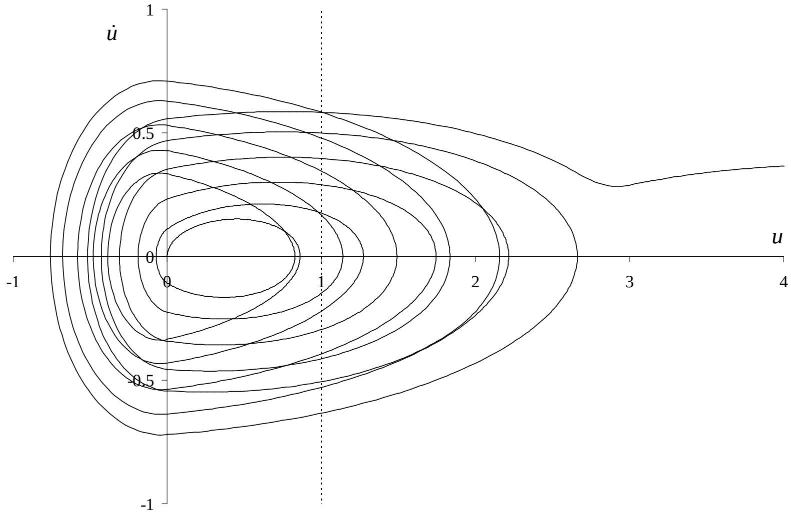


Figure 6. Divergent motion: phase portrait for $f_0 = 0.05$, $\omega = 0.2$, $u_f = 3$, $\upsilon_0 = 2.65$ ($D_0 = 0.934$).

chaos is observed (see further in Figure 9, right). Chaotic vibrations are also observed for higher damage values ($\upsilon_0 \in [2.71, 2.78]$ or $D_0 \in [0.946, 0.960]$) or smaller damage values ($\upsilon_0 \in [2.26, 2.32]$ or $D_0 \in [0.836, 0.853]$; $\upsilon_0 \in [2.48, 2.49]$ or $D_0 \in [0.895, 0.898]$; $\upsilon_0 \in [2.55, 2.58]$ or $D_0 \in [0.912, 0.919]$). These intermittent characteristic damage parameters are close to 1, that is close to the failure value. Figures 7 and 8 show phase portraits and Poincaré maps of the motion numerically observed before the transition to chaotic motion. In fact, the crisis is sudden. A subharmonic motion of order 3 is recognized on Figure 7, right, and a subharmonic of order 43 for Figure 8, right. The phase portrait of Figure 9, left, is difficult to analyze, whereas chaotic motion is clearly exhibited in Figure 9, right, and Figure 10. A symmetry is

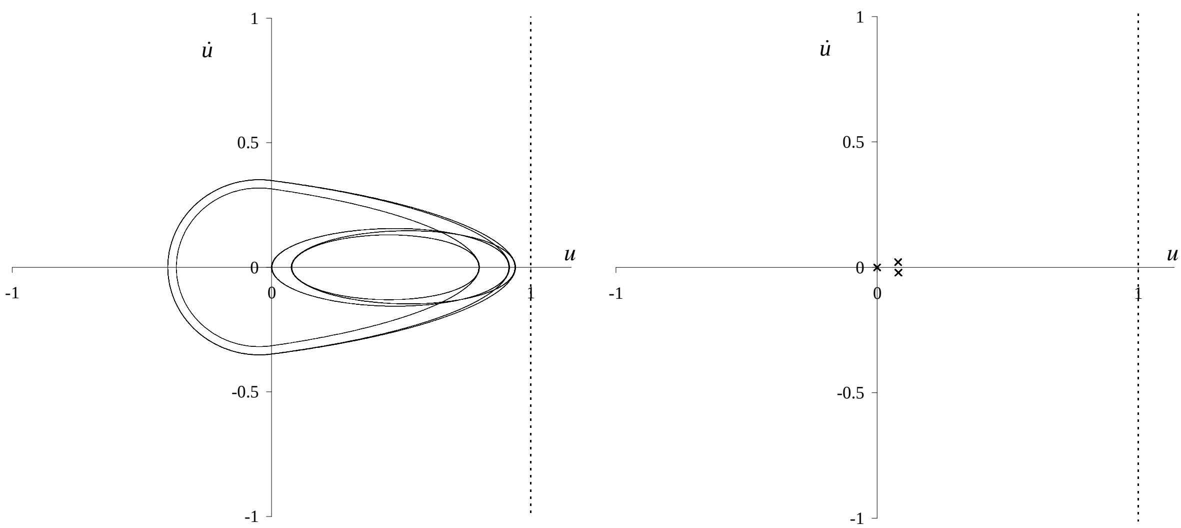


Figure 7. Periodic motion: phase portrait (left) and Poincaré map (right) for $f_0 = 0.05$, $\omega = 0.2$, $u_f = 3$, $\upsilon_0 = 2.68$ ($D_0 = 0.940$).

