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ADVANTAGES OF FORMULATING EVOLUTION EQUATIONS FOR ELASTIC-VISCOPLASTIC MATERIALS IN TERMS OF THE VELOCITY GRADIENT INSTEAD OF THE SPIN TENSOR

M. B. RUBIN AND O. PAPES

Dedicated to Charles and Marie-Louise Steele, who advanced the field of mechanics with their wise editorial leadership

Evolution equations for tensors that characterize elastic-viscoplastic materials are often formulated in terms of a Jaumann derivative based on the spin tensor. Typically, numerical integration algorithms for such equations split the integration operation by first calculating the response due to rate of deformation, followed by a finite rotation. Invariance under superposed rigid body motions of algorithms, incremental objectivity and strong objectivity are discussed. Specific examples of steady-state simple shear at constant rate and steady-state isochoric extension relative to a rotating coordinate system are used to analyze the robustness and accuracy of different algorithms. The results suggest that it is preferable to reformulate evolution equations in terms of the velocity gradient instead of the spin tensor, since strongly objective integration algorithms can be developed using the relative deformation gradient. Moreover, this relative deformation gradient can be calculated independently of the time dependence of the velocity gradient during a typical time step.

1. Introduction

Evolution equations with finite rotations occur naturally in continuum mechanics when history-dependent variables are expressed in terms of the present deforming configuration. Researchers in continuum mechanics typically focus attention on integrating rotations and use a representation of the rotation tensor attributed to Euler and Rodrigues; see, e.g., [Argyris 1982; Simo and Vu-Quoc 1988; Argyris and Poterasu 1993; Govindjee 1997; Becker 2006; Rubin 2007]. One of the objectives of this paper is to discuss fundamental and practical reasons for considering evolution equations based on the velocity gradient instead of on the spin tensor.

To be more specific, it is recalled that within the context of the three-dimensional theory the velocity gradient $\boldsymbol{L}$ separates into a symmetric rate of deformation tensor $\boldsymbol{D}$ and a skew-symmetric spin tensor $\boldsymbol{W}$, such that

$$\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}, \quad \boldsymbol{D} = \tfrac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T), \quad \boldsymbol{W} = \tfrac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T). \tag{1-1}$$

Within the context of hypoelastic formulations of the elastic response for elastic-viscoplastic materials it is common to propose a constitutive equation for the time rate of change of Cauchy stress $\boldsymbol{T}$. For example, a typical constitutive structure based on the Jaumann derivative of stress suggests that

$$\overset{W}{\boldsymbol{T}} = \dot{\boldsymbol{T}} - \boldsymbol{W}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{W}^T = \hat{\boldsymbol{K}}_W(\boldsymbol{T}, \boldsymbol{D}), \tag{1-2}$$

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where a superposed dot denotes material time differentiation and the function $\hat{\boldsymbol{K}}_W(\boldsymbol{T}, \boldsymbol{D})$ characterizes elastic and inelastic rates of deformation.

Integration of (1-2) over the time period $t_1 \leq t \leq t_2$ with time increment $\triangle t = t_2 - t_1$ is usually performed in two steps. First, the initial value $\boldsymbol{T}(t_1)$ of stress is augmented by an increment of stress that is determined by the quantity $\hat{\boldsymbol{K}}_W$ and then the result is rotated with an orthogonal tensor associated with the spin $\boldsymbol{W}$. This procedure can be formalized by introducing a proper orthogonal tensor $\boldsymbol{\Lambda}_W$ defined by the evolution equation and initial condition

$$\dot{\boldsymbol{\Lambda}}_W = \boldsymbol{W}\boldsymbol{\Lambda}_W, \quad \boldsymbol{\Lambda}_W(t_1) = \boldsymbol{I}, \tag{1-3}$$

and by introducing the auxiliary tensor $\overline{\boldsymbol{T}}_W$ defined by

$$\boldsymbol{T} = \boldsymbol{\Lambda}_W \overline{\boldsymbol{T}}_W \boldsymbol{\Lambda}_W^T, \tag{1-4}$$

which satisfies the evolution equation and initial condition

$$\dot{\overline{\boldsymbol{T}}}_W = \boldsymbol{\Lambda}_W^T \hat{\boldsymbol{K}}_W(\boldsymbol{T}, \boldsymbol{D})\boldsymbol{\Lambda}_W, \quad \overline{\boldsymbol{T}}_W(t_1) = \boldsymbol{T}(t_1). \tag{1-5}$$

In particular, this procedure requires two approximations, one for the integral of the stress (1-5) and another for the integral of the rotation (1-3). Hughes and Winget [1980] developed an approximate solution of (1-2), which will be discussed in Section 2. Also, for constant spin, (1-3) can be solved exactly in terms of the exponential map; see, e.g., [Govindjee 1997].

