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SINGULAR HARMONIC PROBLEMS AT A WEDGE VERTEX: MATHEMATICAL ANALOGIES BETWEEN ELASTICITY, DIFFUSION, ELECTROMAGNETISM, AND FLUID DYNAMICS

ALBERTO CARPINTERI AND MARCO PAGGI

Dedicated to the memory of Marie-Louise Steele.

Multimaterial wedges are frequently observed in composite materials. They consist of two or more sectors of dissimilar materials joined together, whose interfaces converge at the same vertex. Due to the mismatch in material properties such as Young's modulus, thermal conductivity, dielectric permittivity, or magnetic permeability, these geometrical configurations can lead to singular fields at the junction vertex. This paper discusses mathematical analogies, focused on singular harmonic problems, between antiplane shear problem in elasticity due to mode III loading or torsion, the steady-state heat transfer problem, and the diffraction of waves in electromagnetism. In the case of a single material wedge, a mathematical analogy between elasticity and fluid dynamics is also outlined. The proposed unified mathematical formulation is particularly convenient for the identification of common types of singularities (power-law or logarithmic type), the definition of a standardized method to solve nonlinear eigenvalue problems, and the determination of common geometrical and material configurations allowing the relief or removal of different singularities.

1. Introduction

Singular stress states occur in many boundary value problems of linear elasticity where different materials are present (see [England 1971; Paggi and Carpinteri 2008; Sinclair 2004a; 2004b] for a broad overview). In this context, problems involving multimaterial wedges or junctions have received a great attention from the scientific community, since they are commonly observed in composite materials. In linear elasticity, most research has been directed toward the characterization of stress singularities for in-plane loading, where the problem is governed by a biharmonic equation. Out-of-plane loading, also referred to as the antiplane shear problem, is governed by a simpler harmonic equation. Stress singularities due to antiplane loading were firstly addressed by Rao [1971]. Afterwards, Fenner [1976] examined the mode III loading problem of a crack meeting a bimaterial interface using the eigenfunction expansion method proposed by Williams [1952]. More recently, Ma and Hour [1989; 1990] analyzed bimaterial wedges using the Mellin transform technique, whereas Pageau et al. [1995] investigated the singular stress field of bonded and debonded tri-material junctions according to the eigenfunction expansion method.

Mathematical analogies among elasticity, electromagnetism, and conductivity have been known and exploited for a long time (see [Hashin and Shtrikman 1962; Duan et al. 2006], for instance). Sinclair [1980] pointed out the mathematical analogy between the singular steady-state heat transfer and the

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singular antiplane loading of composite regions (see also [Paggi and Carpinteri 2008] for a detailed discussion of the boundary conditions). In [Paggi et al. 2009; 2010] we have established an analogy between elasticity and electromagnetism in the case of singular fields. In the solution of diffraction problems, in fact, Bouwkamp [1946] and Meixner [1972] found that the electromagnetic field vectors may become infinite at the sharp edges of a diffracting obstacle. For in-plane problems, a mathematical analogy between elasticity and dynamics of viscous fluids also exists; see [Dean and Montagnon 1949; Paggi and Carpinteri 2008; Carpinteri and Paggi 2009] for more details.

This paper presents mathematical analogies between antiplane shear problem in elasticity due to mode III loading or torsion, the steady-state heat transfer problem, and the diffraction of waves in electromagnetism. In the case of a single material wedge, a mathematical analogy between antiplane elasticity and fluid dynamics of incompressible fluids characterized by a potential flow will also be outlined. The proposed unified mathematical formulation will be based on the eigenfunction expansion method, which has been proven in [Paggi and Carpinteri 2008] to be mathematically equivalent to the Muskhelishvili complex function representation and to the Mellin transform technique for the characterization of elastic singularities at multimaterial junctions. As a main outcome, the order of the stress singularities of various geometrical and mechanical configurations already determined in the literature can be adopted for the analogous diffusion, electromagnetic, and fluid dynamics problems, without the need of performing new calculations. Finally, the possibility to extend the dimensionless numbers used in elasticity to the other analogous physical problems is discussed. In particular, as the *brittleness number* related to the stress-intensity factor rules the competition between brittle crack propagation and plastic flow collapse, a *turbulence number* that rules the competition between laminar and turbulent flow is proposed for fluid dynamics. The use of this new dimensionless number, in addition to the classical Reynolds number, is expected to be of paramount importance.

