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REFLECTION OF TRANSIENT PLANE STEP-STRESS WAVES: SOME CONSIDERATIONS OF ORTHOTROPY AND THERMOELASTICITY

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To Charles and Marie-Louise Steele

Governing transient equations and (dimensionless) wave speeds for isotropic and orthotropic solids, and for isothermal and thermoelastic cases, are presented. For the orthotropic solid, propagation occurs in a principal plane. The thermoelastic cases treat Fourier heat flow and thermal relaxation, and stress- and temperature-step waves of a class that does not exhibit attenuation and dispersion. Reflection of incident step-waves by a half-space surface is then treated. Situations vary in the combinations of model features noted above. Isotropic limit cases of orthotropic results are also examined, for isothermal and thermoelastic situations. Finally, restrictions on angles of incidence and reflection due to anisotropy are identified, and some related calculations presented.

1. Introduction

The reflection of transient plane waves by traction-free surfaces of elastic solids is an important consideration in seismology [Cagniard 1962] and in models for layers and layered media [Brekhovskikh 1957; Achenbach 1973; Miklowitz 1978]. The reflection process is more complicated in the orthotropic elastic solid [Lindsay 1960] because elastic wave speeds depend on propagation direction. Similarly, isotropic thermoelastic solids [Chadwick 1960; Lord and Shulman 1960; Green and Lindsay 1972; Ignaczak and Ostoja-Starzewski 2010] exhibit waves both with and without dispersion and attenuation.

Whether isothermal [Scott and Miklowitz 1967] or thermoelastic [Sharma and Sidhu 1986], anisotropic solids are often studied in terms of the plane harmonic wave. Thus effects of step-stress (shock) signals in a reflection process are not as readily discerned as in transient analyses of, for example, isotropic isothermal solids [Achenbach 1973; Miklowitz 1978]. Plane wave propagation without dispersion can occur in an isotropic solid subject to thermal relaxation [Ignaczak and Ostoja-Starzewski 2010]. For the orthotropic solid, a class of plane waves exhibits neither dispersion nor attenuation [Brock 2010], whether Fourier conduction [Chadwick 1960] or thermal relaxation [Lord and Shulman 1960; Green and Lindsay 1972] holds. The class includes the step-stress. For plane wave propagation in an arbitrary direction, displacement and stress are in effect defined by the temperature change. For propagation in a principal plane, out-of-plane displacement uncouples from temperature, and travels as a shear wave of arbitrary form. In the isotropic limit both displacement components parallel to the wave fronts travel as shear waves of arbitrary form.

This study examines problems of reflection of a transient plane step-wave by the traction-free surface of a half-space. Two problems involve isotropic solids that are governed by the thermal relaxation

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models of [Lord and Shulman 1960] and [Green and Lindsay 1972], respectively. The former exhibits a single relaxation time, while the latter has two such times. A third problem involves an orthotropic solid governed by the Fourier model [Chadwick 1960], and a fourth problem concerns an isothermal orthotropic situation. For simplicity, orthotropic problems treat plane wave propagation in a principal plane, and include the corresponding isotropic cases as limits. The incident wave in the fourth problem is a step-stress. The first three problems consider incident stress- and temperature-step waves without attenuation or dispersion [Brock 2010].

It should be noted, for the Fourier model in particular, that several problems involve restrictions on the particular combination of natural boundary conditions imposed. Nevertheless, the solutions *de facto* represent nonconventional thermoelastic processes [Ignaczak and Ostoja-Starzewski 2010], and allow insight into propagation without dispersion and attenuation for plane waves with temperature or stress steps.

As indicated above, a substantial literature exists for isothermal wave propagation. For purposes of illustration and the use of uniform definitions of parameters and functions, however, some key results for these problems are included. It is also noted that a plane wave in an infinite isothermal solid is obtained from the solution to an eigenvalue problem. That is, the solution defines the wave speeds and couples components of displacement (and also stress). The work in [Brock 2010] differs only in that a class of solutions — including the step-stress (shock) case — does not exhibit the typical attenuation and dispersion of coupled thermoelasticity.

The study begins with a presentation of governing equations for the orthotropic isothermal and the orthotropic thermoelastic solid. Corresponding equations for propagation of plane waves in a principal plane are then extracted, and characteristic wave speeds for the isothermal and thermoelastic cases examined. The isotropic limit speeds are obtained, and asymptotic formulas for the speeds when orthotropy is weak are then presented. Consideration of the four problems and isotropic limit cases follows.

2. Governing equations for orthotropic elasticity

In terms of principal Cartesian basis $\boldsymbol{x} = (x_1, x_2, x_3)$ and time t, linear momentum balance requires that

$$\partial_k T_{ki} = \bar{\mu} D^2 u_i, \quad D = \frac{\partial_t}{\bar{v}}, \quad \bar{v} = \sqrt{\bar{\mu}/\rho} \qquad (i, k = 1, 2, 3). \tag{1}$$

Here ∂_i and ∂_t signify derivatives with respect to x_i and t, and u_i, T_{ik} are the components of displacement vector $\boldsymbol{u}$ and stress tensor $\boldsymbol{T}(\boldsymbol{x}, t)$. For the isothermal case the components of $\boldsymbol{T}$ are

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \end{bmatrix} = \bar{\mu} \begin{bmatrix} C_1 & C_{12} & C_{13} \\ C_{12} & C_2 & C_{23} \\ C_{13} & C_{23} & C_3 \end{bmatrix} \begin{bmatrix} \partial_1 u_1 \\ \partial_2 u_2 \\ \partial_3 u_3 \end{bmatrix}, \tag{2a}$$

$$T_{23} = \bar{\mu} C_4 (\partial_2 u_3 + \partial_3 u_2), \quad T_{31} = \bar{\mu} C_5 (\partial_3 u_1 + \partial_1 u_3), \quad T_{12} = \bar{\mu} C_6 (\partial_1 u_2 + \partial_2 u_1), \tag{2b}$$

$$\bar{\mu} C_i = c_{ii}, \quad \bar{\mu} C_{ik} = c_{ik} \quad (i \neq k). \tag{2c}$$

Here ρ is mass density, the c_{ik} are the nine elastic constants for orthotropic elasticity [Sokolnikoff 1956; Jones 1999], $\bar{\mu}$ is a reference shear modulus chosen for convenience from the set (c_{44}, c_{55}, c_{66}), and $\bar{v}$ is a corresponding reference speed that becomes the shear (rotational) wave speed in the isotropic limit [Achenbach 1973]. A positive-definite elastic strain energy requires [Jones 1999] that the determinant of the coefficient matrix in (2a) be positive, and that

$$C_k > 0 \qquad (k = 1, 2, 3, 4, 5, 6), \tag{3a}$$

$$C_i C_k - C_{ik}^2 > 0 \qquad (i, k = 1, 2, 3, \quad i \neq k), \tag{3b}$$

$$|C_{ij} C_{ik} - C_i C_{jk}| < \sqrt{(C_i C_j - C_{ij}^2)(C_i C_k - C_{ik}^2)} \qquad (i, j, k = 1, 2, 3 \text{ all distinct}). \tag{3c}$$

The summation convention does not hold in (3c). Equations (1), (2b), (2c) and (3) hold for a thermoelastic solid initially at uniform (absolute) temperature T_0, but coupling of $(\boldsymbol{u}, \boldsymbol{T})$ with the change in absolute temperature $\theta(\boldsymbol{x}, t)$ requires that (1) be augmented by

$$(h_i \partial_i^2 - \tilde{D} D)\theta - \bar{D} D\Big(\frac{\varepsilon_1}{K_1}\partial_1 u_1 + \frac{\varepsilon_2}{K_2}\partial_2 u_2 + \frac{\varepsilon_3}{K_3}\partial_3 u_3\Big) = 0, \tag{4}$$

with $\bar{D}$, $\tilde{D}$, ε_i, h_i defined below. The gradient column matrix in (2a) and (3) are also modified:

$$\begin{bmatrix} \partial_1 u_1 - K_1 \hat{D}\theta \\ \partial_2 u_2 - K_2 \hat{D}\theta \\ \partial_3 u_3 - K_3 \hat{D}\theta \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} C_1 & C_{12} & C_{13} \\ C_{12} & C_2 & C_{23} \\ C_{13} & C_{23} & C_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}, \tag{5a}$$

$$\alpha_i, K_i > 0 \quad (i = 1, 2, 3). \tag{5b}$$

For the Fourier model [Chadwick 1960], denoted by F, and the thermal relaxation models of [Lord and Shulman 1960] and [Green and Lindsay 1972], denoted by I and II, respectively, the operators $\bar{D}$, $\tilde{D}$, $\hat{D}$ are

$$\mathrm{F}: \quad \bar{D}, \tilde{D}, \hat{D} = 1, \tag{6a}$$

$$\mathrm{I}: \quad \bar{D}, \tilde{D} = 1 + h_{\mathrm{I}} D, \quad \hat{D} = 1, \tag{6b}$$

$$\mathrm{II}: \quad \bar{D} = 1, \quad \tilde{D} = 1 + h_{\mathrm{I}} D, \quad \hat{D} = 1 + h_{\mathrm{II}} D. \tag{6c}$$