The Jaumann derivative is one of a number of stress rates which is properly invariant under superposed rigid body motions (SRBM) and which has been used in the literature to develop evolution equations for stress. For example, Dienes [1979] used the skew-symmetric tensor associated with the material derivative of the rotation tensor $\boldsymbol{R}$ in the polar decomposition (see [Malvern 1969], for instance) of the total deformation gradient $\boldsymbol{F}$. Also, Rashid [1993] used the skew-symmetric tensor associated with the material derivative of the rotation tensor $\boldsymbol{R}_r$ in the polar decomposition of the relative deformation gradient $\boldsymbol{F}_r$.

The same material response as that characterized by (1-2) can be obtained using different stress rates as long as the term on the right-hand side of (1-2) is modified appropriately. For example, the Oldroyd rate (see, e.g., [Holzapfel 2000]) can be used to obtain the evolution equation

$$\overset{L}{\boldsymbol{T}} = \dot{\boldsymbol{T}} - \boldsymbol{L}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{L}^T = \hat{\boldsymbol{K}}_L(\boldsymbol{T}, \boldsymbol{D}), \tag{1-6}$$

where the function $\hat{\boldsymbol{K}}_L$ is defined by

$$\hat{\boldsymbol{K}}_L(\boldsymbol{T}, \boldsymbol{D}) = \hat{\boldsymbol{K}}_W(\boldsymbol{T}, \boldsymbol{D}) - \boldsymbol{D}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{D}. \tag{1-7}$$

Next, it is recalled that the relative deformation gradient $\boldsymbol{F}_r$ can be defined in terms of the total deformation gradient by the expression

$$\boldsymbol{F}_r(t) = \boldsymbol{F}(t)\boldsymbol{F}^{-1}(t_1). \tag{1-8}$$

Moreover, it can be shown that $\boldsymbol{F}_r$ satisfies the evolution equation and initial condition

$$\dot{\boldsymbol{F}}_r = \boldsymbol{L}\boldsymbol{F}_r, \quad \boldsymbol{F}_r(t_1) = \boldsymbol{I}. \tag{1-9}$$

It then follows that the solution of (1-6) can be written in the form

$$\boldsymbol{T} = \boldsymbol{F}_r \overline{\boldsymbol{T}}_L \boldsymbol{F}_r^T, \tag{1-10}$$

where the auxiliary tensor $\overline{\boldsymbol{T}}_L$ satisfies the evolution equation and initial condition

$$\dot{\overline{\boldsymbol{T}}}_L = \boldsymbol{F}_r^{-1} \hat{\boldsymbol{K}}_L(\boldsymbol{T}, \boldsymbol{D}) \boldsymbol{F}_r^{-T}, \quad \overline{\boldsymbol{T}}_L(t_1) = \boldsymbol{T}(t_1). \tag{1-11}$$

Since the material response characterized by (1-2) can be formulated using different invariant stress rates, there is no fundamental advantage of one properly invariant formulation over another. However, from a practical point of view, details of the approximate integration algorithms for $\{\boldsymbol{\Lambda}_W, \overline{\boldsymbol{T}}_W\}$ or $\{\boldsymbol{F}_r, \overline{\boldsymbol{T}}_L\}$ may present advantages due to accuracy or invariance under SRBM.

Another important consideration was discussed in [Rashid 1993]. In computer codes for integrating the equations of motion in continuum mechanics the positions of material points are known at the beginning of the time step and are determined at the end of the time step. For implicit integration algorithms the final positions are determined by iteration, whereas for explicit integration algorithms they are determined by estimations based on the positions, velocities and accelerations during the time step. Rashid argued that the time dependences of the rate of deformation tensor $\boldsymbol{D}$ and spin tensor $\boldsymbol{W}$ during the time step are never known. This means that integration of the Jaumann formulation necessarily requires approximations associated with the specification of the time dependences of $\boldsymbol{D}$ and $\boldsymbol{W}$ as well as additional approximations due to the specific integration algorithms for both the rotation (1-3) and stress (1-5).