2. Stress singularities in antiplane elasticity

The geometry of a plane elastostatic problem consisting of $n-1$ dissimilar isotropic, homogeneous sectors of arbitrary angles perfectly bonded along their interfaces converging to the same vertex O is

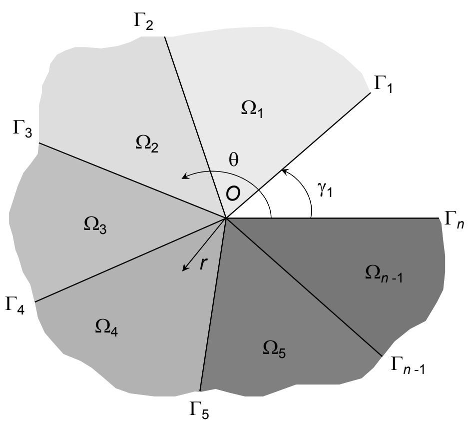

Figure 1. Geometry of a multimaterial wedge.

shown in Figure 1. Each of the material regions is denoted by Ω_i with $i = 1, \ldots, n-1$, and it is comprised between the interfaces Γ_i and Γ_{i+1}.

Antiplane shear (mode III) due to out-of-plane loading on composite wedges can lead to stresses that can be unbounded at the junction vertex O. When out-of-plane deformations only exist, the following displacements in cylindrical coordinates can be considered with the origin at the vertex O:

$$u_r = 0, \quad u_\theta = 0, \quad u_z = u_z(r, \theta), \tag{2-1}$$

where u_z is the out-of-plane displacement, which depends on r and θ. For such a system of displacements, the strain field components become

$$\varepsilon_r = \varepsilon_\theta = \varepsilon_z = \gamma_{r\theta} = 0, \quad \gamma_{rz} = \frac{\partial u_z}{\partial r}, \quad \gamma_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta}. \tag{2-2}$$

From the application of the Hooke's law, the stress field components are given by:

$$\sigma_r^i = \sigma_\theta^i = \sigma_z^i = \tau_{r\theta}^i = 0, \quad \tau_{rz}^i = G_i \gamma_{rz}^i = G_i \frac{\partial u_z^i}{\partial r}, \quad \tau_{\theta z}^i = G_i \gamma_{\theta z}^i = \frac{G_i}{r}\frac{\partial u_z^i}{\partial \theta}, \tag{2-3}$$

where G_i is the shear modulus of the i-th material region. The equilibrium equations in absence of body forces reduce to a single relationship between the tangential stresses:

$$\frac{\partial \tau_{rz}^i}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}^i}{\partial \theta} + \frac{1}{r}\tau_{rz}^i = 0 \quad \text{for all } (r, \theta) \in \Omega_i. \tag{2-4}$$

Introducing (2-3) into (2-4), the Laplace condition upon u_z is derived:

$$\frac{\partial^2 u_z^i}{\partial r^2} + \frac{1}{r}\frac{\partial u_z^i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z^i}{\partial \theta^2} = \nabla^2 u_z^i = 0 \quad \text{for all } (r, \theta) \in \Omega_i. \tag{2-5}$$

In the framework of the eigenfunction expansion method [Paggi and Carpinteri 2008], the following separable variable form for the longitudinal displacement u_z^i can be adopted, for all $(r, \theta) \in \Omega_i$:

$$u_z^i(r, \theta) = \sum_j r^{\lambda_j} f_{i,j}(\theta, \lambda_j), \tag{2-6}$$

where the λ_j are the eigenvalues and $f_{i,j}$ the eigenfunctions of the problem. Summation over j is introduced in (2-6) since it is possible to have more than one eigenvalue and the superposition principle can be applied.

Introducing (2-6) into (2-5), we find the following relation, holding for each eigenvalue λ_j:

$$r^{\lambda_j - 2}\left(\frac{\mathrm{d}^2 f_{i,j}}{\mathrm{d}\theta^2} + \lambda_j^2 f_{i,j}\right) = 0. \tag{2-7}$$

Hence, the coefficients of the term in $r^{\lambda_j - 2}$ must vanish, implying that the eigenfunctions $f_{i,j}$ are a linear combination of trigonometric functions:

$$f_{i,j}(\theta, \lambda_j) = A_{i,j}\sin(\lambda_j \theta) + B_{i,j}\cos(\lambda_j \theta). \tag{2-8}$$

If we introduce the series expansion (2-6) into (2-3), the longitudinal displacement and the tangential stresses can be expressed in terms of the eigenfunction and its first derivative:

$$u_z^i = r^{\lambda_j} f_{i,j} = r^{\lambda_j} \left[A_{i,j} \sin(\lambda_j \theta) + B_{i,j} \cos(\lambda_j \theta) \right], \tag{2-9a}$$

$$\tau_{rz}^i = G_i \lambda_j r^{\lambda_j - 1} f_{i,j} = G_i \lambda_j r^{\lambda_j - 1} \left[A_{i,j} \sin(\lambda_j \theta) + B_{i,j} \cos(\lambda_j \theta) \right], \tag{2-9b}$$

$$\tau_{\theta z}^i = G_i r^{\lambda_j - 1} f'_{i,j} = G_i \lambda_j r^{\lambda_j - 1} \left[A_{i,j} \cos(\lambda_j \theta) - B_{i,j} \sin(\lambda_j \theta) \right]. \tag{2-9c}$$