Here ε_i, h_i are dimensionless thermal coupling constants and h_{I}, h_{II} are thermal characteristic lengths; their expressions are

$$\varepsilon_i = \frac{T_0}{c_V}(\bar{v} K_i)^2, \quad h_i = \frac{k_i}{c_V \sqrt{\bar{\mu}\rho}}, \quad h_{\mathrm{I}} = \bar{v}\, t_{\mathrm{I}}, \quad h_{\mathrm{II}} = \bar{v}\, t_{\mathrm{II}}, \tag{7}$$

where t_{I}, t_{II} are thermal relaxation times, with $t_{\mathrm{II}} > t_{\mathrm{I}}$, while k_i, α_i, c_V are the conductivity, coefficient of linear expansion, and specific heat at constant volume. In the isotropic limit, $\bar{\mu} = \mu$ reduces to the shear modulus μ, while $\bar{v}$ reduces to the isotropic shear (rotational) wave speed $v_S = \sqrt{\mu/\rho}$; we then have, with λ the first Lamé constant and ν Poisson's ratio,

$$C_1, C_2, C_3 = \frac{\lambda}{\mu} + 2, \quad C_4, C_5, C_6 = 1, \quad C_{12}, C_{13}, C_{23} = \frac{\lambda}{\mu} = \frac{2\nu}{1-2\nu}, \tag{8a}$$

$$K_1, K_2, K_3 = K = \Big(3\frac{\lambda}{\mu} + 2\Big)\alpha, \quad \varepsilon_1, \varepsilon_2, \varepsilon_3 = \varepsilon = \frac{T_0}{c_V}\Big[v_r\Big(3\frac{\lambda}{\mu} + 2\Big)\alpha\Big]^2, \tag{8b}$$

$$h_1, h_2, h_3 = h = \frac{k}{c_V \sqrt{\mu\rho}}. \tag{8c}$$

3. Plane wave propagation in arbitrary direction: formulation

The formulation for plane wave propagation in an arbitrary direction is given in [Brock 2010] for the thermoelastic case. For purposes of this article, it is sufficient to consider propagation in the x_1x_2-principal plane. Thus the Cartesian basis (x, y, z) is defined with respect to the principal basis $\boldsymbol{x}$ by the transformation

$$x = x_1 \cos\phi - x_2 \sin\phi, \quad y = x_1 \sin\phi + x_2 \cos\phi, \quad z = x_3 \qquad (|\phi| < \pi/2). \tag{9}$$

Because propagation occurs in the x_1x_2-plane it is convenient to choose

$$\bar{\mu} = c_{66}, \quad \bar{v} = v_6 = \sqrt{\frac{c_{66}}{\rho}}, \quad \tau = v_6 t, \quad D = \partial_\tau. \tag{10}$$

The temporal variable τ has dimensions of length, and D now signifies differentiation with respect to τ. When propagation is in the x-direction and (y, z)-dependence is suppressed, (1) gives

$$\partial_x T_{xx} = c_{66} D^2 u_x, \quad \partial_x T_{yx} = c_{66} D^2 u_y, \quad \partial_x T_{zx} = c_{66} D^2 u_z. \tag{11}$$

Similarly, in place of (2),

$$\begin{bmatrix} T_{xx} \\ T_{yx} \\ T_{zx} \end{bmatrix} = c_{66} \begin{bmatrix} C_x & C_{xy} & 0 \\ C_{xy} & C_y & 0 \\ 0 & 0 & C_{zx} \end{bmatrix} \begin{bmatrix} \partial_x u_x \\ \partial_x u_y \\ \partial_x u_z \end{bmatrix}, \tag{12a}$$

$$\begin{bmatrix} T_{yy} \\ T_{zz} \\ T_{yz} \end{bmatrix} = c_{66} \begin{bmatrix} C_y^x & C_y^y & 0 \\ C_z^x & C_z^y & 0 \\ 0 & 0 & C_{yz} \end{bmatrix} \begin{bmatrix} \partial_x u_x \\ \partial_x u_y \\ \partial_x u_z \end{bmatrix}, \tag{12b}$$

$$T_{xy} = T_{yx}, \quad T_{yz} = T_{zy}, \quad T_{xz} = T_{zx}. \tag{12c}$$

In (12a) the dimensionless coefficients are

$$C_x = C_1 \cos^4\phi + C_2 \sin^4\phi + (1 + \tfrac{1}{2} C_{12}) \sin^2 2\phi, \tag{13a}$$

$$C_y = \cos^2 2\phi + \tfrac{1}{4}(C_1 + C_2 - 2C_{12}) \sin^2 2\phi, \tag{13b}$$

$$C_{xy} = \tfrac{1}{2}[C_2 \sin^2\phi - C_1 \cos^2\phi + (2 + C_{12}) \cos 2\phi] \sin 2\phi, \tag{13c}$$

$$C_{zx} = C_5 \cos^2\phi + C_4 \sin^2\phi. \tag{13d}$$

In (12b) the coefficients are

$$C_y^x = C_{12}(\cos^4\phi + \sin^4\phi) + [\tfrac{1}{4}(C_1 + C_2) - 1] \sin^2 2\phi, \tag{14a}$$

$$C_y^y = \tfrac{1}{2}[C_2 \cos^2\phi - C_1 \sin^2\phi - (2 + C_{12}) \cos 2\phi] \sin 2\phi, \tag{14b}$$

$$C_z^x = C_{13} \cos^2\phi + C_{23} \sin^2\phi, \quad C_z^y = (C_{23} - C_{13}) \sin\phi \cos\phi, \tag{14c}$$

$$C_{yz} = (C_4 - C_5) \sin\phi \cos\phi. \tag{14d}$$

The matrix in (12a) is symmetric but that in (12b) is not, and

$$C_{xy} + C_y^y = (C_2 - C_1) \sin\phi \cos\phi, \quad C_x + C_y = 1 + C_1 \cos^2\phi + C_2 \sin^2\phi. \tag{15}$$

For the thermoelastic case, (11) is coupled with

$$(h_x \partial_x^2 - \tilde{D} D)\theta - D\bar{D}\partial_x\left(\frac{\varepsilon_x}{K_x} u_x + \frac{\varepsilon_y}{K_y} u_y\right) = 0. \tag{16}$$

The modifications of the gradient column matrices in (12a) and (12b) are

$$\begin{bmatrix} \partial_x u_x - K_x \hat{D}\theta \\ \partial_x u_y - K_{yx} \hat{D}\theta \\ \partial_x u_z \end{bmatrix}, \quad \begin{bmatrix} \partial_x u_x - K_y^x \hat{D}\theta \\ \partial_x u_y - K_3 \hat{D}\theta \\ \partial_x u_z \end{bmatrix}. \tag{17}$$

In (16) and (17),

$$h_x = h_1 \cos^2\phi + h_2 \sin^2\phi, \quad \varepsilon_x = \frac{T_0}{c_V}(v_6 K_x)^2, \quad \varepsilon_y = \frac{T_0}{c_V}(v_6 K_{yx})^2. \tag{18}$$

The K coefficients in (16)–(18) are given by

$$K_x = K_1 \cos^2\phi + K_2 \sin^2\phi, \qquad K_y^x = K_1 \sin^2\phi + K_2 \cos^2\phi, \tag{19a}$$

$$K_{yx} = (K_2 - K_1)\sin\phi\cos\phi, \qquad K_x + K_{yx} = K_1 + K_2. \tag{19b}$$

4. Isothermal plane waves

Studies of isothermal waves in isotropic, transversely isotropic and orthotropic solids are available in [Achenbach 1973; Payton 1983; Lindsay 1960; Scott and Miklowitz 1967]. For the sake of transparency and notational consistency, however, basic results are presented here: (11) can be uncoupled, and (12a) then gives

$$(C_{zx}\partial_x^2 - D^2)u_z = 0, \tag{20a}$$

$$\begin{bmatrix} C_x \partial_x^2 - D^2 & C_{xy}\partial_x^2 \\ C_{xy}\partial_x^2 & C_y \partial_x^2 - D^2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{20b}$$

For plane wave propagation in the x-direction, (20) admits the general solutions

$$u_\xi = U_{\xi+} + (c_+\tau - x) + U_{\xi-}(c_-\tau - x), \qquad \xi = (x, y), \tag{21a}$$

$$u_z = U_z(c_z\tau - x), \quad c_z = c_z(\phi) = \sqrt{C_5 \cos^2\phi + C_4 \sin^2\phi}\,. \tag{21b}$$

The functions U_z and $U_{\xi\pm}$ are arbitrary but $U_{\xi\pm}$ is subject to either of two restrictions:

$$(C_x - c_\pm^2)U''_{x\pm} + C_{xy}U''_{y\pm} = 0, \quad C_{xy}U''_{x\pm} + (C_y - c_\pm^2)U''_{y\pm} = 0 \tag{22}$$