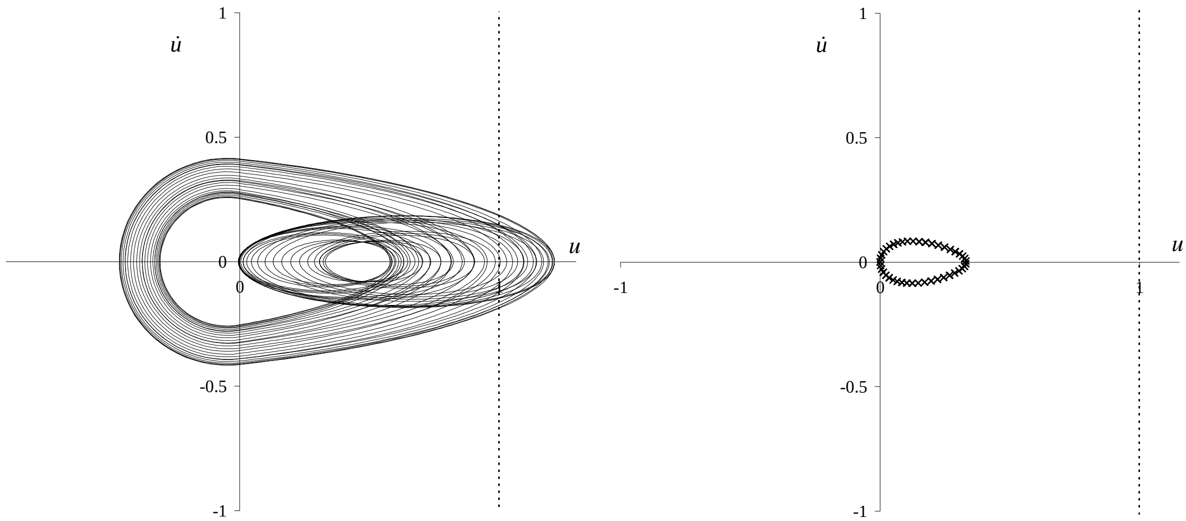


Figure 8. Periodic motion: phase portrait (left) and Poincaré map (right) for $f_0 = 0.05$, $\omega = 0.2$, $u_f = 3$, $v_0 = 2.70$ ($D_0 = 0.944$).

observed in the phase space. Twenty-four thousand cycles have been considered for the Poincaré section applied to chaotic motion. Hence, it is numerically shown that chaos is observed in the vicinity of the divergence zone (see Figure 9, right, and Figure 10, for instance). This closeness of both behaviors, chaos and divergence, is probably related to the perturbation of the homoclinic orbit, associated with the critical energy. The Appendix details this argument, and the possible application of the Melnikov method to an analogous elastic oscillator.

For the simulations considered, chaos is found for large values of the damage: chaos can be considered as a route to collapse. Nevertheless, chaos has been difficult to observe from the initial state ($D_0 = 0$), meaning that this phenomenon is generally a transient phenomenon. Mahfouz and Badrakhan [1990] show that chaos can appear for large stiffness ratios. The asymptotic case is here the limit where the