The determination of the power of the stress singularity, $\lambda_j - 1$, can be performed by imposing the boundary conditions (BCs) along the edges Γ_1 and Γ_n and at the bimaterial interfaces Γ_i, with $i = 2, \ldots, n-1$. Along the edges Γ_1 and Γ_n, defined by the angles γ_1 and γ_n, we consider two possibilities: one corresponding to unrestrained stress-free edges

$$\tau_{\theta z}^i(r, \gamma_1) = 0, \qquad \tau_{\theta z}^i(r, \gamma_n) = 0, \tag{2-10}$$

and the other for fully restrained (clamped) edges

$$u_z^i(r, \gamma_1) = 0, \qquad u_z^i(r, \gamma_n) = 0. \tag{2-11}$$

At the interfaces, the following continuity conditions of displacements and stresses have to be imposed $(i = 1, \ldots, n-2)$:

$$u_z^i(r, \gamma_{i+1}) = u_z^{i+1}(r, \gamma_{i+1}), \qquad \tau_{\theta z}^i(r, \gamma_{i+1}) = \tau_{\theta z}^{i+1}(r, \gamma_{i+1}). \tag{2-12}$$

In this way, a set of $2n-2$ homogeneous equations in the $2n-1$ unknowns $A_{i,j}$, $B_{i,j}$, and λ_j can be symbolically written as:

$$\boldsymbol{\Lambda}\, \boldsymbol{v} = \boldsymbol{0}, \tag{2-13}$$

where $\boldsymbol{\Lambda}$ denotes the coefficient matrix which depends on the eigenvalue, and $\boldsymbol{v}$ represents the vector that collects the unknowns $A_{i,j}$ and $B_{i,j}$. More specifically, the coefficient matrix appearing in (2-13) is characterized by a sparse structure:

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{N}_{\gamma_1}^1 & & & & & & \\ \boldsymbol{M}_{\gamma_2}^1 & -\boldsymbol{M}_{\gamma_2}^2 & & & & & \\ & \boldsymbol{M}_{\gamma_3}^2 & -\boldsymbol{M}_{\gamma_3}^3 & & & & \\ & & \ddots & \ddots & & & \\ & & & \boldsymbol{M}_{\gamma_i}^{i-1} & -\boldsymbol{M}_{\gamma_i}^i & & \\ & & & & \ddots & \ddots & \\ & & & & & \boldsymbol{M}_{\gamma_{n-1}}^{n-2} & -\boldsymbol{M}_{\gamma_{n-1}}^{n-1} \\ & & & & & & \boldsymbol{N}_{\gamma_n}^{n-1} \end{bmatrix} \tag{2-14}$$

where the nonnull elementary matrix $\boldsymbol{M}_\theta^i$ related to the interface BCs is given by

$$\boldsymbol{M}_\theta^i = \begin{bmatrix} \sin(\lambda_j \theta) & \cos(\lambda_j \theta) \\ G_i \cos(\lambda_j \theta) & -G_i \sin(\lambda_j \theta) \end{bmatrix} \tag{2-15}$$

and the components of the vector $\boldsymbol{v}$ are

$$\boldsymbol{v} = \{\boldsymbol{v}^1, \boldsymbol{v}^2, \dots, \boldsymbol{v}^i, \dots, \boldsymbol{v}^{n-2}, \boldsymbol{v}^{n-1}\}, \tag{2-16}$$

with $\boldsymbol{v}^i = \{A_{i,j}, B_{i,j}\}^T$. The two remaining terms $\boldsymbol{N}_\theta^i$ depend on the BCs along the edges Γ_1 and Γ_n. For stress-free edges we have

$$\boldsymbol{N}_\theta^i = \{\cos(\lambda_j\theta), -\sin(\lambda_j\theta)\}, \tag{2-17}$$

whereas for clamped edges it is given by

$$\boldsymbol{N}_\theta^i = \{\sin(\lambda_j\theta), \cos(\lambda_j\theta)\}. \tag{2-18}$$

A nontrivial solution of the equation system (2-13) exists if and only if the determinant of the coefficient matrix vanishes. This condition yields an eigenequation which has to be solved for the eigenvalues λ_j that, in the most general case, do depend on the elastic properties of the materials.

3. Heat flux singularities in diffusion problems

The analogy between steady-state heat transfer and antiplane shear in composite regions was discovered by Sinclair [1980]. In both problems, the field equations for the longitudinal displacement, u_z^i, and for the temperature, T^i, are harmonic. As a result, the following correspondences between these two problems can be set down:

$$\begin{aligned} \nabla^2 T^i &= 0 & &\Longleftrightarrow & \nabla^2 u_z^i &= 0, \\ q_r^i &= -k_i \frac{\partial T^i}{\partial r} & &\Longleftrightarrow & \tau_{rz}^i &= G_i \frac{\partial u_z^i}{\partial r}, \\ q_\theta^i &= \frac{k_i}{r}\frac{\partial T^i}{\partial \theta} & &\Longleftrightarrow & \tau_{\theta z}^i &= \frac{G_i}{r}\frac{\partial u_z^i}{\partial \theta}, \end{aligned} \tag{3-1}$$

where q_r^i and q_θ^i are the heat flux in the radial and circumferential directions and k_i is the thermal conductivity in the i-th material region. Therefore, the analogy is straightforward: the temperature field is analogous to the out-of-plane displacement field, whereas the heat flux components are the analogous counterparts of the stress field components, diverging to infinity as $r \to 0$.