Here a prime signifies differentiation. The dimensionless speed c_z is defined in (21b); the dimensionless speeds $c = c_\pm$ come from the roots of the secular equation

$$c^4 - (C_x + C_y)c^2 + C_x C_y - C_{xy}^2 = 0. \tag{23}$$

Properties of quadratics [Abramowitz and Stegun 1972], together with (13), (14) and (15), give

$$\begin{aligned} c_+^2 + c_-^2 &= C_x + C_y = 1 + C_1 \cos^2\phi + C_2 \sin^2\phi, \\ c_+^2 c_-^2 &= C_x C_y - C_{xy}^2 = C_1 \cos^2\phi + C_2 \sin^2\phi + \Omega \sin^2\phi \cos^2\phi. \end{aligned} \tag{24}$$

Solution of (23) gives

$$2c_{\pm} = 2c_{\pm}(\phi) = \Gamma^{+} \pm \Gamma^{-}, \tag{25a}$$

$$\Gamma^{\pm} = \sqrt{1 + C_1 \cos^2\phi + C_2 \sin^2\phi \pm 2\sqrt{C_1 \cos^2\phi + C_2 \sin^2\phi + \Omega \sin^2\phi \cos^2\phi}}, \tag{25b}$$

$$\Omega = (C_1 - 1)(C_2 - 1) - m^2 = \gamma - C_1 - C_2, \quad \gamma = 1 + C_1 C_2 - m^2, \quad m = 1 + C_{12}. \tag{25c}$$

It is noted that $c_{\pm}$ and c_z are symmetric in ϕ. Parameters (γ, m) and Ω, respectively, are used by [Payton 1983] and [Brock and Georgiadis 2007] for transversely isotropic materials, but prove useful in the present study as well. In the isotropic limit $\Omega = 0$ and (21b) and (25) give

$$v_6 = v_r, \quad c_z = c_- = c_S = 1, \quad c_+ = c_D = \sqrt{\frac{\lambda}{\mu} + 2} > 1 \quad (|\phi| < \pi/2). \tag{26}$$

Subscript D and S signify quantities associated with isotropic dilatational and shear waves. For orthotropic materials, $c_+ > c_- > 0$ in any of these situations:

$$\text{A1}: \quad \gamma^2 - 4C_1C_2 < 0, \quad |\phi| < \pi/2; \tag{27a}$$

$$\text{A2}: \quad \gamma^2 - 4C_1C_2 > 0, \quad \gamma > 0, \quad |\phi| < \pi/2; \tag{27b}$$

$$\text{A3}: \quad \gamma^2 - 4C_1C_2 > 0, \quad \gamma < 0, \quad |\phi| < \Phi_{A-}, \quad \Phi_{A+} < |\phi| < \pi/2, \tag{27c}$$

where

$$\Phi_{A\pm} = \tan^{-1} \frac{1}{\sqrt{2C_2}} \sqrt{-\gamma \mp \sqrt{\gamma^2 - 4C_1C_2}}. \tag{27d}$$

If in A3 we take instead $|\phi| = \Phi_{A\pm}$, then $\Gamma^+ = \Gamma^- > 0$ and $c_+ > 0$, $c_- = 0$. If $\Phi_{A-} < |\phi| < \Phi_{A+}$, then Γ^+, Γ^- are complex conjugates with positive real parts, and c_+ is positive real, but c_- is imaginary.

5. Thermoelastic plane waves

Equation (1) again uncouples to produce (20a) and (21b). However, (u_x, u_y) and θ are coupled by (16); as a consequence [Chadwick 1960; Achenbach 1973; Ignaczak and Ostoja-Starzewski 2010] we get for isotropic materials a secular equation that gives complex dimensionless speeds c; that is, u_x, u_y, θ exhibit dispersion and exponential decay.

However one can consider a restricted class of forms [Brock 2010]

$$\theta = \sum_{k=1}^{N} \Theta_{k-1}(c\tau - x)^{k-1} \quad (c\tau \geq x) \tag{28}$$

with $N = 4$ for F and $N = 3$ for I, II. Here the Θ_k are constants; the Θ_0 term represents a propagating temperature step. The secular equations then become

$$\text{F}: \quad c^4 - I_1^F c^2 + I_2^F = 0, \tag{29a}$$

$$\text{I}: \quad c^6 - (l_x + I_1^F)c^4 + [l_x(c_+^2 + c_-^2) + I_2^F]c^2 - l_x c_+^2 c_-^2 = 0, \tag{29b}$$

$$\text{II}: \quad c^6 - (l_x + \bar{I}_1)c^4 + [l_x(c_+^2 + c_-^2) + \bar{I}_2]c^2 - l_x c_+^2 c_-^2 = 0. \tag{29c}$$

The coefficients in (29a) are

$$I_1^F = 1 + F_1 \cos^2\phi + F_2 \sin^2\phi, \quad I_2^F = F_1 \cos^2\phi + F_2 \sin^2\phi + \Omega_F \sin^2\phi \cos^2\phi, \tag{30a}$$

with

$$\Omega_F = (F_1 - 1)(F_2 - 1) - m_F^2 = \gamma_F - F_1 - F_2, \quad \gamma_F = 1 + F_1 F_2 - m_F^2, \quad m_F = 1 + F_{12}, \tag{30b}$$

$$F_1 = C_1 + \varepsilon_1, \quad F_2 = C_2 + \varepsilon_2, \quad F_{12} = C_{12} + \sqrt{\varepsilon_1 \varepsilon_2}. \tag{30c}$$

In (29b) and (29c), c_+ and c_- are given by (25), and

$$\bar{I}_1 = l I_1^F + (1-l)(c_+^2 + c_-^2), \quad \bar{I}_2 = l I_2^F + (1-l) c_+^2 c_-^2, \quad l_x = \frac{h_x}{h_\mathrm{I}} > 1, \quad l = \frac{h_\mathrm{II}}{h_\mathrm{I}} > 1. \tag{31}$$

The inequalities in (31) are based on data [Ignaczak and Ostoja-Starzewski 2010; Brock 2009]. The roots of (29a), (29b) and (29c) give dimensionless speeds $c = c_{F\pm}$, $c = c_{1\mathrm{I}}, c_{2\mathrm{I}}, c_{3\mathrm{I}}$ and $c = c_{1\mathrm{II}}, c_{2\mathrm{II}}, c_{3\mathrm{II}}$. Properties of quadratic and cubic equations [Abramowitz and Stegun 1972] show that

$$c_{F+}^2 + c_{F-}^2 = I_1^F, \quad c_{F+}^2 c_{F-}^2 = I_2^F, \tag{32a}$$

$$c_{1\mathrm{I}}^2 + c_{2\mathrm{I}}^2 + c_{3\mathrm{I}}^2 = l_x + I_1^F, \quad c_{1\mathrm{II}}^2 + c_{2\mathrm{II}}^2 + c_{3\mathrm{II}}^2 = l_x + \bar{I}_1, \tag{32b}$$

$$c_{1\mathrm{I}}^2 c_{2\mathrm{I}}^2 + c_{2\mathrm{I}}^2 c_{3\mathrm{I}}^2 + c_{3\mathrm{I}}^2 c_{1\mathrm{I}}^2 = l_x I_1 + I_2^F, \quad c_{1\mathrm{II}}^2 c_{2\mathrm{II}}^2 + c_{2\mathrm{II}}^2 c_{3\mathrm{II}}^2 + c_{3\mathrm{II}}^2 c_{1\mathrm{II}}^2 = l_x I_1 + \bar{I}_2, \tag{32c}$$

$$c_{1\mathrm{I}}^2 c_{2\mathrm{I}}^2 c_{3\mathrm{I}}^2 = c_{1\mathrm{II}}^2 c_{2\mathrm{II}}^2 c_{3\mathrm{II}}^2 = l_x I_2. \tag{32d}$$

For model F, (29a) and (30) give symmetric real functions of ϕ:

$$2c_{F\pm} = 2c_{F\pm}(\phi) = \Gamma_F^+ \pm \Gamma_F^-, \tag{33a}$$

$$\Gamma_F^\pm = \sqrt{1 + F_1 \cos^2\phi + F_2 \sin^2\phi \pm 2\sqrt{F_1 \cos^2\phi + F_2 \sin^2\phi + \Omega_F \sin^2\phi \cos^2\phi}}. \tag{33b}$$

In the isotropic limit, Ω_F vanishes and (30) and (33) give, for all $|\phi| < \pi/2$,

$$v_6 = v_r, \quad c_z = c_{F-} = c_S = 1, \quad c_{F+} = c_F = \sqrt{c_D^2 + \varepsilon} > 1. \tag{34}$$

The parameter set $(\gamma_F, m_F, \Omega_F)$ is the thermoelastic counterpart of (γ, m, Ω). For the orthotropic material, the behavior of $c_{F\pm}$ is governed by conditions that can be obtained from (27) in Section 4 by replacing $c_\pm, \Gamma^\pm, \gamma, m, \Omega, C_1, C_2$ with $c_{F\pm}, \Gamma_F^\pm, \gamma_F, m_F, \Omega_F, F_1, F_2$, respectively.