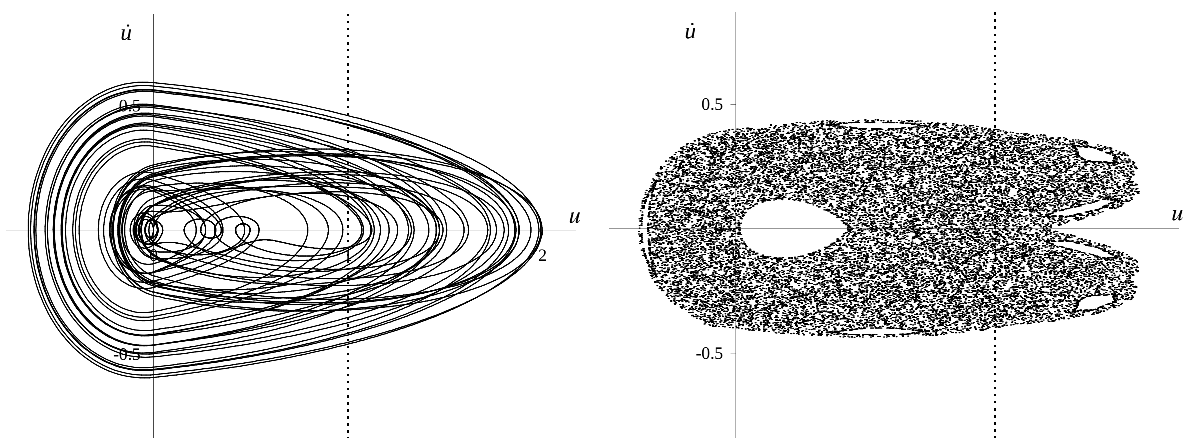


Figure 9. Chaotic motion: phase portrait (left) and Poincaré map (right) for $f_0 = 0.05$, $\omega = 0.2$, $u_f = 3$, $v_0 = 2.71$ ($D_0 = 0.946$).

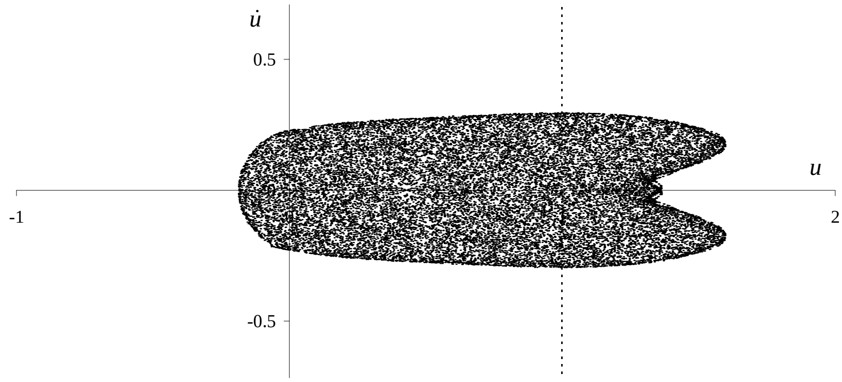


Figure 10. Chaotic motion: Poincaré map for $f_0 = 0.05$; $\omega = 0.2$; $u_f = 3$; $v_0 = 2.75$ ($D_0 = 0.955$).

stiffness ratio vanishes (the case treated by [Thompson et al. 1983] for instance). Understanding damage shakedown is also important for investigating the safety margins of concrete structures. Figure 11 shows a case of successful damage shakedown. The oscillator started in its initial state ($D_0 = 0$) and the stationary damage value associated with the final periodic motion is equal to $\bar{D} = 0.423$. In view of this result and of those in Figures 6–10, we see that damage shakedown is strongly dependent on initial conditions. For instance, a divergent evolution can be achieved for a sufficiently perturbed damage oscillator. This is different from the results observed for an elastic, perfectly plastic oscillator, where elastoplastic shakedown does not depend on initial conditions [Challamel 2005; Challamel et al. 2005].

This study was restricted to an undamped system, but existing results from [Shaw and Holmes 1983; Thompson et al. 1983; Mahfouz and Badrakhan 1990] suggest that the main phenomena exhibited in this