As far as the BCs are concerned, the free-edge conditions (2-10) correspond to insulated edges in diffusion problems, provided that the elastic variables are replaced by the steady-state heat transfer variables according to (3-1). Similarly, the clamped BCs (2-11) in elasticity correspond to zero temperature prescribed along the edges. Finally, the continuity of the longitudinal displacement u_z and of the tangential stress $\tau_{\theta z}$ in (2-12) at the interfaces corresponds to the continuity of temperature, T, and heat-flux, q_θ. The eigenvalue problem for the diffusion problem has therefore the same coefficient matrix as in (2-13).

4. Singularities in the electromagnetic fields

Consider the multimaterial wedge shown in Figure 2. Each material is isotropic and has a dielectric permittivity ϵ_i and a magnetic permeability μ_i. We also admit the presence of a perfect electric conductor (PEC) in the region labeled 1 and defined by the interfaces Γ_1 and Γ_n.

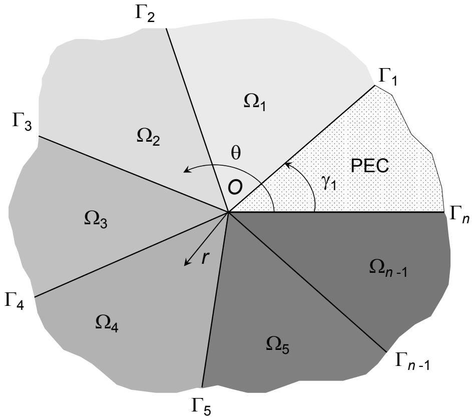

Figure 2. Geometry of a multimaterial wedge including one made of a perfect electric conductor.

For periodic fields with circular frequency ω, the Maxwell's equations for each homogeneous angular domain read as follows [van Bladel 1991]:

$$\mathrm{j}\omega\epsilon_i \boldsymbol{E}^i = \nabla \times \boldsymbol{H}^i, \qquad -\mathrm{j}\omega\mu_i \boldsymbol{H}^i = \nabla \times \boldsymbol{E}^i, \tag{4-1}$$

where $\boldsymbol{E}^i$ and $\boldsymbol{H}^i$ are, respectively, the electric and magnetic fields, and the symbol j stands for the imaginary unit.

In cylindrical coordinates r, θ, z, with the z axis perpendicular to the plane of the wedge, and considering electromagnetic fields independent of z, Maxwell's equations reduce to the following conditions upon the components of the electric and magnetic fields:

$$\begin{aligned} \mathrm{j}\omega\epsilon_i E_r^i &= \frac{1}{r}\frac{\partial H_z^i}{\partial\theta}, & \mathrm{j}\omega\epsilon_i E_\theta^i &= -\frac{\partial H_z^i}{\partial r}, & \mathrm{j}\omega\epsilon_i E_z^i &= \frac{1}{r}\frac{\partial}{\partial r}(r H_\theta^i) - \frac{1}{r}\frac{\partial H_r^i}{\partial\theta}, \\ -\mathrm{j}\omega\mu_i H_r^i &= \frac{1}{r}\frac{\partial E_z^i}{\partial\theta}, & -\mathrm{j}\omega\mu_i H_\theta^i &= -\frac{\partial E_z^i}{\partial r}, & -\mathrm{j}\omega\mu_i H_z^i &= \frac{1}{r}\frac{\partial}{\partial r}(r E_\theta^i) - \frac{1}{r}\frac{\partial E_r^i}{\partial\theta}. \end{aligned} \tag{4-2}$$

It is easy to verify that the E_z^i and H_z^i components satisfy the Helmholtz equation [van Bladel 1991]:

$$\begin{aligned} \frac{\partial^2 E_z^i}{\partial r^2} + \frac{1}{r}\frac{\partial E_z^i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z^i}{\partial\theta^2} + k_i^2 E_z^i &= \nabla^2 E_z^i + k_i^2 E_z^i = 0, \\ \frac{\partial^2 H_z^i}{\partial r^2} + \frac{1}{r}\frac{\partial H_z^i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_z^i}{\partial\theta^2} + k_i^2 H_z^i &= \nabla^2 H_z^i + k_i^2 H_z^i = 0, \end{aligned} \tag{4-3}$$

where $k_i = \omega^2 \epsilon_i \mu_i$.