Formal expressions for the roots $c_{1\mathrm{I}}^2, c_{2\mathrm{I}}^2, c_{3\mathrm{I}}^2$ of (29b) and basic inequalities that guarantee positive real values are given in [Brock 2010] for Model I. The isotropic case is of interest here, and it can be shown that (29b) gives, for all $|\phi| < \pi/2$,

$$v_6 = v_S, \quad c_{1\mathrm{I}} = c_{\mathrm{I}+}, \quad c_{2\mathrm{I}} = c_{\mathrm{I}-}, \quad c_{3\mathrm{I}} = c_z = c_S = 1, \tag{35a}$$

$$2c_{\mathrm{I}\pm} = \Gamma_\mathrm{I}^+ \pm \Gamma_\mathrm{I}^-, \quad \Gamma_\mathrm{I}^\pm = \sqrt{(c_D \pm \sqrt{l_x})^2 + \varepsilon}. \tag{35b}$$

Available data [Brock 2009] suggest that $1 < c_{\mathrm{I}-} < c_D < c_F < c_{\mathrm{I}+}$. Equation (35a) shows that the components of u_x, u_y corresponding to $c_{3\mathrm{I}}$ uncouple from θ, i.e., are shear waves defined by arbitrary functions of $\tau - x$.

Equation (29c) for Model II yields roots $c_z, c_{1\mathrm{II}}, c_{2\mathrm{II}}, c_{3\mathrm{II}}$. Formal expressions for these, and basic inequalities that guarantee positive real values, are given in [Brock 2010]. As with Model I the isotropic case is of interest here:

$$v_6 = v_S, \quad c_{1\mathrm{II}} = c_{\mathrm{II}+}, \quad c_{2\mathrm{II}} = c_{\mathrm{II}-}, \quad c_{3\mathrm{II}} = c_z = c_S = 1, \tag{36a}$$

$$2c_{\mathrm{II}\pm} = \Gamma_{\mathrm{II}}^{+} \pm \Gamma_{\mathrm{II}}^{-}, \quad \Gamma_{\mathrm{II}} = \sqrt{(c_D \pm \sqrt{l_x})^2 + l\varepsilon} \tag{36b}$$

Available data [Brock 2009] suggest that $1 < c_{\mathrm{II}-} < c_D < c_{\mathrm{F}} < c_{\mathrm{II}+}$.

The components of u_x, u_y that correspond to $c_{\mathrm{F}-}, c_{3\mathrm{I}}, c_{3\mathrm{II}}$ are seen from equations (34), (35a), (36a), respectively, to uncouple from θ, and become shear waves that are arbitrary functions of $\tau - x$. Moreover (35) and (36) give the dimensionless speeds in transient two-dimensional studies that are valid for short times ($\tau/h_{\mathrm{I}} \ll 1$ and $\tau/h_{\mathrm{II}} \ll 1$ respectively); see [Brock 2009].

6. Speeds in weakly orthotropic solids

The limit (26) of (25) can be defined in terms of dimensionless parameters C_1 and Ω, and likewise the limit (34) of (33) in terms of F_1 and Ω_{F}:

$$C_1 \to C_2, \quad \Omega \to 0: \quad c_+(\phi) = c_D, \quad c_-(\phi) = 1, \tag{37a}$$

$$F_1 \to F_2, \quad \Omega_{\mathrm{F}} \to 0: \quad c_{\mathrm{F}+}(\phi) = c_{\mathrm{F}}, \quad c_{\mathrm{F}-}(\phi) = 1, \tag{37b}$$

In similar fashion, results from [Brock 2010] show that

$$C_1 \to C_2, \quad F_1 \to F_2, \quad h_1 \to h_2, \quad \Omega, \Omega_{\mathrm{F}} \to 0: \begin{cases} c_{1k\mathrm{I}}(\phi) = c_{\mathrm{I}+}, \quad c_{2\mathrm{I}}(\phi) = c_{\mathrm{I}-}, \quad c_{3\mathrm{I}}(\phi) = 1, & (38\mathrm{a}) \\ c_{1\mathrm{II}}(\phi) = c_{\mathrm{II}+}, \; c_{2\mathrm{II}}(\phi) = c_{\mathrm{II}-}, \; c_{3\mathrm{II}}(\phi) = 1, & (38\mathrm{b}) \end{cases}$$

When orthotropy is weak, i.e., when $C_1, F_1, h_1, \Omega, \Omega_{\mathrm{F}}$ are close to the limits indicated in (37) and (38), asymptotic formulas can be derived. The derivation does not require explicit formulas such as (25a) and (33a). A first-order variation of (24) and (29), under constraints (23) and (32), gives

$$c_+(\phi) \approx c_D + \frac{\delta C_1}{2c_D}\cos^2\phi - \frac{\Omega}{8mc_D}\sin^2 2\phi, \qquad c_-(\phi) \approx 1 + \frac{\Omega}{8m}\sin^2 2\phi, \tag{39a}$$

$$c_{\mathrm{F}+}(\phi) \approx c_{\mathrm{F}} + \frac{\delta F_1}{2c_{\mathrm{F}}}\cos^2\phi - \frac{\Omega_{\mathrm{F}}}{8m_{\mathrm{F}}c_{\mathrm{F}}}\sin^2 2\phi, \qquad c_{\mathrm{F}-}(\phi) \approx 1 + \frac{\Omega_{\mathrm{F}}}{8m_{\mathrm{F}}c_{\mathrm{F}}}\sin^2 2\phi, \tag{39b}$$

$$[c_{1\mathrm{I}}(\phi), c_{2\mathrm{I}}(\phi)] \approx c_{\mathrm{I}\pm} + \frac{1}{2c_{\mathrm{I}\pm}}\frac{\cos^2\phi}{P_{\mathrm{I}}c_{\mathrm{I}\pm}^2 + Q_{\mathrm{I}}}(l_0\Omega - c_{\mathrm{I}\pm}^2\Omega_{\mathrm{F}})\sin^2\phi + (c_{\mathrm{I}\pm}^2 - 1)\Big[c_{\mathrm{I}\pm}^2\delta F_1 - l_0\delta C_1 + (c_{\mathrm{I}\pm}^2 - c_D^2)\frac{\delta h_1}{h_{\mathrm{I}}}\Big], \tag{39c}$$

$$\begin{aligned}[c_{1\mathrm{II}}(\phi), c_{2\mathrm{II}}(\phi)] \approx c_{\mathrm{II}\pm} + \frac{1}{2c_{\mathrm{II}\pm}}\frac{\cos^2\phi}{P_{\mathrm{II}}c_{\mathrm{II}\pm}^2 + Q_{\mathrm{II}}} &\times [\Omega(l_0 + c_{\mathrm{II}\pm}^2(l-1)) - lc_{\mathrm{II}\pm}^2\Omega_{\mathrm{F}}]\sin^2\phi \\ &+ (c_{\mathrm{II}\pm}^2 - 1)\Big[lc_{\mathrm{II}\pm}^2\delta F_1 + (c_{\mathrm{II}\pm}^2(1-l) - l_0)\delta C_1 + (c_{\mathrm{II}\pm}^2 - c_D^2)\frac{\delta h_1}{h_{\mathrm{I}}}\Big],\end{aligned} \tag{39d}$$

$$c_{3\mathrm{I}}(\phi) \approx 1 + \frac{l_0\Omega - \Omega_\mathrm{F}}{1 - c_\mathrm{F}^2 + ml_0} \frac{1}{8} \sin^2 2\phi, \tag{39e}$$

$$c_{3\mathrm{II}}(\phi) \approx 1 + \frac{(l_0 + l - 1)\Omega - l\Omega_\mathrm{F}}{1 - lc_\mathrm{F}^2 + ml_0 + 2c_D^2(l-1)} \frac{1}{8} \sin^2 2\phi, \tag{39f}$$

where $l_0 = h/h_\mathrm{I}$ is the limit value of l_x, the parameters P_I, P_II, Q_I, Q_II are defined by

$$P_\mathrm{I} = 1 + c_\mathrm{F}^2 - 2l_0, \quad P_\mathrm{II} = 1 + c_D^2 + l\varepsilon - 2l_0, \tag{40a}$$

$$Q_\mathrm{I} = 3l_0^2 + l_0[c_\mathrm{F}^2 + 2(\varepsilon - 1)] - 2c_\mathrm{F}^2, \tag{40b}$$

$$Q_\mathrm{II} = 3l_0[l_0 + \varepsilon(l-1)] + l_0[c_\mathrm{F}^2 + 2(\varepsilon - 1)] - 2(c_D^2 + l\varepsilon), \tag{40c}$$

and $\delta C_1, \delta F_1, \delta h_1, \Omega, \Omega_\mathrm{F}$ are small:

$$\delta C_1 = C_1 - C_2, \quad \delta F_1 = F_1 - F_2, \quad \delta h_1 = h_1 - h_2, \tag{41a}$$

$$\left|\frac{C_1}{C_2} - 1\right| \ll 1, \quad \left|\frac{F_1}{F_2} - 1\right| \ll 1, \quad \left|\frac{h_1}{h_2} - 1\right| \ll 1, \quad \left|\frac{\gamma}{C_1 + C_2} - 1\right| \ll 1, \quad \left|\frac{\gamma_\mathrm{F}}{F_1 + F_2} - 1\right| \ll 1. \tag{41b}$$