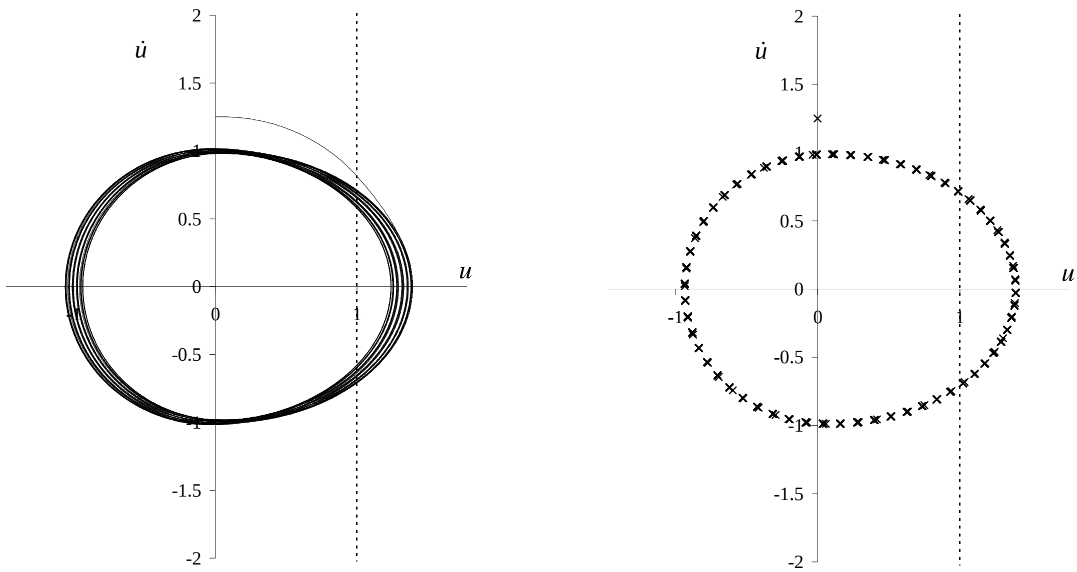


Figure 11. Stationary periodic motion (damage shakedown): phase portrait (left) and Poincaré map (right) for $f_0 = 0.05$, $\omega = 0.2$, $u_f = 3$, $u_0 = 0$, $\dot{u}_0 = 1.25$, $v_0 = 0$ ($D_0 = 0$).

paper may be also observed for a weakly damped system, where a damping term is added to (10):

$$\begin{cases} \hat{E}^+ : & \ddot{u}+2\zeta\dot{u}+(1-D(\upsilon))u = f_0\cos\omega\tau, & \dot{D}=0, \\ \hat{E}^- : & \ddot{u}+2\zeta\dot{u}+u = f_0\cos\omega\tau, & \dot{D}=0, \\ \hat{D} : & \ddot{u}+2\zeta\dot{u}+\left\langle\dfrac{u-u_f}{1-u_f}\right\rangle = f_0\cos\omega\tau, & \dot{\upsilon}=\dot{u}, \end{cases} \tag{35}$$

Here ζ is a dimensionless damping ratio. Equations (11) and (12) are still valid. The bifurcation diagrams of Figure 12, showing the dependence on the choice of the initial parameter υ_0 or D_0, confirm the appearance of chaotic phenomena for a severely damaged oscillator. Bifurcation diagrams similar to

