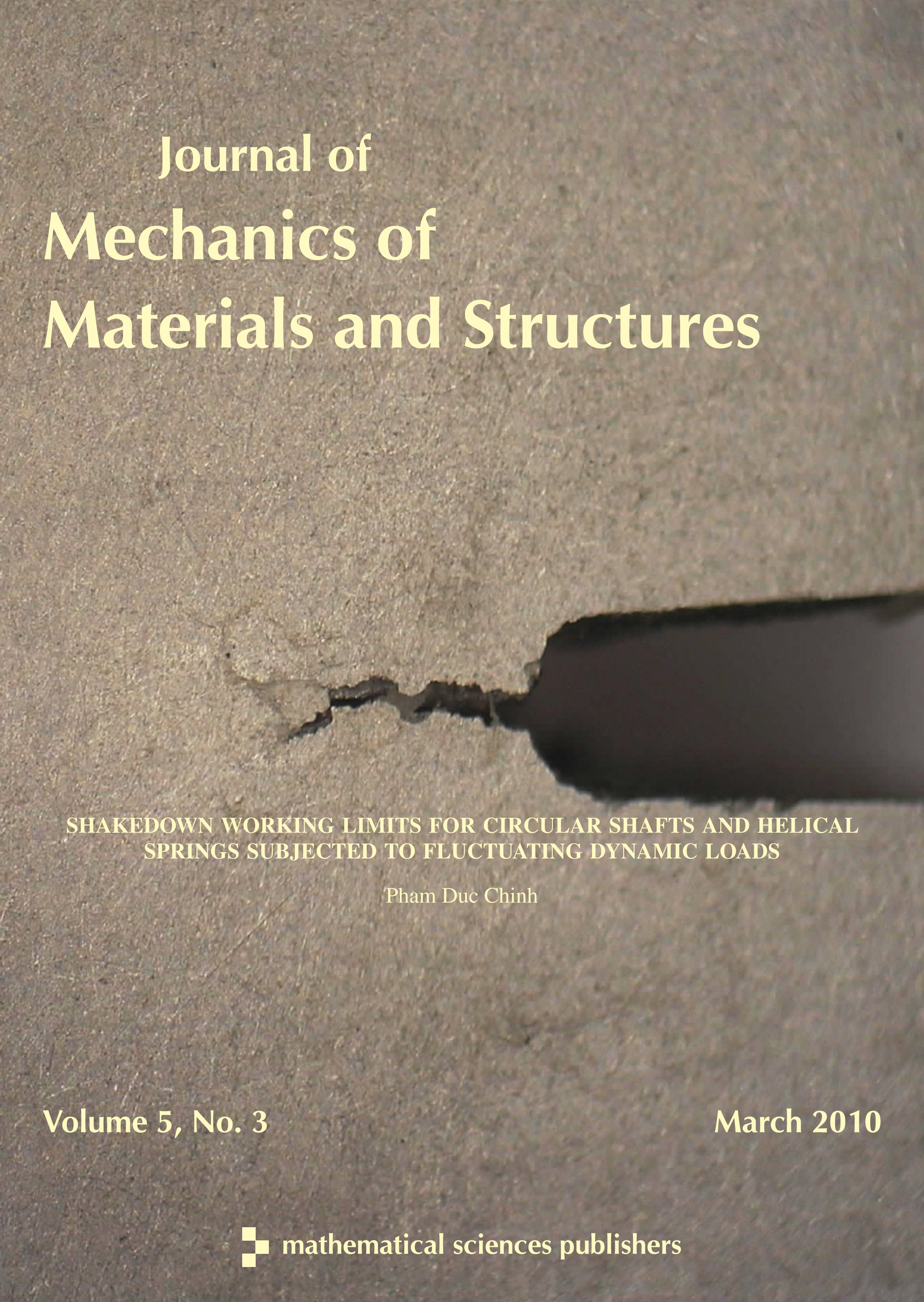

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SHAKEDOWN WORKING LIMITS FOR CIRCULAR SHAFTS AND HELICAL SPRINGS SUBJECTED TO FLUCTUATING DYNAMIC LOADS

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Under quasiperiodic fluctuating dynamic loads, a structure made of elastic plastic material may fail by incremental collapse (ratcheting) or alternating plasticity (fatigue). For the kinematic hardening materials considered, the only two crucial material parameters needed are the initial and ultimate yield stresses, but not the generally deformation-history-dependent hardening curve between them. With the high-cycle loading we suggest taking the fatigue limit as the initial yield stress, and taking the stress corresponding to a certain allowable amount of plastic deformation from the empirical Ramberg–Osgood curve (or the particular cyclic yield strength corresponding to the amount 0.2% of plastic deformation) as the ultimate yield stress in our shakedown analysis of structures. The approach is practical and well founded within our shakedown theory, while the small deformation assumption framework of the classical plasticity theory is kept. As illustrations, we derive explicit expressions of the working load limits for the circular shaft and helical spring, which are based on the shakedown analysis and can be used for safety design of the structures with given loading conditions.