In close analogy with the antiplane problem in linear elasticity, the following separable form for E_z^i and H_z^i can be postulated for all $(r, \theta) \in \Omega_i$ [Meixner 1972]:

$$E_z^i(r,\theta) = \sum_j r^{\lambda_j} f_{i,j}(\theta, \lambda_j), \qquad H_z^i(r,\theta) = \sum_j r^{\lambda_j} F_{i,j}(\theta, \lambda_j) \tag{4-4}$$

where λ_j are the eigenvalues, and $f_{i,j}$, and $F_{i,j}$ are the eigenfunctions.

We can introduce (4-4) into (4-3), obtaining the equalities

$$r^{\lambda_j-2}\left(\frac{d^2 f_{i,j}}{d\theta^2}+\lambda_j^2 f_{i,j}\right)=0, \quad r^{\lambda_j-2}\left(\frac{d^2 F_{i,j}}{d\theta^2}+\lambda_j^2 F_{i,j}\right)=0. \tag{4-5}$$

Hence, we find that the eigenfunctions $f_{i,j}$ and $F_{i,j}$ are linear combinations of trigonometric functions, in perfect analogy with the eigenfunction $f_{i,j}$ in antiplane elasticity (see (2-8)):

$$f_{i,j}(\theta,\lambda_j)=A_i\sin(\lambda_j\theta)+B_i\cos(\lambda_j\theta), \quad F_{i,j}(\theta,\lambda_j)=C_i\sin(\lambda_j\theta)+D_i\cos(\lambda_j\theta). \tag{4-6}$$

These eigenfunctions are responsible for the singular behavior of the components E_r^i, E_θ^i, H_r^i and H_θ^i of the electric and magnetic fields near the wedge apex. In particular, from (4-2), we observe that

$$\begin{aligned}
E_r^i &= \frac{1}{r j\omega\epsilon_i}\frac{\partial H_z^i}{\partial\theta}=\frac{1}{j\omega\epsilon_i}\sum_j \lambda_j r^{\lambda_j-1}F'_{i,j}\sim O(r^{\lambda_j-1}),\\
E_\theta^i &= -\frac{1}{j\omega\epsilon_i}\frac{\partial H_z^i}{\partial r}=-\frac{1}{j\omega\epsilon_i}\sum_j \lambda_j r^{\lambda_j-1}F_{i,j}\sim O(r^{\lambda_j-1}),\\
H_r^i &= -\frac{1}{r j\omega\mu_i}\frac{\partial E_z^i}{\partial\theta}=-\frac{1}{j\omega\mu_i}\sum_j \lambda_j r^{\lambda_j-1}f'_{i,j}\sim O(r^{\lambda_j-1}),\\
H_\theta^i &= \frac{1}{j\omega\mu_i}\frac{\partial E_z^i}{\partial r}=\frac{1}{j\omega\mu_i}\sum_j \lambda_j r^{\lambda_j-1}f'_{i,j}\sim O(r^{\lambda_j-1}).
\end{aligned} \tag{4-7}$$

Hence, $E_z^i\sim O(r^{\lambda_j})$ and $H_z^i\sim O(r^{\lambda_j})$ are the analogous counterparts of u_z^i and remain finite as $r\to 0$. Moreover, the radial components of the electric and magnetic fields, E_r^i and H_r^i, are analogous to $\tau_{\theta z}^i$ and the circumferential components, E_θ^i and H_θ^i, are analogous to τ_{rz}^i. More specifically, we have $E_r^i=\tau_{\theta z}^i/(j\omega\epsilon_i G_i)$, $H_r^i=-\tau_{\theta z}^i/(j\omega\mu_i G_i)$, $E_\theta^i=-\tau_{rz}^i/(j\omega\epsilon_i G_i)$ and $H_\theta^i=\tau_{rz}^i/(j\omega\mu_i G_i)$. All of these components diverge when $r\to 0$ with a power-law singularity of order $-1<(\lambda_j-1)<0$.

Regarding the BCs, the tangential components of the electric field vanish along the edges Γ_1 and Γ_n of the PEC:

$$E_z^1(r,\gamma_1)=0, \quad E_r^1(r,\gamma_1)=0, \quad E_z^{n-1}(r,\gamma_n)=0, \quad E_r^{n-1}(r,\gamma_n)=0, \tag{4-8}$$

On the PEC surface also $H_\theta=0$, but this condition does not need be enforced, since it is a consequence of the previous ones. Along each bimaterial interface ($i=1,\dots,n-2$), the tangential components of the electric and magnetic fields are continuous:

$$\begin{aligned}
E_z^i(r,\gamma_{i+1}) &= E_z^{i+1}(r,\gamma_{i+1}), \quad E_r^i(r,\gamma_{i+1})=E_r^{i+1}(r,\gamma_{i+1}),\\
H_z^i(r,\gamma_{i+1}) &= H_z^{i+1}(r,\gamma_{i+1}), \quad H_r^i(r,\gamma_{i+1})=H_r^{i+1}(r,\gamma_{i+1}).
\end{aligned} \tag{4-9}$$