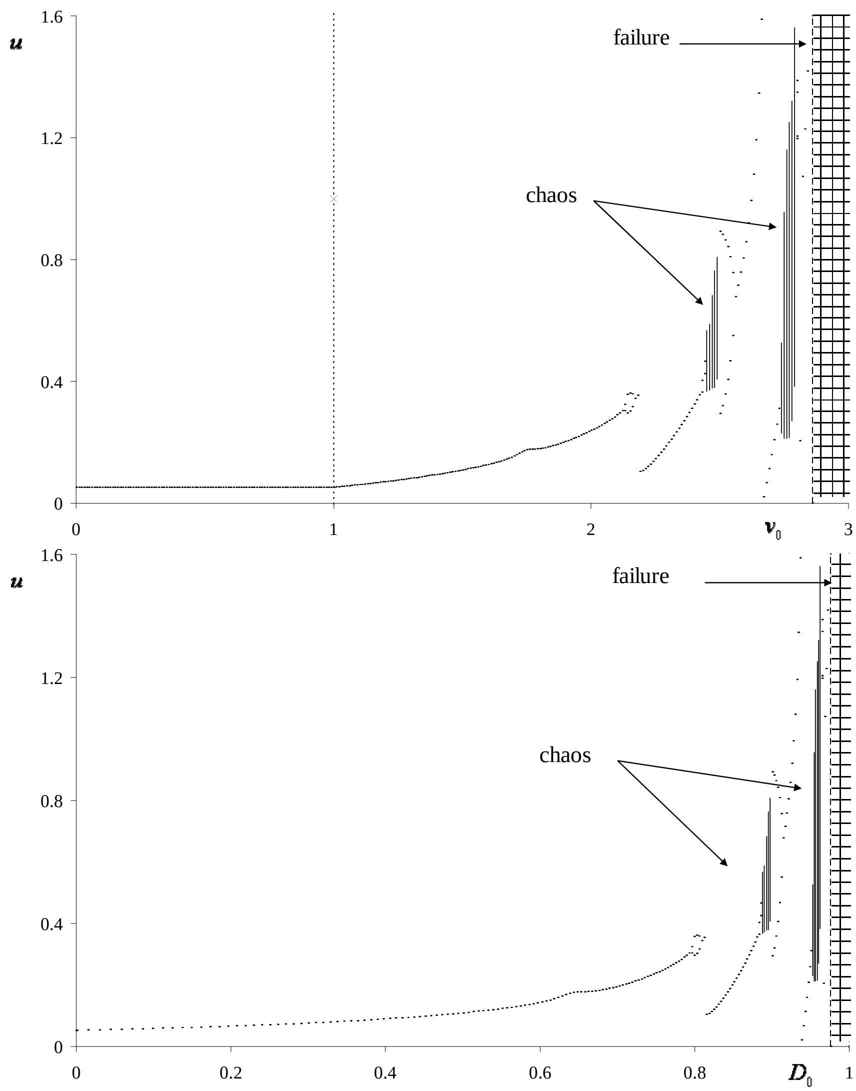


Figure 12. Bifurcation diagram in (υ_0, u) space (top) and in (D_0, u) space: damped system, for $f_0 = 0.05$, $\omega = 0.2$, $u_f = 3$, $u_0 = 0$, $\dot{u}_0 = 0$, $\zeta = 0.01$.

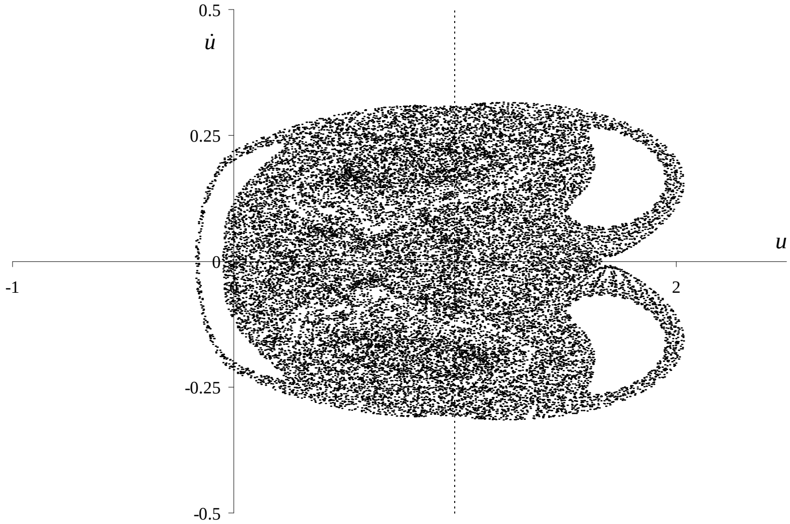


Figure 13. Chaotic motion, Poincaré map, undamped system. $f_0 = 0.05$; $\omega = 0.2$; $u_f = 3$; $u_0 = 0$; $\dot{u}_0 = 0$; $v_0 = 2.77$ ($D_0 = 0.958$); $\zeta = 0$.