1. Introduction

The design of machine elements made of elastic plastic materials, including shafts and springs [Lubliner 1990; Beer and Johnston 1992; Parmley 2000; Okopny et al. 2001; Akiniwa et al. 2008], requires the determination of plastic collapse loads for the structures. A plastic load limit is reached when an entire section of a determinate structure yields plastically, or full plastic yielding happens at a number of sections within an indeterminate structure to make it a mechanism. Many practical machine elements are subjected to fluctuating dynamic loads, whether periodic [Gavarini 1969] or quasiperiodic [Pham 1992; 2008]. Under such fluctuating dynamic loads, a structure would not collapse instantaneously according to the classical plastic limit theory, thanks to the inertia effect, but would fail incrementally (ratcheting mode) or by alternating plasticity (fatigue mode). The problem can be solved in the framework of shakedown theory [Koiter 1963; Gokhfeld and Cherniavski 1980; König 1987; Bree 1989; Pham 1992; 2003; 2005; 2007; 2008; Pham and Stumpf 1994; Pham and Weichert 2001; Weichert and Maier 2002].

Machines and structures are often made of elastic plastic materials that can be described by various sophisticated kinematic hardening models [Prager 1949; Armstrong and Frederick 1966; Ohno and Wang 1993; Pham 2007; Chaboche 2008], in which the hardening curve is generally nonlinear and depends on the plastic deformation history. However it was established in [Pham 2007; 2008] that for the shakedown safety assessment of a structure, the only plastic parameters needed are the initial and ultimate yield

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stresses, not the particular hardening curve between them. Further development of the theory and its practical implementation will be demonstrated as it applies to shaft and spring structures in this study.

2. Circular shaft

Consider a circular shaft along the central axis z ($0 \le z \le L$), attached to a fixed support at one end ($z = 0$), as presented in Figure 1. When the shaft is subjected to torsion, every cross-section remains plane and undistorted. That implies the kinematic assumption for the tangential angular displacement in the shaft's circular cross-section:

$$u_\varphi = C(z)r, \tag{1}$$

where C is a function of z, and r is the radial distance from the neutral axis of the shaft. The respective shear strain is

$$\gamma = \frac{\partial u_\varphi}{\partial z} = \frac{dC}{dz}r = C_1(z)r. \tag{2}$$

As the torque $M(L)$ is applied to the free end ($z = L$) of the shaft, the shaft will twist, with its cross-section at z rotating through an angle $\varphi(z)$, which, in the elastic range, is related to the elastic moment $M^e(z)$ via the differential relation

$$\frac{d\varphi}{dz} = \frac{M^e}{GJ_p}, \tag{3}$$

where G is the elastic shear modulus and J_p is the polar moment of inertia, which for a circular hollow shaft of constant cross-section, with inner and outer radii R_1 and R_2, has the expression

$$J_p = \frac{\pi}{2}(R_2^4 - R_1^4). \tag{4}$$

The elastic shear stress in the shaft is

$$\tau_\varphi^e = \frac{M^e}{J_p}r, \qquad R_1 \le r \le R_2\,. \tag{5}$$

The shaft is made of an elastic plastic kinematic hardening material with initial and ultimate yield stresses τ_Y^i and τ_Y^u [Pham 2007; 2008].

As the torque increases, the maximal shear stress at the outer radius R_2 from (5) reaches the initial yield value τ_Y^i, and the moment over the whole section achieves the initial yield value

$$M_Y^i = \tau_Y^i \frac{J_p}{R_2}. \tag{6}$$

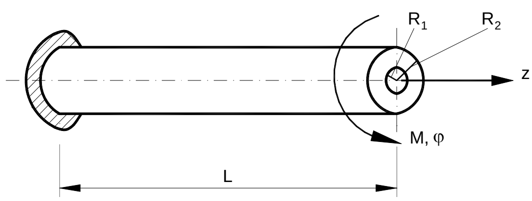

Figure 1. Conventions for a circular shaft.

The whole section of the shaft yields plastically at the ultimate yield moment

$$M_Y^u = \int_{R_1}^{R_2} r\tau_Y^u 2\pi r\, dr = \tau_Y^u \frac{2\pi}{3}(R_2^3 - R_1^3). \tag{7}$$

M_Y^u is considered the plastic collapse limit for the shaft subjected to static torque.