7. Isotropic case: shear wave reflection for model I

The isotropic half-space $x_1 > 0$ is initially at rest at uniform temperature T_0. An incident plane shear step-wave travels in the negative x-direction, and reaches surface point $(x_1, x_2) = 0$ at time $t = 0$ $(\tau = 0)$:

$$(u_x, u_z) = 0, \quad T_{xy} = G_i + G_i'(\tau + x)(\tau + x \geq 0). \tag{42}$$

Here G_i, G_i' are given constants. In view of (9), the wave (42) generates, for $x_1 = 0$ and $\tau + x_2 \sin\phi \geq 0$,

$$\theta = 0, \quad (T_{12}, T_{11}) = (\cos 2\phi, -\sin 2\phi)[G_i + G_i'(\tau + x_2 \sin\phi)] \tag{43}$$

Reflection of (43) generates plane waves governed by (11), (28), (29b) and (34). These travel away from the surface, i.e., in positive x-directions whose angles differ from ϕ. In view of (9), (12) and (17), therefore, we have, for $x_1 = 0$ and $\tau - x_2 \sin\phi_S \geq 0$,

$$\theta = 0, \quad (T_{12}, T_{11}) = (\cos 2\phi_S, -\sin 2\phi_S)[G_S + G_S'(\tau - x_2 \sin\phi_S)]. \tag{44}$$

For $x_1 = 0$, $c_{\mathrm{I}\pm}\tau - x_2 \sin\theta_\pm \geq 0$ we have

$$\theta = \Theta_\pm + \Theta_\pm'(c_{\mathrm{I}\pm}\tau - x_2 \sin\phi_\pm), \tag{45a}$$

$$(T_{12}, T_{11}) = -\frac{\mu K}{d_\pm}(\sin 2\phi_\pm, C_\pm)[\Theta_\pm + \Theta_\pm'(c_{\mathrm{I}\pm}\tau - x_2 \sin\phi_\pm)], \tag{45b}$$

$$C_\pm = c_{\mathrm{I}\pm}^2 - 2\sin^2\phi_\pm, \quad d_\pm = c_{\mathrm{I}\pm}^2 - c_D^2, \quad d_+ d_- = -\varepsilon c_D^2. \tag{45c}$$

Here $G_S, G_S', \Theta_\pm, \Theta_\pm'$ are unknown constants. The half-space surface remains traction-free and is governed by thermal convection [Chadwick 1960]:

$$x_1 = 0: \quad T_{12} = T_{11} = 0, \quad \partial_1\theta - \beta\theta = 0 \tag{46}$$

Parameter β is related to the Biot number [Boley and Weiner 1985]. Satisfaction of (46) by the summation of (43)–(45) requires that

$$\phi_S = -\phi, \quad \phi_\pm = -\sin^{-1}(c_{\mathrm{I}\pm}\sin\phi) \quad (c_{\mathrm{I}\pm}\sin\phi \le 1). \tag{47}$$

It follows that (46) produces the equations

$$\Theta'_\pm = \pm\frac{\beta}{\Delta}(\Theta_+ + \Theta_-), \qquad \Delta = \cos\phi_+ - \cos\phi_-, \tag{48a}$$

$$\mu K\left(\frac{\Theta_+}{d_+}\sin 2\phi_+ + \frac{\Theta_-}{d_-}\sin 2\phi_-\right) + G_S\cos 2\phi_S + G_i\cos 2\phi = 0, \tag{48b}$$

$$-\mu K\left(\frac{\Theta_+}{d_+}C_+ + \frac{\Theta_-}{d_-}C_-\right) + G_S\sin 2\phi_S - G_i\sin 2\phi = 0, \tag{48c}$$

$$\mu K\frac{\beta}{\Delta}\left(\frac{c_{\mathrm{I}+}}{d_+}\sin 2\phi_+ - \frac{c_{\mathrm{I}-}}{d_-}\sin 2\phi_-\right)(\Theta_+ + \Theta_-) + G'_S\cos 2\phi_S + G'_i\cos 2\phi = 0, \tag{48d}$$

$$-\mu K\frac{\beta}{\Delta}\left(\frac{c_{\mathrm{I}+}}{d_+}C_+ - \frac{c_{\mathrm{I}-}}{d_-}C_-\right)(\Theta_+ + \Theta_-) + G'_S\sin 2\phi_S - G'_i\sin 2\phi = 0. \tag{48e}$$

Equation (45a) implies a spike (Dirac) function $\delta(\tau + x_2\sin\phi)$ at the wave intersection in the heat flux term in (46). Therefore (48) is subject to the restriction

$$c_{\mathrm{I}-}\Theta_+\cos\phi_+ + c_{\mathrm{I}+}\Theta_-\cos\phi_- = 0. \tag{49}$$

The equation set (48b)–(48e) is solved for $(G_S, G'_S, \Theta_\pm)$, whereupon (48a) yields $\Theta'_\pm$. The surface temperature change generated by reflection is of particular interest, and it can be shown that, for $x_1 = 0$ and $\tau + x_2\sin\phi \ge 0$,

$$\theta = \frac{\varepsilon c_D^2}{\mu K}\frac{G'_i N_{2\mathrm{I}}\sin 4\phi}{R_{2\mathrm{I}}R_{3\mathrm{I}}}\left[\frac{\Delta}{\beta} + (c_{\mathrm{I}+} - c_{\mathrm{I}-})(\tau + x_2\sin\phi)\right], \tag{50a}$$

$$R_{k\mathrm{I}} = d_- c^k_{\mathrm{I}+}R_+ - d_+ c^k_{\mathrm{I}-}R_-, \qquad N_{k\mathrm{I}} = d_- c^k_{\mathrm{I}+}R_+ + d_+ c^k_{\mathrm{I}-}R_- \qquad (k = 1, 2, 3), \tag{50b}$$

$$R_\pm = 2\sin 2\phi\sin\phi\sqrt{s^2_{\mathrm{I}\pm} - \sin^2\phi} - \cos^2 2\phi, \qquad s_{\mathrm{I}\pm} = \frac{1}{c_{\mathrm{I}\pm}}. \tag{50c}$$

The function $R_\pm$ is of the Rayleigh type [Achenbach 1973] in the isothermal case, and $R_{2\mathrm{I}}$ is a thermoelastic counterpart. Investigation of possible roots of functions $(N_{k\mathrm{I}}, R_{k\mathrm{I}})$ is beyond the scope of this article, but is necessary to complete this analysis.

Equation (50a) represents the effects of heat production from a shear wave by mode conversion. It is noted that the step-stress term G_i does not contribute to surface temperature. Moreover, restriction (49) is satisfied only when the incident shear wave parameters (G_i, G'_i) are related by

$$\left[\beta\frac{G_i}{G'_i}(c_{\mathrm{I}+}\cos\phi_- - c_{\mathrm{I}-}\cos\phi_+) + 1 - \cos\phi_+\cos\phi_-\right]R_{3\mathrm{I}} - c^2_{\mathrm{I}+}c^2_{\mathrm{I}-}R_{1\mathrm{I}}\sin^2\phi = 0 \tag{51}$$

That is, the surface in general exhibits a spike in the heat flux.

The restriction in (47) indicates that reflections moving at speed $c_{\mathrm{I}-}v_S$ travel parallel to the surface when $c_{\mathrm{I}+}\sin\phi = 1$. Therefore $\phi = \sin^{-1} s_{\mathrm{I}+}$ is the minimum grazing angle of incidence. For this angle

$\cos\phi_+ = 0$ and (50) gives

$$R_+ = -(1-2s_{\mathrm{I}+}^2)^2, \quad R_- = 4s_{\mathrm{I}+}^2\sqrt{1-s_{\mathrm{I}+}^2}\sqrt{s_{\mathrm{I}-}^2 - s_{\mathrm{I}+}^2} - (1-2s_{\mathrm{I}+}^2)^2. \tag{52}$$

Use of (52) causes distinct changes in the forms of (50a) and (51). However, this does not give correspondingly distinctive behavior.