In contrast, the value $\boldsymbol{F}_r(t_2)$ of the relative deformation gradient at the end of the time step is a unique function of the positions at the beginning and end of the time step and therefore is known. This means that the evolution equation (1-9) can be integrated exactly without assuming an approximation of the velocity gradient. Consequently, the only approximation associated with the Oldroyd formulation appears in the specified integration algorithm for the auxiliary stress value $\overline{\boldsymbol{T}}_L$ in (1-11). For this reason the Oldroyd seems to have a practical advantage over the Jaumann formulation.

Eckart [1948] seems to be the first to have proposed an evolution equation directly for elastic deformation for large deformations of elastically isotropic elastic-plastic materials. In particular, using [Flory 1961] it is possible to introduce a symmetric unimodular tensor $\boldsymbol{B}_e'$,

$$\det \boldsymbol{B}_e' = 1, \tag{1-12}$$

which is a pure measure of elastic distortional deformation. Then, for elastic-viscoplastic response $\boldsymbol{B}_e'$ can be determined by the evolution equation

$$\dot{\boldsymbol{B}}_e' = \boldsymbol{L}\boldsymbol{B}_e' + \boldsymbol{B}_e'\boldsymbol{L}^T - \tfrac{2}{3}(\boldsymbol{D}\cdot\boldsymbol{I})\boldsymbol{B}_e' - \Gamma \boldsymbol{A}_p, \quad \boldsymbol{A}_p = \boldsymbol{B}_e' - \frac{3}{\boldsymbol{B}_e'^{-1}\cdot\boldsymbol{I}}\boldsymbol{I}, \tag{1-13}$$

where Γ and $\boldsymbol{A}_p$ characterize the rate of relaxation due to plasticity. This equation must be integrated subject to the initial condition

$$\boldsymbol{B}_e' = \boldsymbol{B}_e'(t_1) \quad \text{for } t = t_1. \tag{1-14}$$

This idea was used in [Leonov 1976] for polymeric liquids, and in [Simo 1992; Rubin 1994] for elastic-plastic and elastic-viscoplastic solids. Besseling [1968] and Rubin [1994] proposed generalizations of (1-13) for elastically anisotropic response and a number of physical aspects of constitutive equations for plasticity have been discussed in [Rubin 1994; Rubin 1996; Rubin 2001]. Also, a simple integration algorithm for (1-13) was discussed in [Rubin and Attia 1996].

Again, using [Flory 1961], it can be shown that the unimodular part $\boldsymbol{F}_r'$ of the relative deformation tensor $\boldsymbol{F}_r$ defined by

$$\boldsymbol{F}_r' = (\det \boldsymbol{F}_r)^{-1/3}\boldsymbol{F}_r, \quad \det \boldsymbol{F}_r' = 1, \tag{1-15}$$

satisfies the evolution equation and initial condition

$$\dot{\boldsymbol{F}}_r' = \boldsymbol{L}\boldsymbol{F}_r' - \tfrac{1}{3}(\boldsymbol{D}\cdot\boldsymbol{I})\boldsymbol{F}_r', \quad \boldsymbol{F}_r'(t_1) = \boldsymbol{I}. \tag{1-16}$$

Consequently, the solution of (1-13) can be written in terms of the auxiliary unimodular symmetric tensor $\bar{\boldsymbol{B}}_e'$ defined by

$$\boldsymbol{B}_e' = \boldsymbol{F}_r'\bar{\boldsymbol{B}}_e'\boldsymbol{F}_r'^T, \quad \det \bar{\boldsymbol{B}}_e' = 1, \tag{1-17}$$

with $\bar{\boldsymbol{B}}_e'$ satisfying the evolution equation and initial condition

$$\dot{\bar{\boldsymbol{B}}}_e' = -\Gamma \boldsymbol{F}_r'^{-1}\boldsymbol{A}_p\boldsymbol{F}_r'^{-T}, \quad \bar{\boldsymbol{B}}_e'(t_1) = \boldsymbol{B}_e'(t_1). \tag{1-18}$$

In general, Γ is a nonlinear function of state variables that can include hardening. Also, it is noted that for an elastically isotropic hyperelastic material the Cauchy stress can be obtained from derivatives of a strain energy function that depends on the dilatation $J = \det \boldsymbol{F}$ and the two nontrivial invariants of $\boldsymbol{B}_e'$.