Assume that the free end of the shaft is subjected to the quasiperiodic dynamic torque

$$M(L) = M_p(\underline{t}) + M_q(\underline{t})\sin[\omega(\underline{t})t], \tag{8}$$

where $M_p(\underline{t})$, $M_q(\underline{t})$ and $\omega(\underline{t})$, with an underlined time variable, are slowly-varying functions of time (so their time derivatives can be neglected in comparison with the functions themselves), varying within the limits

$$M_p^- \le M_p(\underline{t}) \le M_p^+, \quad |M_p^-| \le M_p^+, \quad 0 \le M_q(\underline{t}) \le M_q^+, \quad 0 \le \omega(\underline{t}) \le \omega^+ < \omega_I = \frac{\pi}{2L}\sqrt{\frac{G}{m}}, \tag{9}$$

Here ω_I is the principal natural frequency of the shaft (in twisting vibration), and m is the shaft's mass density.

We need to determine the collapse load limits for the shaft in the space of external load parameters M_p^-, M_p^+, M_q^+, ω^+.

The equilibrium equation for the problem is

$$\frac{dM}{dz} = mJ_p \frac{d^2\varphi}{dt^2}. \tag{10}$$

The elastic moment solution of the problem (10)+(3) with boundary conditions (8) and $\varphi(0) = 0$ is

$$M^e = M_p + M_q \frac{\cos(\sqrt{m/G}\omega z)}{\cos(\sqrt{m/G}\omega L)} \sin(\omega t)\;. \tag{11}$$

The shakedown kinematic theorem [Pham 2007; 2008], applied to the problem and expressed through the shakedown safety factor k_s (at $k_s < 1$ the structure collapses, at $k_s > 1$ it is safe, and $k_s = 1$ determines the shakedown boundary in the space of external load parameters), has the form

$$k_s^{-1} = \max\{I, A\}, \tag{12}$$

where I and A describe respectively the incremental and alternating plasticity collapse modes:

$$\begin{aligned} I &= \sup_{\tau_\varphi^e;\gamma} \frac{\int_0^L \int_{R_1}^{R_2} \max_t(\tau_\varphi^e \gamma) 2\pi r\, dr\, dz}{\int_0^L \int_{R_1}^{R_2} \tau_Y^u |\gamma| 2\pi r\, dr\, dz} = \sup_{M^e;C_1} \frac{J_p^{-1}\int_0^L \int_{R_1}^{R_2} \max_t M^e C_1 r^3\, dr\, dz}{\tau_Y^u \int_0^L \int_{R_1}^{R_2} |C_1| r^2\, dr\, dz} \\ &= \sup_{C_1} \frac{\int_0^L \max_t \Big[M_p + M_q \dfrac{\cos(\sqrt{m/G}\omega z)}{\cos(\sqrt{m/G}\omega L)}\sin(\omega t)\Big] C_1(z) \int_{R_1}^{R_2} r^3\, dr\, dz}{J_p \tau_Y^u \int_0^L |C_1(z)| \int_{R_1}^{R_2} r^2\, dr\, dz} \\ &= \sup_{C_1} \frac{\int_0^L \Big[M_p^+ + M_q^+ \dfrac{\cos(\sqrt{m/G}\omega^+ z)}{\cos(\sqrt{m/G}\omega^+ L)}\Big] C_1(z)\, dz}{M_Y^u \int_0^L |C_1(z)|\, dz} = \frac{1}{M_Y^u}\Big[M_p^+ + \frac{M_q^+}{\cos(\sqrt{m/G}\omega^+ L)}\Big], \end{aligned} \tag{13}$$

$$A = \sup_{z,r,t_1,t_2} \frac{\tau^e(z,r,t_1) - \tau^e(z,r,t_2)}{2\tau_Y^i}$$
$$= \sup_{z,r} \frac{\Big[M_p^+ - M_p^- + 2M_q^+ \dfrac{\cos(\sqrt{m/G}\omega^+ z)}{\cos(\sqrt{m/G}\omega^+ L)}\Big] r}{2\tau_Y^i J_p} = \frac{1}{2M_Y^i}\Big[M_p^+ - M_p^- + \frac{2M_q^+}{\cos(\sqrt{m/G}\omega^+ L)}\Big]. \quad (14)$$

Here γ is the compatible plastic strain increment of the type (2), and τ^e is the elastic stress from (5) and (11). To obtain the last equality in (13), we applied a theorem on the norm of a linear functional. The optimal field $C_1(z)$ in (13) is proportional to $\delta(z)$, the Dirac delta function, which means the incremental collapse mode (13) happens at the section $z = 0$. The alternating plasticity collapse mode (14) takes place at $z = 0$ and $r = R_2$.