8. Isotropic case: shear wave reflection for model II

Equations (29c) and (34) now govern. Thus (42)–(44) and (45a) hold, but with $c_{\mathrm{I}\pm}$ replaced by $c_{\mathrm{II}\pm}$. In place of (45b) and (45c) we have, for $x_1 = 0$ and $c_{\mathrm{II}\pm}\tau - x_2\sin\phi_\pm \geq 0$,

$$(T_{12}, T_{11}) = -\frac{\mu K}{d_\pm}(\sin 2\phi_\pm, C_\pm)[\Theta_\pm + \Theta'_\pm(h_{\mathrm{II}}c_{\mathrm{II}\pm} + c_{\mathrm{II}\pm}\tau - x_2\sin\phi_\pm)], \tag{53a}$$

$$d_\pm = c_{\mathrm{II}\pm}^2 - c_D^2, \quad d_+d_- = -\varepsilon\frac{h_{\mathrm{II}}}{h_{\mathrm{I}}}c_D^2 \tag{53b}$$

Equations (48a), (48d), (48e) and (49) still hold, with $c_{\mathrm{I}\pm}$ replaced by $c_{\mathrm{II}\pm}$. Equations (47), (48b) and (48c) are replaced by

$$\phi_r = -\phi, \quad \phi_\pm = -\sin^{-1}(c_{\mathrm{II}\pm}\sin\phi)(c_{\mathrm{II}+}\sin\phi \leq 1), \tag{54a}$$

$$\mu K\left(p_+\frac{\Theta_+}{d_+}\sin 2\phi_+ + p_-\frac{\Theta_-}{d_-}\sin 2\phi_-\right) + G_S\cos 2\phi_S + G_i\cos 2\phi = 0, \tag{54b}$$

$$-\mu K\left(p_+\frac{\Theta_+}{d_+}C_+ + p_-\frac{\Theta_-}{d_-}C_-\right) + G_S\sin 2\phi_S - G'_i\sin 2\phi = 0, \tag{54c}$$

$$p_+ = 1 + \frac{\beta}{\Delta}c_{\mathrm{II}+}h_{\mathrm{II}}, \quad p_- = 1 - \frac{\beta}{\Delta}c_{\mathrm{II}-}h_{\mathrm{II}}. \tag{54d}$$

In this case, for $x_1 = 0$ and $\tau + x_2\sin\phi \geq 0$, we have

$$\theta = \frac{\varepsilon c_D^2}{\mu K}\frac{h_{\mathrm{II}}}{h_{\mathrm{I}}}\frac{G'_i N_{2\mathrm{II}}\sin 4\phi}{R_{2\mathrm{II}}R_{3\mathrm{II}}}\left[\frac{\Delta}{\beta} + (c_{\mathrm{II}+} - c_{\mathrm{II}-})(\tau + x_2\sin\phi)\right], \tag{55a}$$

$$R_{k\mathrm{II}} = d_-p_+c_{\mathrm{II}+}^kR_+ - d_+p_-c_{\mathrm{II}-}^kR_-, \quad N_{k\mathrm{II}} = d_-p_+c_{\mathrm{II}+}^kR_+ + d_+p_-c_{\mathrm{II}-}^kR_- \qquad (k = 1, 2, 3). \tag{55b}$$

Equation (50c) still holds, with $c_{\mathrm{I}\pm}$ replaced by $c_{\mathrm{II}\pm}$. The counterpart to (51) does not hold and a surface heat flux spike will arise unless

$$\beta\frac{G_i}{G'_i}(c_{\mathrm{II}-}\cos\phi_+ - c_{\mathrm{II}+}\cos\phi_-)R_{3\mathrm{I}} + (1 - \cos\phi_+\cos\phi_-)R_{3\mathrm{II}} - c_{\mathrm{II}+}^2c_{\mathrm{II}-}^2R_{1\mathrm{II}}\sin^2\phi = 0. \tag{56}$$

Here $R_{3\mathrm{I}}$, $R_{3\mathrm{II}}$ both appear, with $c_{\mathrm{I}\pm}$ replaced by $c_{\mathrm{II}\pm}$ in (50b). Completion of this analysis will require study of possible roots of $N_{k\mathrm{II}}$, $R_{k\mathrm{II}}$. Equation (54a) shows that the minimum grazing angle is $\phi = \sin^{-1}s_{\mathrm{II}+}$. Then (52) holds, with $s_{\mathrm{I}+}$ replaced by $s_{\mathrm{II}+}$. Equations (55a) and (56) do not exhibit distinctive behavior for this angle.

9. Orthotropic case: thermal wave reflection for model F

Consider the temperature change $\theta = \Theta_i$, traveling as a plane step-wave with speed $c_{F+}v_6$ in the negative x-direction toward the surface $x_1 = 0$ of half-space $x_1 > 0$. Arrival at surface point $(x_1, x_2) = 0$ occurs at time $t = 0$ $(\tau = 0)$. The material is orthotropic and satisfies the Fourier model equations, and (33a) for $c_{F\pm} = c_{F\pm}(\phi)$ in particular. This problem is treated in [Brock 2010], so only key steps are presented:

For $x_1 = 0$, $c_{F+}(\phi)\tau + x_2 \sin\phi \geq 0$ the incident wave generates

$$\theta = \Theta_i, \quad T_{12} = -c_{66}\frac{Q(c_{F+}, \phi)}{S(c_{F+}, \phi)}\Theta_i, \quad T_{11} = -c_{66}\frac{P(c_{F+}, \phi)}{S(c_{F+}, \phi)}\Theta_i \tag{57}$$

In (57) the functions Q, P are defined by

$$Q(c, \phi) = (K_1 + K_2)c^2(\phi)\sin\phi\cos\phi + Q_{12}\sin^2\phi(1 + 2\cos 2\phi) + Q_{21}\cos^2\phi(1 - 2\cos 2\phi), \tag{58a}$$

$$Q_{12} = K_1 m\cos^2\phi - K_2(C_1\cos^2\phi + \sin^2\phi), \tag{58b}$$

$$Q_{21} = K_2 m\sin^2\phi - K_1(C_2\sin^2\phi + \cos^2\phi), \tag{58c}$$

$$P(c, \phi) = [F_1K_1\cos^2\phi + (m_F - 1)K_2\sin^2\phi]c^2(\phi) + [F_1P_1\cos^2\phi + (m_F - 1)P_2\sin^2\phi]\sin\phi\cos\phi, \tag{59a}$$

$$P_1 = K_1(m\cos 2\phi + 1 + 2C_2\sin^2\phi) - K_2(C_1\cos 2\phi + 2m\sin^2\phi), \tag{59b}$$

$$P_2 = K_2(m\cos 2\phi - 1 - 2C_1\cos^2\phi) - K_1(C_2\cos 2\phi - 2m\cos^2\phi). \tag{59c}$$

The function S is defined by

$$S(c, \phi) = [c^2(\phi) - 1](\varepsilon_1\cos^2\phi + \varepsilon_2\sin^2\phi) + [(C_2 - 1)\sqrt{\varepsilon_1} - m\sqrt{\varepsilon_2}]\sqrt{\varepsilon_1} + [(C_1 - 1)\sqrt{\varepsilon_2} - m\sqrt{\varepsilon_1}]\sqrt{\varepsilon_2}. \tag{60}$$

Here only the condition that the half-space surface remains traction-free is imposed. In accordance with (28) and (29a), reflected plane waves travel in positive x-directions that form angles $\phi_\pm$ with the positive x_1-axis at speed $c_{F\pm}(\phi_\pm)v_6$. The temperature steps are $\Theta_\pm$, Equations (58)–(60) hold with ϕ replaced by $\phi_\pm$, and for $x_1 = 0$ and $c_{F\pm}(\phi_\pm)\tau - x_2\sin\phi_\pm \geq 0$ we have

$$\theta = \Theta_\pm, \quad T_{12} = -c_{66}\frac{Q(c_{F\pm}, \phi_\pm)}{S(c_{F\pm}, \phi_\pm)}\Theta_\pm, \quad T_{11} = -c_{66}\frac{P(c_{F\pm}, \phi_\pm)}{S(c_{F\pm}, \phi_\pm)}\Theta_\pm. \tag{61}$$

Reflection requires that

$$c_{F+}(\phi)\sin\phi_+ + c_{F+}(\phi_+)\sin\phi = 0, \quad c_{F+}(\phi)\sin\phi_- + c_{F-}(\phi_-)\sin\phi = 0. \tag{62}$$

This equation is satisfied when

$$\phi_+ = -\phi, \quad \phi_- = -\sin^{-1}\frac{\sin\phi}{c_{12}^F(\phi)}, \tag{63a}$$

$$c_{12}^F(\phi) = \frac{1}{\sqrt{F_1}}\sqrt{(1 + F_1)c_{F+}^2(\phi) - F_1\cos^2\phi - (F_2 + \Omega_F)\sin^2\phi}. \tag{63b}$$

Imposing a traction-free surface gives

$$\Theta_+ = -\Theta_i \frac{Q(c_{F+},\phi)P(c_{F-},\phi_-) - P(c_{F+},\phi)Q(c_{F-},\phi_-)}{Q(c_{F+},-\phi)P(c_{F-},\phi_-) - P(c_{F+},-\phi)Q(c_{F-},\phi_-)}, \tag{64a}$$

$$\Theta_- = -\Theta_i \frac{S(c_{F-},\phi_-)}{S(c_{F+},\phi)} \frac{P(c_{F+},\phi)Q(c_{F+},-\phi) - Q(c_{F+},\phi)P(c_{F+},-\phi)}{Q(c_{F+},-\phi)P(c_{F-},\phi_-) - P(c_{F+},-\phi)Q(c_{F-},\phi_-)}. \tag{64b}$$