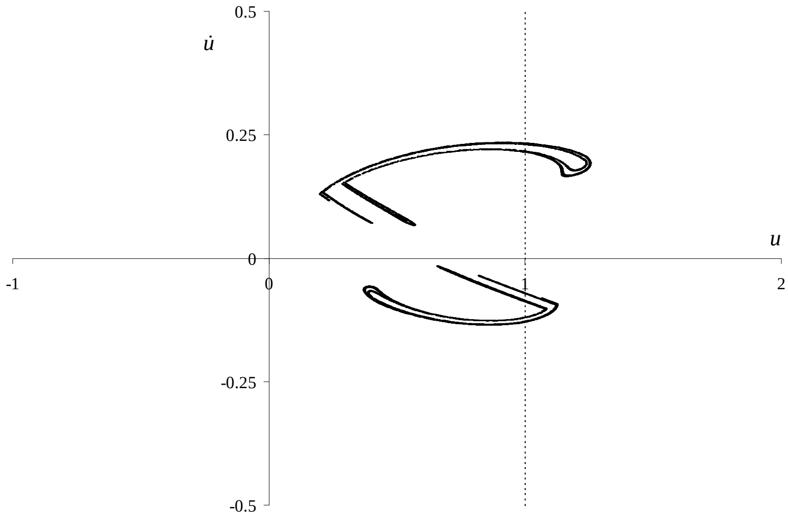


Figure 14. Chaotic motion: Poincaré map, damped system, for $f_0 = 0.05$, $\omega = 0.2$, $u_f = 3$, $u_0 = 0$, $\dot{u}_0 = 0$, $v_0 = 2.77$ ($D_0 = 0.958$), $\zeta = 0.01$.

those in Figure 12 were obtained in [Foong et al. 2003] from a fatigue-testing rig (without the divergence phenomenon). The structure of the chaotic attractor is affected by the damping ratio; compare Figures 13 and 14. The attractor can deform in shape, depending on the strength of damping, and can even become a chaotic sea with islands of quasiperiodic trajectories for the undamped system, as also observed in [Cao et al. 2008]. It is worth mentioning that the chaotic attractor associated with the damaged system is very analogous to the Hénon's attractor; see, for example, [Thompson 1982].

5. Conclusions

This paper deals with the stability of a single-DOF damage softening oscillator with crack closure effect. This system can be understood as an archetypal damage oscillator for concrete structures. Using appropriate internal variables (a damage variable or an equivalent memory variable), the free dynamics of such a nonlinear system can be written as a nonsmooth autonomous system with three regimes describing elasticity, elastic stiffness recovery, and energy dissipation due to damage growth, respectively. It is shown that the free vibrations of such an oscillator are reduced to stationary periodic regimes, attractive trajectories, or divergent motion.

The stability of this free oscillator is investigated with the direct Lyapunov method extended to nonsmooth systems. Following physical arguments, the Lyapunov function of the problem is chosen as the nonsmooth energy of the system. A critical energy that the oscillator can support in order to remain stable is obtained (induced by seismic loads for instance). For energy levels higher than this critical value, a divergent evolution is observed, leading to structural collapse. This result may have some implications in seismic design applications.

The behavior of the forced harmonic damage oscillator is much more complex. It is worth mentioning that the present damage model is different from a nonlinear elastic one essentially because the damage behavior is irreversible in nature. As a consequence, the inelastic (damage) problem has additional regimes (that the nonlinear elastic problem does not have), loading-unloading conditions, an energy threshold associated with the elastic domain, numerical analysis to compute the transition from the elastic state to the damage state and vice versa, and so on. The dynamics of a forced elastodamage oscillator with the crack closure effect have been numerically investigated. Periodic, quasiperiodic, chaotic, and divergent motions are observed. Chaotic motions are observed for severely damaged oscillators, in the vicinity of the divergence area. This oscillator is typically a simple physical model associated with the coexistence of chaos and divergence.

Damage shakedown, a fundamental feature related to the structural integrity, means that the damage value is stationary after a critical time. Damage shakedown is typically controlled by initial conditions and structural parameters (for example, stiffness ratios). For instance, a divergent evolution can be achieved for a sufficiently perturbed damage oscillator. In the case of damage shakedown, the stationary response is the same as that of an elastic oscillator with different stiffnesses in tension and in compression. Chaotic motions are observed for severely damaged oscillators (for sufficiently small stiffness in tension). Furthermore, it is numerically shown that chaos is observed in the vicinity of the divergence zone. This closeness of both behaviors, chaos and divergence, is probably related to the perturbation of the homoclinic orbit, associated with the critical energy. Therefore, chaos may be understood as a route to collapse. A more theoretical analysis would be probably needed to understand the specific route to chaos, for example, using the Melnikov method (even if the application of the Melnikov method to nonsmooth systems is a mathematically difficult problem [Kukučka 2007]).

Classically, chaotic systems are extremely sensitive to initial conditions and numerical simulations require specific attention when devising computational schemes [Symonds and Yu 1985; Tongue 1987]. The structure of the chaotic attractor is notably affected by the damping ratio. The attractor can deform in shape, depending on the strength of damping, and can even become a chaotic sea with islands of quasiperiodic trajectories for the undamped system, as also observed in [Cao et al. 2008]. It is worth mentioning that the chaotic attractor associated with the damaged system is very analogous to the Hénon's attractor [Thompson 1982]. One of the characteristics of the inelastic system considered here is that the dynamic collapse of concrete structures may involve chaotic phenomena. It is expected that large-scale models of concrete structures may also reveal the complex phenomena highlighted in this paper from an archetypal damage oscillator.