Two-surface models for kinematic hardening materials involving the initial and ultimate yield stresses have been used widely in literature; see, among others, [Halphen and Nguyen 1975; Mandel 1976; Weichert and Gross-Weege 1988; Polizzotto et al. 1991; Stein et al. 1992; Corigliano et al. 1995; Pham and Weichert 2001; Nguyen 2003; Pham 2005]. Our model leaves unspecified the hardening curve, which generally depends on the plastic deformation history, but assumes it satisfies the positive hysteresis postulate ($\oint \boldsymbol{\alpha}\, d\boldsymbol{\varepsilon}^p \geq 0$ for any closed cycle, where $\boldsymbol{\alpha}$ is the back stress and $\boldsymbol{\varepsilon}^p$ the plastic deformation). This postulate seems to be supported by the experimental data in the literature [Pham 2007].

For our particular problem, the plastic deformation does change proportionally at every point within the structure over loading cycles; hence the expressions (12)–(14) are exact, not just an upper bound, for the shakedown safety factor k_s. For more details, consult [Pham and Stumpf 1994].

From the relations (12)–(14), the safety criterion against incremental plastic collapse (ratcheting) of the shaft can be represented as

$$M_p^+ + \frac{M_q^+}{\cos(\sqrt{m/G}\omega^+ L)} \leq M_Y^u \quad \text{(i.e., } I \leq 1), \quad (15)$$

while the safety against alternating plasticity collapse (fatigue) requires

$$M_p^+ - M_p^- + \frac{2M_q^+}{\cos(\sqrt{m/G}\omega^+ L)} \leq 2M_Y^i \quad \text{(i.e., } A \leq 1). \quad (16)$$

At $M_q^+ = 0$, (15) reduces to the known criterion for safety of the shaft against static plastic collapse stated in (7).

The shear initial yield stress τ_Y^i appearing in the expression (6) for the initial yield moment M_Y^i in (14) and (16), which is responsible for the alternating plasticity mode, is generally not the convenient one corresponding to the amount 0.2% of plastic deformation, but may take its value as small as the fatigue stress limit τ_Y^f, since (16) determines the alternating plasticity collapse at the high-number of loading cycles (about 10^6–10^7 cycles). For particular loading processes with smaller numbers of cycles, τ_Y^i (and hence M_Y^i) may be given larger values, up to the ultimate shear yield stress τ_Y^u (corresponding to M_Y^u), and can be taken from the fatigue curve for the particular material making the shaft.

Similarly, the shear ultimate yield stress τ_Y^u appearing in the expression (7) for the ultimate yield moment M_Y^u in (13) and (15), which is responsible for the incremental mode, is generally not that determined from a monotonic loading experiment, but may be smaller and can be taken from ratcheting experiments on high number of cycles corresponding to those met in particular loading conditions of the material. The ratcheting (ultimate yield) stress may also be taken as that corresponding to a certain amount of allowable plastic deformation. Fatigue and ratcheting are observed widely in experiments on the mechanical properties of materials.

We need the experimental ratcheting curve (yield stress versus number of cycles) for a material, like the known fatigue curve, for application in our shakedown safety assessment procedure, in particular for the incremental mode.

Introduce the dimensionless parameters and variables

$$a = \frac{M_p^-}{M_p^+}, \quad f = \frac{M_Y^i}{M_Y^u} = \frac{3(R_2^4 - R_1^4)\tau_Y^i}{4R_2(R_2^3 - R_1^3)\tau_Y^u}, \quad P = \frac{M_p^+}{M_Y^u}, \quad Q = \frac{M_q^+}{M_Y^u}, \quad W = \sqrt{\frac{m}{G}}\omega^+ L \frac{2}{\pi}. \tag{17}$$

Then (15) and (16) can be represented as

$$P + \frac{Q}{\cos(W\pi/2)} \leq 1 \qquad (\text{i.e., } I \leq 1), \tag{18}$$

$$P(1-a) + \frac{2Q}{\cos(W\pi/2)} \leq 2f \quad (\text{i.e., } A \leq 1). \tag{19}$$

The incremental collapse curve $I = 1$, through the W-Q relation, can be written as

$$Q = (1-P)\cos(W\pi/2), \tag{20}$$

while the alternating plasticity collapse curve $A = 1$ is

$$Q = \big(f - P(1-a)/2\big)\cos(W\pi/2). \tag{21}$$

Comparing (20) and (21), one sees that, at

$$P > \frac{2(1-f)}{1+a}, \tag{22}$$

the curve (20) lies under the curve (21); thus, according to (12), the collapse mode is incremental, while at

$$P < \frac{2(1-f)}{1+a} \tag{23}$$

the curve (21) is lower; hence the collapse mode is alternating plasticity.