Using the equations (4-7), the BCs (4-8) become

$$E_z^1(r,\gamma_1)=0, \quad E_z^{n-1}(r,\gamma_n)=0, \quad \frac{\partial H_z^1}{\partial\theta}(r,\gamma_1)=0, \quad \frac{\partial H_z^{n-1}}{\partial\theta}(r,\gamma_n)=0, \tag{4-10}$$

whereas those defined by (4-9) become ($i = 1, \dots, n-2$)

$$
\begin{aligned}
E_z^i(r, \gamma_{i+1}) &= E_z^{i+1}(r, \gamma_{i+1}), \quad \frac{1}{\epsilon_i}\frac{\partial H_z^i}{\partial \theta}(r, \gamma_{i+1}) = \frac{1}{\epsilon_{i+1}}\frac{\partial H_z^{i+1}}{\partial \theta}(r, \gamma_{i+1}) \\
H_z^i(r, \gamma_{i+1}) &= H_z^{i+1}(r, \gamma_{i+1}), \quad \frac{1}{\mu_i}\frac{\partial E_z^i}{\partial \theta}(r, \gamma_{i+1}) = \frac{1}{\mu_{i+1}}\frac{\partial E_z^{i+1}}{\partial \theta}(r, \gamma_{i+1}).
\end{aligned}
\tag{4-11}
$$

It is interesting that (4-3), (4-10) and (4-11) can be separated into two independent sets of equations, one involving only H_z and another involving only E_z. Hence, the electromagnetic field for this problem can be decomposed into two distinct independently evolving fields, the so-called transverse electric (TE) and transverse magnetic (TM) fields. The TE (resp. TM) field has vanishing electric (resp. magnetic) but nonzero magnetic (resp. electric) components parallel to the cylinder axis z.

Considering the series expansion for E_z and H_z, along with the expressions for the eigenfunctions $f_{i,j}$ and $F_{i,j}$, the boundary value problem consists of two sets of $2n-2$ equations in $2n-1$ unknowns, one for E_z and another for H_z. The former equation set (TM case) involves the coefficients $A_{i,j}$, $B_{i,j}$ and λ_j and can be symbolically written as

$$\boldsymbol{\Lambda}\, \boldsymbol{v} = \boldsymbol{0}, \tag{4-12}$$

where $\boldsymbol{\Lambda}$ denotes the coefficient matrix which depends on the eigenvalue and $\boldsymbol{v}$ represents the vector which collects the unknowns $A_{i,j}$ and $B_{i,j}$. The coefficient matrix in (4-12) has exactly the same structure as that for the elasticity problem in (2-13), provided that we consider $\boldsymbol{N}_\theta^i = \{\sin(\lambda_j\theta), \cos(\lambda_j\theta)\}$ and we set $G_i = 1/\mu_i$.

The latter equation set (TE case) involves the coefficients $C_{i,j}$, $D_{i,j}$ and λ_j and can be symbolically written as:

$$\boldsymbol{\Lambda}\, \boldsymbol{w} = \boldsymbol{0}, \tag{4-13}$$

where $\boldsymbol{\Lambda}$ is the coefficient matrix which depends on the eigenvalue and $\boldsymbol{w}$ represents the vector which collects the unknowns $C_{i,j}$ and $D_{i,j}$. Again, the coefficient matrix in (4-13) has exactly the same structure as that for the elasticity problem in (2-13), provided that we consider $\boldsymbol{N}_\theta^i = \{\cos(\lambda_j\theta), -\sin(\lambda_j\theta)\}$ and we set $G_i = 1/\epsilon_i$.

For the existence of nontrivial solutions, the determinants of the coefficient matrices must vanish, yielding two eigenequations that, for given values of ϵ_i and μ_i, determine the eigenvalues λ_j^{TE} and λ_j^{TM}. Hence, this proves that the analysis of the singularities of the electromagnetic field is mathematically analogous to that for the elastic field due to antiplane loading.

5. Singularities in fluid dynamics

In fluid dynamics, a large class of problems can be described by a potential flow. In such cases, a stream function, $\boldsymbol{\Psi}$, can be introduced such that the flow velocity $\boldsymbol{v}$ can be determined from its curl:

$$\boldsymbol{v} = \nabla \times \boldsymbol{\Psi}, \tag{5-1}$$

where, in polar coordinates and for 2D problems, we have $\boldsymbol{v} = (v_r, v_\theta, 0)^T$ and $\boldsymbol{\Psi} = (0, 0, \Psi)^T$. Moreover, if the flow is irrotational, the curl of the velocity is zero [Batchelor 1973]:

$$\nabla \times \boldsymbol{v} = 0. \tag{5-2}$$

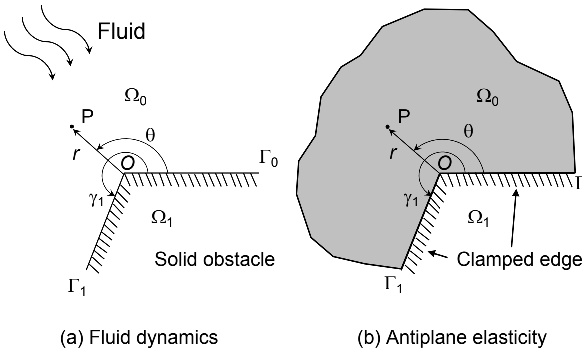

Figure 3. Geometry of a flow meeting a sharp obstacle: analogy between fluid dynamics and antiplane elasticity.