An outline of the paper is as follows. Section 2 summarizes the Hughes–Winget algorithm [Hughes and Winget 1980], Section 3 discusses invariance under SRBM and the notion of incremental objectivity. Section 4 presents integration algorithms for a simple elastic-viscoplastic material. The robustness and accuracy of these algorithms are discussed in Section 5 for the example of steady-state simple shear and in Section 6 for the example of steady-state isochoric extension relative to a rotating coordinate system. Finally, Section 7 presents conclusions.

Above and in the following, boldfaced symbols denote tensors, $\boldsymbol{e}_i$ is a right-handed orthonormal triad of fixed rectangular Cartesian base vectors, the components of all tensors are referred to $\boldsymbol{e}_i$ and $\otimes$ denotes the tensor product. Also, $\boldsymbol{A}\cdot\boldsymbol{B} = \mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^T)$ denotes the inner product between two second-order tensors $\{\boldsymbol{A}, \boldsymbol{B}\}$ and $\boldsymbol{a}\otimes\boldsymbol{b}$ denotes the tensor product between two vectors $\{\boldsymbol{a}, \boldsymbol{b}\}$.

2. Summary of the Hughes–Winget algorithm

Hughes and Winget [1980] developed an expression for the value $\boldsymbol{\Lambda}_W(t_2)$ of a rotation tensor based on the skew-symmetric part of an approximate incremental displacement gradient. Specifically, they introduced a mapping from the position $\boldsymbol{y}^1$ at the beginning of the time step to the position $\boldsymbol{y}^2$ at the end of the time step and a mapping to the position $\boldsymbol{y}^\alpha$ at an intermediate configuration by the expressions

$$\boldsymbol{y}^2 = \boldsymbol{y}^2(\boldsymbol{y}^1), \quad \boldsymbol{y}^\alpha = (1-\alpha)\boldsymbol{y}^1 + \alpha\boldsymbol{y}^2. \tag{2-1}$$

Then, the incremental displacement $\boldsymbol{\delta}$, and displacement gradient $\boldsymbol{G}$ relative to this intermediate configuration are defined by

$$\boldsymbol{\delta} = \boldsymbol{y}^2 - \boldsymbol{y}^1, \quad \boldsymbol{G} = \partial\boldsymbol{\delta}/\partial\boldsymbol{y}^\alpha = (\partial\boldsymbol{y}^2/\partial\boldsymbol{y}^1 - \boldsymbol{I})(\partial\boldsymbol{y}^1/\partial\boldsymbol{y}^\alpha). \tag{2-2}$$

Moreover, with the help of (2-1) it follows that

$$\partial \boldsymbol{y}^{\alpha}/\partial \boldsymbol{y}^{1} = (1-\alpha)\boldsymbol{I} + \alpha(\partial \boldsymbol{y}^{2}/\partial \boldsymbol{y}^{1}). \tag{2-3}$$

Next, using the definition of the relative deformation gradient $\boldsymbol{F}_r(t_2)$ at the end of the time step

$$\boldsymbol{F}_r(t_2) = \partial \boldsymbol{y}^{2}/\partial \boldsymbol{y}^{1}, \tag{2-4}$$

the tensor $\boldsymbol{G}$ can be expressed in the form

$$\boldsymbol{G} = \big(\boldsymbol{F}_r(t_2) - \boldsymbol{I}\big)\big(\boldsymbol{I} + \alpha\{\boldsymbol{F}_r(t_2) - \boldsymbol{I}\}\big)^{-1}. \tag{2-5}$$

Thus, for $\alpha = \frac{1}{2}$ as specified in [Hughes and Winget 1980], the tensor $\boldsymbol{G}$ is given by

$$\boldsymbol{G} = 2\big(\boldsymbol{F}_r(t_2) - \boldsymbol{I}\big)\big(\boldsymbol{F}_r(t_2) + \boldsymbol{I}\big)^{-1}. \tag{2-6}$$