As numerical illustrations, we present in Figure 2 the shakedown curve (20) and (21) in the W-Q coordinate plane for these particular cases (the domain under the curve is the safety domain):

- $a = 0,\ f = \frac{1}{2},\ P = \frac{1}{4}$, fatigue mode ($A = 1$): $Q = \frac{3}{8}\cos(W\pi/2)$;
- $a = 1,\ f = \frac{1}{3},\ P = \frac{7}{10}$, ratcheting mode ($I = 1$): $Q = \frac{3}{10}\cos(W\pi/2)$;
- $a = 1,\ f = \frac{1}{2},\ P = \frac{3}{4}$, ratcheting mode ($I = 1$): $Q = \frac{1}{4}\cos(W\pi/2)$;
- $a = \frac{1}{2},\ f = \frac{1}{3},\ P = \frac{1}{2}$, fatigue mode ($A = 1$): $Q = \frac{5}{24}\cos(W\pi/2)$.

On approaching the principal natural frequency of the structure ($W \to 1$), the safety limit on Q reduces to 0.

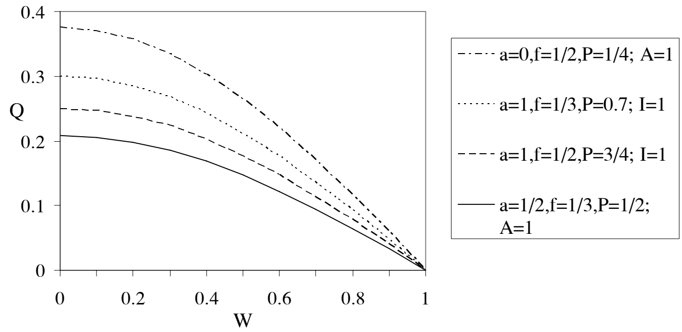

Figure 2. Shakedown curves (and modes) in the plane of external torque's frequency-amplitude parameters, at various values of a, f, P.

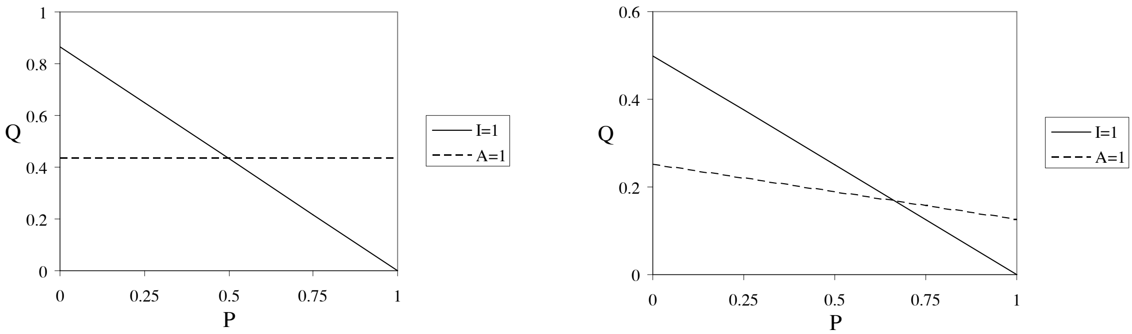

Figure 3. Incremental ($I = 1$) and alternating plasticity ($A = 1$) lines in the plane of external torque's amplitudes' parameters, at $a = 1$, $W = \frac{1}{3}$ (left) and at $a = \frac{1}{2}$, $W = \frac{2}{3}$ (right).

Alternatively, the incremental collapse line $I = 1$ from (18) and the alternating plasticity collapse line $A = 1$ from (19) are plotted in Figure 3 in the P-Q coordinate plane for these two cases:

- $a = 1,\ W = \frac{1}{3}$, corresponding to $I = P + \frac{2}{\sqrt{3}} Q = 1,\ A = \frac{4}{\sqrt{3}} Q = 1$;
- $a = \frac{1}{2},\ W = \frac{2}{3}$, corresponding to $I = P + 2Q = 1,\ A = \frac{1}{2} P + 4Q = 1$.