It is noted that $\Theta_i + \Theta_+ + \Theta_- \neq 0$, i.e., the Fourier model predicts surface temperature change. Moreover a surface heat flux spike occurs in $\partial_1 \theta$ unless

$$(\Theta_i - \Theta_+)\cos\phi - \Theta_- \cos\phi_- = 0. \tag{65}$$

The nature of c_{12}^F depends on material categorization that differs from the counterpart to (27) mentioned in Section 5, e.g., $\gamma_F < 0$ implies $\Omega_F < 0$, and $\Omega_F - m_F^2 > 0$ implies $\Omega_F > 0$. Therefore (63b) gives the positive real results

$$c_{12}^F(\Phi_{A\pm}) = \frac{1}{\sqrt{F_1}}\sqrt{1 + F_1\cos^2\Phi_{A\pm} + F_2\sin^2\Phi_{A\pm}}, \tag{66a}$$

$$c_{12}^F(\Phi_{B\pm}) = \sqrt{F_1\cos^2\Phi_{B\pm} + (F_2 + \Omega_F\cos^2\Phi_{B\pm})\sin^2\Phi_{B\pm}}. \tag{66b}$$

However, when $(F_1 - F_2)^2 - 4\Omega_F < 0$ and $(1+F_1)(F_1-F_2) - 2(F_2+\Omega_F) > 0$, the value of c_{12}^F vanishes when $|\phi| = \Phi_\pm$ and is imaginary for $\Phi_- < |\phi| < \Phi_+$, where

$$\Phi_\pm = \tan^{-1}\sqrt{\frac{F_1}{2}}\sqrt{(1+F_1)[F_1 - F_2 \pm \sqrt{(F_1-F_2)^2 - 4\Omega_F}] - 2(F_2+\Omega_F)} \tag{67}$$

This behavior implies that (63a) is subject to the restriction $\sin^2\phi < [c_{12}^F(\phi)]^2$. It can be shown that the restriction is satisfied except in the following cases:

$$\mathrm{B1:}\quad Q_B > 0, \quad \Phi_{B1} < |\phi| < \pi/2 \tag{68a}$$

$$\mathrm{B2:}\quad P_B^2 + Q_B > 0, \quad P_B < 0, \quad Q_B < 0, \quad \Phi_{2B}^- < |\phi| < \Phi_{2B}^+, \tag{68b}$$

with

$$\Phi_{1B} = \tan^{-1}\frac{1}{\sqrt{Q_B}}\sqrt{P_B + \sqrt{P_B^2 + Q_B}}, \qquad \Phi_{2B}^\pm = \tan^{-1}\frac{1}{\sqrt{-Q_B}}\sqrt{-P_B \mp \sqrt{P_B^2 + Q_B}}, \tag{68c}$$

$$P_B = 1 - \frac{F_1}{2}(1+F_1)(1+F_2), \qquad Q_B = F_1F_2 + \Omega_F(1+F_2). \tag{68d}$$

For angles of incidence ϕ that lie outside of the ranges prescribed by B1 and B2, a reflected wave travels in the negative x_2-direction at speed $c_{F+}v_6$. In view of (68) and the thermoelastic counterpart to (27) in Section 4, (61) and (64) are governed by two cases. A study of the various limits $|\phi| = \Phi_{A\pm}$, Φ_{1B}, $\Phi_{2B}^\pm$, as well as study of possible situations for which (61) and (64) vanish or become unbounded, is beyond the scope of this single paper. Such efforts are planned for a longer format.

As observed above the reflections uncouple as a thermal and a shear wave in the isotropic limit. Thus for $x_1 = 0$ the incident and reflected fields give

$$\text{for } c_F\tau + x_2 \sin\phi \geq 0: \quad \theta = \Theta_i, \quad T_{12} = -\frac{\mu K Q(\phi)}{m_F \varepsilon}\Theta_i, \quad T_{11} = -\frac{\mu K P(\phi)}{m_F \varepsilon}\Theta_i; \tag{69a}$$

$$\text{for } c_F\tau - x_2 \sin\phi_F \geq 0: \quad \theta = \Theta_F, \quad T_{12} = -\frac{\mu K Q(\phi_F)}{m_F \varepsilon}\Theta_F, \quad T_{11} = -\frac{\mu K P(\phi_F)}{m_F \varepsilon}\Theta_F; \tag{69b}$$

$$\text{for } \tau - x_2 \sin\phi_S \geq 0: \quad \theta = 0, \quad T_{12} = T_S \cos 2\phi_S, \quad T_{11} = T_S \sin 2\phi_S. \tag{69c}$$

Here T_S is the unknown stress T_{xy} due to the reflected shear wave, and

$$Q(\phi) = c_F^2 \sin 2\phi - 1 + \cos^2 2\phi, \tag{70a}$$

$$P(\phi) = c_F^2(c_F^2 - 2\sin^2\phi) + 8\sin^3\phi\, \cos^3\phi. \tag{70b}$$

A traction-free surface now requires that

$$\phi_F = -\phi, \quad \phi_S = -\sin^{-1}\frac{\sin\phi}{c_F}, \tag{71a}$$

$$T_S = \frac{2\mu K \Theta_i}{m_F D_0 \varepsilon}[4(1 - 2\cos^2 2\phi) + c_F^2(c_F^2 - 2\sin^2\phi)]\sin 2\phi, \tag{71b}$$

$$\Theta_F = \Theta_i\Big[-1 + \frac{4}{D_0}(c_F^2 \sin\phi_S \cos\phi_S - \sin^2 2\phi \cos 2\phi_S)\sin 2\phi\Big], \tag{71c}$$

$$D_0 = Q(-\phi)\sin 2\phi_S - P(-\phi)\cos 2\phi_S. \tag{71d}$$

On the surface we have $\Theta_F + \Theta_i \neq 0$, and a heat flux spike occurs unless

$$(\Theta_i - \Theta_F)\cos\phi = 0. \tag{72}$$

The parameter $c_{12}(\phi)$ equals c_F when $F_1 = F_2$ and $\Omega_F = 0$, and for the weakly orthotropic case

$$c_{12}^F(\phi) \approx c_F + (1 + \cos^2\phi)\frac{\delta F_1}{2c_F} - \frac{c_F}{m_F}\Omega_F \sin^2\phi. \tag{73}$$

Here δF_1, Ω_F are governed by (41).

10. Orthotropic isothermal case: stress wave reflection

Consider the situation in Section 9, except that material response is isothermal, and an incident plane wave moves with speed $c_+(\phi)v_6$ toward the half-space surface. In terms of the traction T_i, the wave is defined in accordance with (20)–(22) as step-stresses

$$T_{xx} = T_i c_+^2(\phi)[\cos^2 2\phi + (C_1 + C_2 - 2C_{12})\sin^2\phi\cos^2\phi - c_+^2(\phi)], \tag{74a}$$

$$T_{xy} = -T_i c_+^2(\phi)[C_2\sin^2\phi - C_1\cos^2\phi + (2 + C_{12})\cos 2\phi]\sin\phi\, \cos\phi, \tag{74b}$$

$$T_{yy} = T_i c_+^2(\phi)[C_{12}\cos^2 2\phi + (C_1 + C_2 + 2C_{12} - 4)\sin^2\phi\cos^2\phi] - T_i(C_{12} + \Omega\sin^2\phi\cos^2\phi). \tag{74c}$$

For $x_1 = 0$ and $c_+(\phi)\tau + x_2\sin\phi \geq 0$, the step-stresses T_{xx}, T_{xy}, T_{yy} generate traction

$$T_{11} = T_i P(c_+, \phi), \quad T_{12} = T_i Q(c_+, \phi), \tag{75}$$

where P, Q are defined by

$$P(c,\phi)=c^2(\phi)[C_{12}\cos^2 2\phi+2C_2\sin^2\phi-C_1\cos 2\phi-8\sin^2\phi]\sin^2\phi \\ +(C_1\cos^2\phi+C_2\sin^2\phi)\cos^2\phi+\Omega\cos 2\phi\sin^2\phi\cos^2\phi-C_{12}\sin^2\phi, \tag{76a}$$

$$Q(c,\phi)=-c^2(\phi)[(C_1+C_2)\sin^2\phi\cos^2\phi+C_{12}(\cos^4\phi+\sin^4\phi)+\cos^2 2\phi]\sin 2\phi \\ +[C_{12}+C_1\cos^2\phi+C_2\sin^2\phi+2\Omega\sin^2\phi\cos^2\phi]\sin\phi\cos\phi. \tag{76b}$$

(Thus P is symmetric and Q is antisymmetric in ϕ.) Reflection generates plane waves T_{xx}, T_{xy}, T_{yy} that travel away from the surface with speed $c_\pm(\phi_\pm)v_6$, so that for $x_1=0$ and $c_\pm(\phi)\tau-x_2\sin\phi_\pm\geq 0$ we have

$$T_{11}=T_\pm P(c_\pm,\phi_\pm),\quad T_{12}=T_\pm Q(c_\pm,\phi_\pm). \tag{77}$$

In this case a traction-free surface requires that

$$\phi_+=-\phi,\quad \phi_-=-\sin^{-1}\frac{\sin\phi}{c_{12}(\phi)}, \tag{78a}$$

$$T_+=T_i\frac{Q(c_+,\phi)P(c_-,\phi_-)-Q(c_-,\phi_-)P(c_+,\phi)}{Q(c_+,\phi)P(c_-,\phi_-)+Q(c_-,\phi_-)P(c_+,\phi)}, \tag{78b}$$

$$T_-=-2T_i\frac{Q(c_+,\phi)P(c_+,\phi)}{Q(c_+,\phi)P(c_-,\phi_-)+Q(c_-,\phi_-)P(c_+,\phi)}, \tag{78c}$$

$$c_{12}(\phi)=\frac{1}{\sqrt{C_1}}\sqrt{(1+C_1)c_+^2(\phi)-C_1\cos^2\phi-(C_2+\Omega)\sin^2\phi}\,. \tag{78d}$$

The behavior of (78d) is analogous to that of (63b), so that (78a) must be subject to the restriction $\sin^2\phi< c_{12}^2(\phi)$. Condition (68) in Section 9 again holds, but with $(c_{12}^F, \Omega_F, F_1, F_2)$ replaced by $(c_{12}, \Omega, C_1, C_2)$. Consistent with the observations in Sections 7, 8 and 9, study of (78a) in light of (27), (68) and their analogues, and consideration of cases for which (78b) and (78c) vanish or become unbounded, is reserved for future work.