Next, using the lemma quoted in [Hughes and Winget 1980],

$$\boldsymbol{A}(\boldsymbol{A}+\boldsymbol{B})^{-1}\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{A}+\boldsymbol{B})^{-1}\boldsymbol{A}, \tag{2-7}$$

for all square nonsingular matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{A}+\boldsymbol{B}$. Taking $\boldsymbol{A} = \boldsymbol{F}_r^T$ and $\boldsymbol{B} = \boldsymbol{I}$ it follows that

$$\boldsymbol{F}_r^T(\boldsymbol{F}_r^T+\boldsymbol{I})^{-1} = (\boldsymbol{F}_r^T+\boldsymbol{I})^{-1}\boldsymbol{F}_r^T, \quad (\boldsymbol{F}_r^T-\boldsymbol{I})(\boldsymbol{F}_r^T+\boldsymbol{I})^{-1} = (\boldsymbol{F}_r^T+\boldsymbol{I})^{-1}(\boldsymbol{F}_r^T-\boldsymbol{I}), \tag{2-8}$$

so that

$$\boldsymbol{G}^T = 2(\boldsymbol{F}_r^T+\boldsymbol{I})^{-1}(\boldsymbol{F}_r^T-\boldsymbol{I}) = 2(\boldsymbol{F}_r^T-\boldsymbol{I})(\boldsymbol{F}_r^T+\boldsymbol{I})^{-1}. \tag{2-9}$$

Thus, the symmetric part $\boldsymbol{\gamma}$ and skew-symmetric part $\boldsymbol{\omega}$ of $\boldsymbol{G}$ are given by

$$\begin{aligned}\boldsymbol{\gamma} &= \tfrac{1}{2}(\boldsymbol{G}+\boldsymbol{G}^T) = \big(\boldsymbol{F}_r(t_2)-\boldsymbol{I}\big)\big(\boldsymbol{F}_r(t_2)+\boldsymbol{I}\big)^{-1} + \big(\boldsymbol{F}_r^T(t_2)-\boldsymbol{I}\big)\big(\boldsymbol{F}_r^T(t_2)+\boldsymbol{I}\big)^{-1},\\ \boldsymbol{\omega} &= \tfrac{1}{2}(\boldsymbol{G}-\boldsymbol{G}^T) = \big(\boldsymbol{F}_r(t_2)-\boldsymbol{I}\big)\big(\boldsymbol{F}_r(t_2)+\boldsymbol{I}\big)^{-1} - \big(\boldsymbol{F}_r^T(t_2)-\boldsymbol{I}\big)\big(\boldsymbol{F}_r^T(t_2)+\boldsymbol{I}\big)^{-1}.\end{aligned} \tag{2-10}$$

Now, the Hughes–Winget algorithm for the value $\boldsymbol{\Lambda}_W(t_2)$ of the rotation tensor $\boldsymbol{\Lambda}_W$ at the end of the time step is given by

$$\boldsymbol{\Lambda}_W(t_2) = \big(\boldsymbol{I} + \tfrac{1}{2}\boldsymbol{\omega}\big)\big(\boldsymbol{I} - \tfrac{1}{2}\boldsymbol{\omega}\big)^{-1}. \tag{2-11}$$

3. Invariance under superposed rigid body motions and incremental objectivity

Under SRBM the material point $\boldsymbol{x}$ at time t moves to the position $\boldsymbol{x}^+$ at time t^+, such that

$$\boldsymbol{x}^+ = \boldsymbol{c}(t) + \boldsymbol{Q}(t)\boldsymbol{x}, \quad t^+ = t + c, \quad \boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}, \quad \det \boldsymbol{Q} = +1, \quad \dot{\boldsymbol{Q}} = \Omega\boldsymbol{Q}, \quad \Omega^T = -\Omega(t), \tag{3-1}$$

where c is an arbitrary constant, $\boldsymbol{c}(t)$ is an arbitrary function of time characterizing superposed translation, $\boldsymbol{Q}(t)$ is an arbitrary proper orthogonal tensor characterizing rotation and $\Omega(t)$ is the associated spin tensor. Moreover, under SRBM the quantities $\{\boldsymbol{F}, \boldsymbol{F}_r, \boldsymbol{F}_r', \boldsymbol{D}, \boldsymbol{W}, \boldsymbol{\Lambda}_W, \boldsymbol{T}, \boldsymbol{B}_e', \Gamma\}$ transform to $\{\boldsymbol{F}^+, \boldsymbol{F}_r^+, \boldsymbol{F}_r'^+, \boldsymbol{D}^+, \boldsymbol{W}^+, \boldsymbol{\Lambda}_W^+, \boldsymbol{T}^+, \boldsymbol{B}_e'^+, \Gamma^+\}$, respectively, by the relations