The shakedown domain is what lies under both the incremental (ratcheting) $I = 1$ and alternating plasticity (fatigue) $A = 1$ lines. At the intersect of the lines, the collapse mode changes from one mode to the other

3. Helical spring

Consider a cylindrical coiled spring Figure 4 with small angle of lifting of coil $\alpha \ll 1$ and the central axis z ($0 \le z \le L$, not counting the two irregular short ends) along the wire making the spring; D is the pitch diameter of spring; R_1 and R_2 are inner and outer radii of the circular hollow wire. One end of the spring is fixed, while the other end is subjected to the dynamic quasiperiodic fluctuating load

$$F = F_p(\underset{\sim}{t}) + F_q(\underset{\sim}{t}) \sin[\omega(\underset{\sim}{t}) t], \tag{24}$$

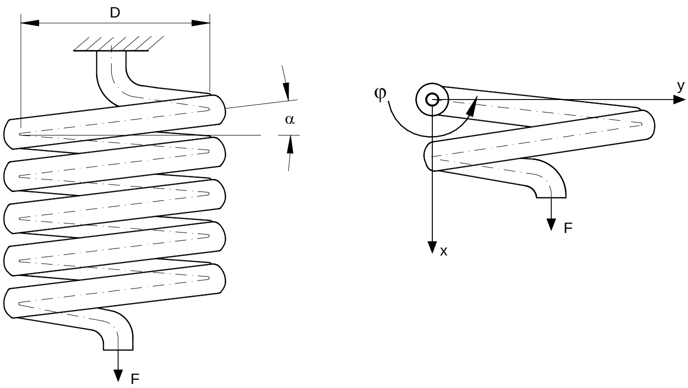

Figure 4. Conventions for a helical spring.

where $F_p(\underline{t})$, $F_q(\underline{t})$, $\omega(\underline{t})$ with underlined time variable are slowly-varying functions of time, which vary within the limits

$$F_p^- \le F_p(\underline{t}) \le F_p^+, \quad |F_p^-| \le F_p^+, \quad 0 \le F_q(\underline{t}) \le F_q^+, \quad 0 \le \omega(\underline{t}) \le \omega^+ < \omega_I, \tag{25}$$

ω_I is the principal natural frequency of the spring. We have to determine the collapse load limits for the spring in the space of external load parameters F_p^-, F_p^+, F_q^+, ω^+.

The elastic shear stress on a cross-section of the spring's wire is composed of the two parts: the torsional shear stress (M^e and F^e are the torque and cutting load acting on the section, r is the radial distance from the section's center)

$$\tau_\varphi^e = \frac{M^e}{J_p} r = \frac{F^e}{2J_p} Dr, \tag{26}$$

and the cutting shear stress

$$\tau_x^e = \frac{F^e}{\pi(R_2^2 - R_1^2)}. \tag{27}$$

Comparing (26) and (27), one sees that under the condition

$$R \ll D, \tag{28}$$

we have

$$|\tau_x^e| \ll |\tau_\varphi^e|, \tag{29}$$

and the component τ_x^e can be disregarded as a small contribution (compared to τ_φ^e). The effect of the two irregular short ends of the spring is also disregarded (they are considered as rigid).

The elastic moment on the wire's sections in response to load (24) is

$$M^e = F_p \frac{D}{2} + F_q \frac{D}{2} \frac{\cos(\sqrt{m/G}\omega z)}{\cos(\sqrt{m/G}\omega L)} \sin(\omega t)\,. \tag{30}$$

Then, similar to the problem of the previous section, the shakedown kinematic theorem applied to the problem has the particular form

$$k_s^{-1} = \max\{I, A\}, \tag{31}$$

with

$$I = \frac{P}{g} + \frac{Q}{g\cos(W\pi/2)}, \qquad A = P\frac{1-a}{2} + \frac{Q}{\cos(W\pi/2)}, \tag{32}$$

where

$$\begin{gathered} P = \frac{F_p^+}{F_Y^i}, \quad Q = \frac{F_q^+}{F_Y^i}, \quad W = \sqrt{\frac{m}{G}}\omega^+ L \frac{2}{\pi}, \quad a = \frac{F_p^-}{F_p^+}, \\ g = \frac{F_Y^u}{F_Y^i} = \frac{4R_2(R_2^3 - R_1^3)\tau_Y^u}{3(R_2^4 - R_1^4)\tau_Y^i}, \quad F_Y^i = \tau_Y^i \frac{\pi}{DR_2}(R_2^4 - R_1^4), \quad F_Y^u = \tau_Y^u \frac{4\pi}{3D}(R_2^3 - R_1^3). \end{gathered} \tag{33}$$

The expressions (31)–(32) are similar to those in (12)–(14), with F_p^+, F_p^-, F_q^+, F_Y^i, F_Y^u replacing M_p^+, M_p^-, M_q^+, M_Y^i, M_Y^u, respectively; hence the shakedown analysis follows the same line.