As a result, introducing (5-1) into (5-2), after some algebra we obtain the Laplace condition upon the stream function Ψ:

$$\nabla \times (\nabla \times \boldsymbol{\Psi}) = \nabla^2 \Psi = 0. \tag{5-3}$$

For a problem where a fluid flow meets a sharp obstacle, as shown in Figure 3, the flow may have singularities in the velocities. The power of the singularity can be determined according to an asymptotic analysis, adopting the eigenfunction expansion method for the stream function Ψ:

$$\Psi(r, \theta) = \sum_j r^{\lambda_j} f_j\,(\theta, \lambda_j), \tag{5-4}$$

where, again, λ_j are the eigenvalues of the problem and $f_j(\theta, \lambda_j)$ are the eigenfunctions. The Laplace condition upon Ψ implies that the functions $f_j(\theta, \lambda_j)$ are a combination of trigonometric functions, as in (2-8):

$$f_j(\theta, \lambda_j) = A_j \sin(\lambda_j \theta) + B_j \cos(\lambda_j \theta). \tag{5-5}$$

The velocity field components in polar coordinates, v_r and v_θ, can be obtained by differentiating the stream function Ψ:

$$v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \sum_j r^{\lambda_j - 1} f_j'(\theta, \lambda_j), \qquad v_\theta = -\frac{\partial \Psi}{\partial r} = -\sum_j \lambda_j r^{\lambda_j - 1} f_j(\theta, \lambda_j), \tag{5-6}$$

where $f_j'(\theta, \lambda_j)$ are the angular derivatives of the eigenfunctions $f_j(\theta, \lambda_j)$.

Hence, the following correspondences between fluid dynamics and antiplane elasticity can be set down:

$$\begin{aligned} \nabla^2 \Psi = 0 \quad &\Longleftrightarrow \quad \nabla^2 u_z = 0, \\ v_\theta = -\frac{\partial \Psi}{\partial r} \quad &\Longleftrightarrow \quad \tau_{rz} = G \frac{\partial u_z}{\partial r}, \\ v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} \quad &\Longleftrightarrow \quad \tau_{\theta z} = \frac{G}{r}\frac{\partial u_z}{\partial \theta}. \end{aligned} \tag{5-7}$$

The boundary conditions along the edges of the obstacle correspond to a vanishing velocity in the direction normal to the interface. According to Figure 3, this corresponds to the following conditions:

$$v_\theta(\theta = 0) = 0, \quad v_\theta(\theta = \gamma_1) = 0. \tag{5-8}$$

Introducing $(5\text{-}6)_2$ into (5-8), as well as the expression of the eigenfunction in (5-5), the boundary conditions (5-8) reduce to

$$f_j(0, \lambda_j) = B_j = 0, \quad f_j(\gamma_1, \lambda_j) = A_j \sin(\lambda_j \gamma_1) + B_j \cos(\lambda_j \gamma_1) = 0. \tag{5-9}$$

Note that the analogous of these boundary conditions in antiplane elasticity corresponds to the clamped boundary conditions (2-11) along Γ_0 and Γ_1 (see Figure 3). Therefore, it is possible to state that the singularities in fluid dynamics correspond to those in antiplane elasticity, provided that clamped-clamped BCs are considered along the edges of the elastic wedge. More specifically, since $(5\text{-}9)_1$ leads to $B_j = 0$ and we are looking for nontrivial solutions, the eigenequation of the problems is

$$\sin(\lambda_j \gamma_1) = 0, \tag{5-10}$$

from which we determine the lowest eigenvalue

$$\lambda_1 = \frac{\pi}{\gamma_1}. \tag{5-11}$$

Hence, a singularity exists ($\lambda_1 < 1$) for $\gamma_1 > \pi$, i.e., when the fluid domain Ω_1 presents a reentrant corner. Some notable eigenvalues can be found in [Batchelor 1973] and match exactly those provided in [Sinclair 1980] for the antiplane problem of a single material wedge. For instance, a flow around a right corner ($\gamma_1 = 3\pi/2$) has $\lambda_1 = \frac{2}{3}$, whereas a uniform flow ($\gamma_1 = \pi$) is nonsingular. The case of a flow around a thin obstacle ($\gamma_1 = 2\pi$) leads to $\lambda_1 = \frac{1}{2}$, as for an anticrack (rigid line inclusion) in antiplane elasticity [dal Corso et al. 2008; Bigoni et al. 2008].