$$\begin{aligned}&\boldsymbol{F}^+ = \boldsymbol{Q}\boldsymbol{F}, &&\boldsymbol{F}_r^+ = \boldsymbol{Q}\boldsymbol{F}_r, &&\boldsymbol{F}_r'^+ = \boldsymbol{Q}\boldsymbol{F}_r',\\ &\boldsymbol{D}^+ = \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T, &&\boldsymbol{W}^+ = \boldsymbol{Q}\boldsymbol{W}\boldsymbol{Q}^T + \Omega, &&\boldsymbol{\Lambda}_W^+ = \boldsymbol{Q}\boldsymbol{\Lambda}_W,\\ &\boldsymbol{T}^+ = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T, &&\boldsymbol{B}_e'^+ = \boldsymbol{Q}\boldsymbol{B}_e'\boldsymbol{Q}^T, &&\Gamma^+ = \Gamma.\end{aligned} \tag{3-2}$$

Since the rates in (1-2) and (1-6) transform under SRBM by

$$\overset{W}{\boldsymbol{T}}{}^{+} = \boldsymbol{Q}\overset{W}{\boldsymbol{T}}\boldsymbol{Q}^T, \quad \overset{L}{\boldsymbol{T}}{}^{+} = \boldsymbol{Q}\overset{L}{\boldsymbol{T}}\boldsymbol{Q}^T, \tag{3-3}$$

it follows that the evolution equations (1-2) and (1-6) are properly invariant under SRBM provided that the functions $\hat{\boldsymbol{K}}_W$ and $\hat{\boldsymbol{K}}_L$ satisfy the restrictions

$$\hat{\boldsymbol{K}}_W(\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T) = \boldsymbol{Q}\hat{\boldsymbol{K}}_W(\boldsymbol{T}, \boldsymbol{D})\boldsymbol{Q}^T, \quad \hat{\boldsymbol{K}}_L(\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T) = \boldsymbol{Q}\hat{\boldsymbol{K}}_L(\boldsymbol{T}, \boldsymbol{D})\boldsymbol{Q}^T, \tag{3-4}$$

for all proper orthogonal $\boldsymbol{Q}$. Moreover, the evolution equation (1-13) is properly invariant under SRBM.

Using these results together with the definitions (1-4), (1-10) and (1-17), it can also be shown that the auxiliary tensors $\{\overline{\boldsymbol{T}}_W, \overline{\boldsymbol{T}}_L, \overline{\boldsymbol{B}}'_e\}$ transform to $\{\overline{\boldsymbol{T}}^+_W, \overline{\boldsymbol{T}}^+_L, \overline{\boldsymbol{B}}'^+_e\}$, so they are unaffected by SRBM:

$$\overline{\boldsymbol{T}}^+_W = \overline{\boldsymbol{T}}_W, \quad \overline{\boldsymbol{T}}^+_L = \overline{\boldsymbol{T}}_L, \quad \overline{\boldsymbol{B}}'^+_e = \overline{\boldsymbol{B}}'_e. \tag{3-5}$$

Hughes and Winget [1980] introduced the notion of an algorithm being incrementally objective. Specifically, they proved that the expressions (2-10) and (2-11) are incrementally objective in the sense that they correctly produce zero strain increment ($\boldsymbol{\gamma} = 0$) and the correct rotation tensor ($\boldsymbol{\Lambda}_W = \boldsymbol{F}_r$) when the deformation during the time step is a pure rotation with $\boldsymbol{F}_r$ being an orthogonal tensor ($\boldsymbol{F}_r^T\boldsymbol{F}_r = \boldsymbol{I}$). It can also be shown that (2-10) and (2-11) correctly produce zero incremental spin ($\boldsymbol{\omega} = 0$) and the correct rotation ($\boldsymbol{\Lambda}_W = \boldsymbol{I}$) when the deformation during the time step is a pure stretch with $\boldsymbol{F}_r$ being a symmetric tensor.