As already stated in our previous works, the initial yield stress should be taken as small as the fatigue limit for the shakedown safety in the general path-independent spirit of the shakedown theorems. The ultimate yield stress is expected to be the lowest limit from those obtained in multicycle loading experiments rather than that obtained in the standard monotonic loading experiment. However the high-cycle ultimate yield strength as well as the monotonic one are often attained at the large plastic deformations, which fall far outside the small deformation assumption framework of the classical plasticity theory and the shakedown theorems. Also the design requirement of many structures would not allow excessive global configuration changes due to the large plastic deformations. Hence we suggest taking for the ultimate yield stress (of the quasistatic or low-cycle processes) - the yield stress corresponding to some allowable small amount of plastic deformation from the standard monotonic loading experiment, such as that from the broadly used Ramberg–Osgood empirical formula (the unnecessary for our purpose elastic part of the relationship is dropped)

$$\sigma_Y = K(\varepsilon^p)^n, \tag{34}$$

where σ_Y is the yield strength, ε^p the plastic deformation, K the strength coefficient, and n the strain hardening exponent. The best known one is the yield strength $\sigma_Y^{(0.2)}$ corresponding to 0.2% of plastic deformation; this is considered as the first significant amount of irreversible strain:

$$\sigma_Y^{(0.2)} = K(0.002)^n. \tag{35}$$

Note that though a local plastic deformation at the amount 0.2% may be insignificant for the global geometry of a structure because of the global compatible strain constraint, when a global incremental mechanism $I = 1$ is formed with $\sigma_Y^u = \sigma_Y^{(0.2)}$, a significant global compatible plastic strain increment arises leading to a significant configuration change of the structure. Still, Equations (34) and (35), obtained from the monotonic loading experiment, are the only first approximations for our ultimate yield stress σ_Y^u of multicycle loading processes, which should be established from high-cycle loading experiments (such as those in the fatigue tests). Ideally we need the stress-controlled cyclic loading experiments leading to certain allowable amount of plastic deformation. For cyclic softening materials, we may rely on the strain-controlled cyclic loading experiments, as those presented in [Tucker et al. 1979; Roessle and Fatemi 2000; Li et al. 2009]. From the multicycle tests they got the Ramberg–Osgood type relationship

$$\sigma_{Yc} = K'(\varepsilon^p)^{n'}, \tag{36}$$

Alloy steel	K'	n'	σ_{Yf}	$\sigma_{Yc}^{(0.2)}$	$\sigma_{Y}^{(0.2)}$	σ_{Yb}	ε_{Yb}
SAE 1141*	1127	0.124	433	591	814	925	0.88
SAE 1038	1009	0.208	248	364	410	649	1.10
SAE 1541	1622	0.194	228	424	475	783	0.80
SAE 1090	1310	0.174	350	545	735	1090	0.15
08 Si_2Mn	524	0.110	195	248	400	414	1.02
20 Si_2Mn	772	0.180	152	241	262	441	0.96
40 Si_2Mn	1434	0.140	403	600	883	931	1.02
60 Si_2Mn	1358	0.120	381	648	789	1000	0.41

* Aluminum fine grain

Table 1. Strength properties of some alloy steels. K' is the cyclic strength coefficient, n' the cyclic strain hardening exponent, σ_{Yf} the fatigue limit, $\sigma_{Yc}^{(0.2)}$ the cyclic yield strength (0.2%); $\sigma_{Y}^{(0.2)}$ the yield strength (0.2%), σ_{Yb} the ultimate yield strength, and ε_{Yb} the corresponding ultimate plastic strain. All strength parameters are given in MPa, except the dimensionless parameters n' and ε_{Yb}.

where σ_{Yc} is the cyclic yield stress amplitude, ε^p is the cyclic plastic deformation amplitude, K' is the cyclic strength coefficient, and n' is the cyclic strain hardening exponent. The most significant strength parameter might be the cyclic yield strength $\sigma_{Yc}^{(0.2)}$ corresponding to the amount 0.2% of plastic deformation, which we could adopt as the ultimate yield stress σ_Y^u for the incremental collapse mode

$$\sigma_{Yc}^{(0.2)} = K'(0.002)^{n'}. \tag{37}$$

It designates the critical point, beyond which the excessive global compatible plastic deformation increments of the structure are expected. Because the small plastic deformation assumption is kept, application of the classical plasticity theory and our path-independent shakedown theorems is legitimate.