6. Discussion and conclusion

In the present paper, we have compared and unified the mathematical formulations for the asymptotic characterization of the singular fields at multimaterial wedges in antiplane elasticity, diffusion problems, and electromagnetic diffraction. For a single and homogeneous material sector, we have also established the analogy between fluid dynamics and antiplane elasticity.

The asymptotic analysis of the stress singularities at the vertex of multimaterial wedges and junctions in antiplane elasticity is perfectly analogous to the corresponding diffusion problem. The temperature field plays the same role as the out-of-plane displacement field and the heat fluxes correspond to the tangential stresses. On the other hand, the analogy with electromagnetism is more complex. In particular, when an isotropic multimaterial wedge with PEC boundaries is considered, we have shown that two independent problems can be defined, one for TE fields, associated to an eigenequation for H_z, and one for TM fields, associated to an eigenequation for E_z. The eigenequation for E_z corresponds exactly to that obtained for the same geometrical configuration in antiplane elasticity by setting $G_i = 1/\mu_i$ and replacing the PEC region with an infinitely stiff material leading to clamped edge BCs along Γ_1 and Γ_n. Similarly, the other eigenequation for H_z can be obtained in antiplane elasticity for the same geometrical configuration by setting $G_i = 1/\epsilon_i$ and replacing the PEC region with an infinitely soft material leading to stress-free BCs along Γ_1 and Γ_n.

As far as the analogy for fluid dynamics is concerned, we have found that the stream function Ψ plays the same role as u_z and that the singularities in fluid dynamics correspond to those in antiplane elasticity, provided that clamped-clamped BCs are considered along the edges of the elastic wedge. These analogical results are also important for stress concentration problems. For instance, in the case of a flux around a circular cylinder, the velocity concentration factor is equal to 2, that is the velocity of the fluid at the border of the cylinder is twice higher than the velocity at infinity. Analogously, the stress concentration factor for a plate in uniaxial tension with an infinitely stiff round inclusion tends also to 2 [Duan et al. 2005].

Finally, the presence of singularities in diffusion, electromagnetism, and fluid dynamics suggests extending to these fields the dimensional analysis considerations we have proposed for the scaling of structural strength in [Carpinteri 1981; 1982a; 1982b; 1983; 1987; Carpinteri and Paggi 2006; 2009]. In solid mechanics, the presence of a stress singularity of power $\lambda - 1$ requires the use of a generalized stress-intensity factor, K^*, which has anomalous physical dimensions:

$$\sigma_{ij} = K^* r^{\lambda-1} f_{ij}(\theta, \lambda) \;\Rightarrow\; K^* = \sigma b^{1-\lambda} g, \tag{6-1}$$

where σ is a nominal applied stress, b is the characteristic structural size, and g is a shape factor depending on the geometry of the structure and the topology of the junction. The same reasoning can therefore be applied to the other analogous fields, defining generalized heat-intensity, electromagnetic-intensity, and velocity-intensity factors. In structural mechanics, the anomalous physical dimensions of K^* lead to size-scale effects on the nominal strength, i.e., the material strength becomes a function of the structural size [Carpinteri 1987; Carpinteri and Paggi 2006]. Therefore, size-scale effects in diffusion, electromagnetism, and fluid dynamics are also expected and can be analyzed using the same mathematical formalism as in structural mechanics.

Finally, it is well-known that elastic singularities are usually a mathematical artifact and that plasticity relieves the singularities in elastoplastic materials. Similarly to plasticity, a saturation of the electromagnetic fields at the sharp tip of an antenna is experimentally found in electromagnetism. In fluid dynamics, a possible analogous mechanism could be the occurrence of turbulence. In structural mechanics, the competition between crack propagation and plastic flow collapse is ruled by the brittleness number s (see [Carpinteri 1981; 1982a; 1982b] for more details):

$$s = \frac{K_{IC}}{\sigma_c b^{1/2}}, \tag{6-2}$$

where K_{IC}, σ_c and b are, respectively, the fracture toughness, the material strength, and the structural size. Brittle failure, characterized by *small scale yielding*, takes place when the brittleness number is lower than a threshold value. Above that, *large scale yielding* takes place and plastic flow collapse prevails over brittle failure. Generalizing these concepts to fluid dynamics, it is possible to define a similar dimensionless number in the case of a fluid against a thin obstacle. We shall call it *turbulence number*, s_t, and it is given by:

$$s_t = \frac{K_C}{v_c b^{1/2}}, \tag{6-3}$$

where K_C and v_c are, respectively, a critical velocity-intensity factor and a critical velocity for the appearance of turbulence. This resembles the Reynolds number, $\mathrm{Re} = v_c b/\nu$, where ν is the kinematic viscosity of the fluid, although the proposed turbulence number comes directly from the presence of singularities. The use of s_t in addition to the Reynolds number has never been explored so far. Therefore, further developments of this work will regard the assessment of the applicability of these concepts to interpret the laminar-turbulent flow transition in fluid mechanics, a still open problem nowadays.

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