However, for some insight into both (27) in Section 4 and the isothermal analogue to (68) given in Section 9, Table 2 in the Appendix provides calculations for four orthotropic wood materials (see Table 1) under isothermal conditions. The table entries show that $c_\pm$ exists for these materials, i.e., there are no angles $\Phi_{A\pm}$ that restrict angle of incidence ϕ. Similarly, the entries show that only the isothermal counterpart of restriction B1 governs reflection; that is, Φ_{B1} exists, but $\Phi_{2B}^{\pm}$ does not.

The isotropic limit case is a standard problem [Achenbach 1973]. However, for completeness some results are presented here:

$$\phi_D=-\phi,\quad \phi_S=-\sin^{-1}\frac{\sin\phi}{c_D},\quad T_D=T_i\frac{N}{R},\quad T_S=-\frac{T_i}{R}\sin 2\phi(c_D^2-2\sin^2\phi), \tag{79a}$$

$$N=2\sin 2\phi\sin\phi\sqrt{c_D^2-\sin^2\phi}-(c_D^2-2\sin^2\phi)^2, \tag{79b}$$

$$R=2\sin 2\phi\sin\phi\sqrt{c_D^2-\sin^2\phi}+(c_D^2-2\sin^2\phi)^2. \tag{79c}$$

The isothermal counterpart to (73) is

$$c_{12}(\phi) \approx c_D + (1+\cos^2\phi)\frac{\delta C_1}{2c_D} - \frac{c_D}{m}\Omega\sin^2\phi. \tag{80}$$

Here δC_1, Ω are governed by (41).

11. Some observations

The results of Sections 7 and 8 illustrate the alteration in half-space temperature that occurs when an isothermal (shear) step-stress wave is reflected from its surface, and only the wave class studied in [Brock 2010] is considered. In Section 7 thermal relaxation with a single relaxation time [Lord and Shulman 1960] governs. The [Green and Lindsay 1972] model, with an additional thermal relaxation time, governs in Section 8, and this time is coupled with the convection parameter β in the solution. A minimum grazing angle of incidence arises in both Sections, but distinctive changes in solution behavior do not seem to occur at this angle.

Section 9 treats an incident temperature step-wave that propagates without dispersion or attenuation [Brock 2010] in an orthotropic half-space governed by the Fourier law [Chadwick 1960]. Only two corresponding reflection waves arise, so that only the requirement of a traction-free surface is met. Formulas for reflection angles are also presented. In the isotropic limit one reflection becomes an isothermal shear wave, and an asymptotic reflection angle formula valid for weak orthotropy is given.

Section 10 involves the commonly studied isothermal wave reflection process in an orthotropic half-space. For comparison with Section 9 the incident wave is a step-stress. In the isotropic limit the two waves generated by reflection reduce to the standard dilatational/shear wave pair. As in Section 9, distinctive behavior does not seem to occur at the minimum grazing angle. The reflection angle, its isotropic limit, and an asymptotic form for weak orthotropy are also given.

These results can in general be predicted by work in Sections 4 and 5. There governing equations and associated (dimensionless) speeds for plane wave propagation in a principal plane of isothermal and thermoelastic orthotropic solids are examined. Only the wave class discussed in [Brock 2010] is treated in the latter instance. Isotropic limits for the dimensionless wave speeds are also given, as well as asymptotic formulas for weakly orthotropic solids.

Sections 7, 8 and 9 demonstrate the limited applicability of thermoelastic plane wave ensembles that travel without dispersion or attenuation [Brock 2010], both for isotropic and orthotropic solids. Because the Fourier model allows only two signals, stress-free surfaces can result only if a prescribed uniform temperature and associated heat flux spite form the thermal boundary conditions. General time-harmonic [Chadwick 1960] or transient [Brock 2005; Brock and Hanson 2006] studies admit three, so the "missing" signal corresponds to the Fourier paradox of infinite speed.

However, Sections 7, 8 and 9 also demonstrate that there are combinations of speed, wave profile and angle of incidence that do not couple prescribed stress and thermal boundary conditions. As implied at the outset, moreover, problems that do couple conditions represent in effect special cases of nonconventional thermoelastic processes [Ignaczak and Ostoja-Starzewski 2010]. Such boundary conditions are artificial. Nevertheless, the problems discussed here do illustrate that the transient response of thermoelastic solids subject to surface reflection can differ from that described by analyses based on time-harmonic waves, dispersion and attenuation.

Finally, it is noted again that the restrictions imposed on wave travel by orthotropic elastic solids, whether thermoelastic or isothermal, need to be examined in more detail. The work by Kraut [1963] and Payton [1983] in isothermal transversely isotropic cases, and by Ignaczak and Ostoja-Starzewski [2010] in isotropic thermoelasticity with relaxation, are models in this regard. At present work is also proceeding on thermoelastic plane waves that exhibit a particular form of attenuation, but without dispersion.

Appendix

Data for four orthotropic wood materials — balsa (B), yellow birch (YB), Douglas fir (DF) and Sitka spruce (SS) — are taken from [Crandall and Dahl 1959, pp. 224–228] and summarized here:

	S_{22}/S_{11}	S_{33}/S_{11}	S_{12}/S_{11}	S_{23}/S_{11}	S_{13}/S_{11}	S_{44}/S_{11}	S_{55}/S_{11}	S_{66}/S_{11}
B	20	70	−0.3	−15	−0.5	18	200	27
DF	13	20	−0.5	−9	−0.5	14	60	15
YB	15	20	−0.4	−7	−0.5	16	140	13
SS	13	23	−0.4	−6	−0.5	16	20	16

Table 1. Compliance ratios.

The dimensionless constants C_1, C_2, C_{12} can be obtained from [Jones 1999]:

$$C_1 = \frac{S_{66}}{S_{11}s}\left[\frac{S_{22}}{S_{11}}\frac{S_{33}}{S_{11}} - \left(\frac{S_{23}}{S_{11}}\right)^2\right], \quad C_2 = \frac{S_{66}}{S_{11}s}\left[\frac{S_{33}}{S_{11}} - \left(\frac{S_{13}}{S_{11}}\right)^2\right], \quad C_{12} = \frac{S_{66}}{S_{11}s}\left[\frac{S_{13}}{S_{11}}\frac{S_{23}}{S_{11}} - \frac{S_{33}}{S_{11}}\frac{S_{12}}{S_{11}}\right],$$

$$s = \frac{S_{22}}{S_{11}}\frac{S_{33}}{S_{11}} - \left(\frac{S_{23}}{S_{11}}\right)^2 - \frac{S_{22}}{S_{11}}\left(\frac{S_{13}}{S_{11}}\right)^2 - \frac{S_{33}}{S_{11}}\left(\frac{S_{12}}{S_{11}}\right)^2 + 2\frac{S_{12}}{S_{11}}\frac{S_{13}}{S_{11}}\frac{S_{23}}{S_{11}}.$$

Equations (25), (27) and the isothermal analogue of (68) then give results in Table 2 for the various dimensionless parameters.

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	C_1	C_2	m	Ω	$c_\pm$ $(\phi = 15°)$		$c_\pm$ $(\phi = 45°)$		$c_\pm$ $(\phi = 75°)$		Φ_{1B}
B	27.37	1.63	4.67	−5.32	5.07	0.99	3.82	0.95	1.87	0.93	1.27°
DF	16.15	1.78	2.31	6.49	3.898	0.996	3.004	0.97	1.42	0.94	2.07°
YB	13.53	1.06	1.62	−1.82	3.56	1.02	2.65	1.13	1.57	1.13	2.85°
SS	16.59	1.43	2.16	−2.64	3.95	0.99	3.05	0.84	1.598	0.95	2.15°

Table 2. Dimensionless parameters. The conditions for existence of $\Phi_{A\pm}$ and $\Phi_{2B}^{\pm}$ are not met.

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