Appendix: A possible application of the Melnikov method to the analogous elastic oscillator

We study in this Appendix a piecewise elastic system with softening restoring force, somewhat reminiscent of the single-DOF system associated with a simple arch model (which includes snap-through buckling). This later system was considered in [Cao et al. 2008] as an archetypal oscillator exhibiting chaotic phenomena.

The constitutive elastic law of our dimensionless piecewise elastic system is

$$\ddot{u}+2\xi\dot{u}+f(u)=f_0\cos\omega\tau,\text{ where}\qquad f(u)=\begin{cases} u & \text{if } u\le 1,\\ \dfrac{u-u_f}{1-u_f} & \text{if } u\ge 1,\end{cases}$$

with $u_f \geq 1$. This oscillator is similar to the damage oscillator (inelastic system) studied in the paper, but differs from it in that the restoring force depends only on the position. This piecewise elastic system has limited tension strength.

Cao et al. [2008] also investigated a symmetrical piecewise elastic constitutive law and used Melnikov's method to detect the homoclinic tangling under the perturbation of damping and driving. The Melnikov method can be used to show the possible occurrence of chaos, for certain range of parameters of the system. We mention the application of the Melnikov method to nonsmooth dynamics [Awrejcewicz and Lamarque 2003; Awrejcewicz and Holicke 2007; Kukučka 2007]. As shown by Kukučka [2007], the mathematical background of the Melnikov method applied to nonsmooth systems is very recent and complex.

The Hamiltonian function associated with the free undamped piecewise elastic system is given by

$$H(u,\dot{u})=\frac{1}{2}\dot{u}^2+\begin{cases} \frac{1}{2}u^2 & \text{if } u\le 1,\\ \dfrac{1}{2}\dfrac{(u-u_f)^2}{1-u_f}+\dfrac{1}{2}u_f & \text{if } u\ge 1.\end{cases}$$

This system is very close to the system studied by Cao et al. [2008] who investigated a piecewise linear model, based on the following constitutive elastic law written with respect to the unstable equilibrium solution:

$$f(u)=\begin{cases} -\omega_1^2 u & \text{if } |u|\le u_0,\\ \omega_2^2\big(u-\operatorname{sgn}(u)u_f\big) & \text{if } |u|\ge u_0,\end{cases}\qquad u_f=\omega_2,$$

which is a symmetrical piecewise elastic model with a softening restoring force (the equilibrium solution at the origin is unstable). There is a correspondence between both systems for the set of parameters:

$$\omega_1^2=\frac{1}{u_f-1}=\frac{1}{u_0},\qquad \omega_2=1.$$

The reasoning concerning the application of the Melnikov method is identical to the case treated by Cao et al. [2008], as the symmetrical part of the constitutive law does not change the result of the perturbed orbit around the homoclinic orbit.

The equation of the homoclinic orbit is given by

$$u^2 + \dot{u}^2 = u_f \quad \text{if } u \leq 1,$$
$$\frac{u}{u_f} + \frac{\sqrt{u_f - 1}}{u_f}|\dot{u}| - 1 = 0 \quad \text{if } u \geq 1.$$

The Melnikov function is defined by the formula for the undamped system:

$$M(\tau_0) = \int_{-\infty}^{\infty} \dot{u}(\tau) f_0 \cos \omega(\tau + \tau_0)\, d\tau.$$

If the function $M(\tau_0)$ has simple zeros, then for a sufficiently small parameter f_0, the motion governed by the forced dynamic system can be chaotic. The reader is referred to [Cao et al. 2008] for the technical calculation of the Melnikov integral. It is shown that chaos may occur for a certain range of parameters for the equivalent elastic oscillator. This result cannot be strictly used in the case of the damage oscillator, for two reasons. First, the dimension of the phase space is augmented in the presence of damage (additional state variable). Secondly, the homoclinic nature of the orbit is lost for the damaged system. Therefore, the proximity of the chaotic response of the damage oscillator with the divergence area is actually a numerical finding, without a rigorous theoretical proof, based on the Melnikov method.

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