The cyclic yield strength $\sigma_{Yc}^{(0.2)}$, cyclic strength coefficient K', cyclic strain hardening exponent n', as well as the fatigue limit σ_{Yf}, yield strength $\sigma_{Y}^{(0.2)}$, ultimate yield strength σ_{Yb} and corresponding ultimate plastic strain ε_{Yb} for a number of alloy steels are given in [Tucker et al. 1979; Roessle and Fatemi 2000; Li et al. 2009], some of which are presented in Table 1. All the strength parameters are given in MPa, except the dimensionless parameters n' and ε_{Yb}. Note that the ultimate yield strength σ_{Yb} generally is reached at the large amount of plastic deformation ε_{Yb}, and the cyclic yield strength $\sigma_{Yc}^{(0.2)}$ may be much different from the yield strength $\sigma_{Y}^{(0.2)}$.

As numerical illustrations of Equations (31)–(33) we choose SAE-1090 steel, $a = \frac{3}{4}$, $W = \frac{2}{3}$, and the following situations:

- The alternating plasticity collapse mode $A = 1$ with $\tau_Y^i = \frac{1}{2}\sigma_Y^i = \frac{1}{2}\sigma_{Yf} = 175\,\text{MPa}$,
- The incremental collapse mode $I = 1$ with $\tau_Y^u = \frac{1}{2}\sigma_Y^u = \frac{1}{2}\sigma_{Yc}^{(0.2)} = 272.5\,\text{MPa}$,
- The incremental collapse mode $(I =)\ I' = 1$ with $\tau_Y^u = \frac{1}{2}\sigma_Y^u = \frac{1}{2}\sigma_{Y}^{(0.2)} = 367.5\,\text{MPa}$,
- The incremental collapse mode $(I =)\ I'' = 1$ with $\tau_Y^u = \frac{1}{2}\sigma_Y^u = \frac{1}{2}\sigma_{Yb} = 545\,\text{MPa}$.

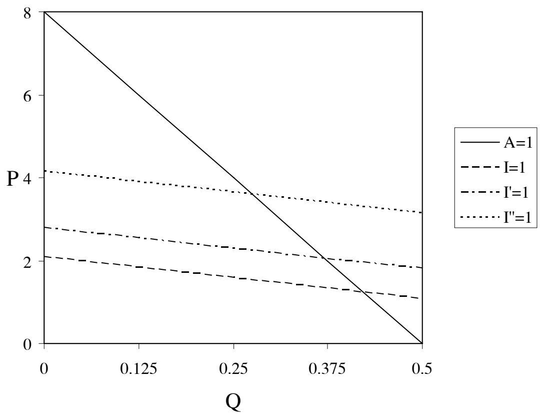

Figure 5. Alternating plasticity line $A = 1$; Incremental lines $I = 1$ ($\sigma_Y^u = \sigma_{Yc}^{(0.2)}$), $I' = 1$ ($\sigma_Y^u = \sigma_Y^{(0.2)}$), and $I'' = 1$ ($\sigma_Y^u = \sigma_{Yb}$) in the plane of external load's amplitude parameters.

The results of calculations are presented in the plane of dimensionless load amplitude parameters Q-P in Figure 5. The shakedown working domain is bounded above by the lower envelope of the ratcheting (incremental) and fatigue (alternating plasticity) lines. On the boundary of the domain, the collapse mode changes from the ratcheting one at small values of Q to the fatigue one at sufficiently high values of Q.

One should keep in mind that, in the light of shakedown analysis, the alternating plasticity (fatigue) mode is local, while the incremental (ratcheting) mode is global [Pham 2000].

4. Conclusion

Our shakedown theory has been applied to determine the working load limits for some typical elements of machines subjected to quasiperiodic dynamic loads. In the static limit, the results reduce to those plastic limit ones often used in design of the structures. It is clear that for dynamic loading, the load amplitude limits may decrease significantly, especially when the frequency of the acting load approaches the natural frequencies of a structure. The two distinct nonshakedown collapse modes: the incremental and alternating plasticity ones are separated.

In shakedown safety analysis for elastic plastic kinematic hardening materials, the only plastic parameters required are the initial and ultimate yield stresses. However for high-cycle loadings, which are usual for structures under working dynamic fluctuating loads, the initial and ultimate yield stresses should not be taken as the convenient and usual ones from monotonic loading experiments, but may be much lower and are to be taken from the fatigue and ratcheting curves experimentally constructed for the materials under high-cycle loadings (up to 10^6–10^7 cycles). Application of shakedown theorems should be kept within the framework of the small plastic deformation assumption. Here, in particular, we take the fatigue limit as the initial yield stress, and the cyclic yield strength (corresponding to 0.2% of plastic deformation) as the ultimate yield stress for the shakedown safety assessment of the structures subjected to dynamic high-cycle loading.

Though used here to analyze only simple shaft and spring structures as illustrations, our approach involving two nonshakedown modes (and the corresponding recommendations) applies to general elastic plastic kinematic hardening structures. The approach is supported by the shakedown theorems in [Pham 2007; 2008].

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