Rashid [1993] extended this notion of incremental objectivity by demanding that the integrator computes a stretching part that is independent of the input rotation when the incremental motion involves both stretch and rotation. Actually, the notion of strong objectivity, as discussed in [Papes and Mazza 2009], requires the estimates of all variables at the end of the time step to satisfy the same invariance properties under SRBM as their exact values. In this regard, it is noted that the tensors $\{\boldsymbol{\gamma}, \boldsymbol{\omega}\}$ are not properly invariant under SRBM since they retain an unphysical dependence on the arbitrary rotation tensor $\boldsymbol{Q}$:

$$\begin{aligned}
\boldsymbol{\gamma}^+ &= \boldsymbol{Q}\big[\big(\boldsymbol{F}_r(t_2)\boldsymbol{Q} - \boldsymbol{I}\big)\big(\boldsymbol{F}_r(t_2)\boldsymbol{Q} + \boldsymbol{I}\big)^{-1} + \big(\boldsymbol{Q}^T\boldsymbol{F}_r^T(t_2) - \boldsymbol{I}\big)\big(\boldsymbol{Q}^T\boldsymbol{F}_r^T(t_2) + \boldsymbol{I}\big)^{-1}\big]\boldsymbol{Q}^T, \\
\boldsymbol{\omega}^+ &= \boldsymbol{Q}\big[\big(\boldsymbol{F}_r(t_2)\boldsymbol{Q} - \boldsymbol{I}\big)\big(\boldsymbol{F}_r(t_2)\boldsymbol{Q} + \boldsymbol{I}\big)^{-1} - \big(\boldsymbol{Q}^T\boldsymbol{F}_r^T(t_2) - \boldsymbol{I}\big)\big(\boldsymbol{Q}^T\boldsymbol{F}_r^T(t_2) + \boldsymbol{I}\big)^{-1}\big]\boldsymbol{Q}^T.
\end{aligned} \tag{3-6}$$

Consequently, the approximation (2-11) for $\boldsymbol{\Lambda}_W$ is also not properly invariant:

$$\boldsymbol{\Lambda}^+_W \neq \boldsymbol{Q}\boldsymbol{\Lambda}_W. \tag{3-7}$$

Moreover, it is natural to consider the approximation of incremental strain

$$\Delta t\, \boldsymbol{D} \approx \boldsymbol{\gamma}. \tag{3-8}$$

However, this expression is also not properly invariant since

$$\Delta t\, \boldsymbol{D}^+ \neq \boldsymbol{Q}(\Delta t\, \boldsymbol{D})\boldsymbol{Q}^T. \tag{3-9}$$

4. Integration algorithms for a simple elastic-viscoplastic material

In the remainder of this paper attention will be focused on an elastic-viscoplastic material which is characterized by the equations (1-13), (1-14), (1-17) and (1-18) with the simplification that the scalar Γ is constant. In this section three algorithms are discussed which yield approximate solutions of these evolution equations. Sections 5 and 6 will discuss examples to test the accuracy and robustness of these algorithms.

Algorithm 1. This algorithm is based on the formulation (1-17) and (1-18). This formulation has the simplicity that any approximation of $\bar{\boldsymbol{B}}'_e$ that is unaffected by SRBM will yield a solution for $\boldsymbol{B}'_e$ that transforms appropriately under SRBM. To motivate an approximate solution of (1-18) consider the value determined by the fully implicit equation

$$\bar{\boldsymbol{B}}'_e(t_2) = \bar{\boldsymbol{B}}'_e(t_1) - \Delta t\, \Gamma \Big(\bar{\boldsymbol{B}}'_e(t_2) - \frac{3}{\boldsymbol{B}'^{-1}_e(t_2)\cdot \boldsymbol{I}} \boldsymbol{C}'^{-1}_r(t_2) \Big), \quad \boldsymbol{C}'^{-1}_r(t_2) = \boldsymbol{F}'^{-1}_r(t_2)\boldsymbol{F}'^{-T}_r(t_2). \tag{4-1}$$

Next, use is made of the approximation that $\boldsymbol{B}'^{-1}_e(t_2)\cdot \boldsymbol{I} \approx 3$ to deduce that

$$\bar{\boldsymbol{B}}'_e(t_2) = \frac{1}{1+\Delta t\, \Gamma} \big(\boldsymbol{B}'_e(t_1) + \Delta t\, \Gamma \boldsymbol{C}'^{-1}_r(t_2) \big). \tag{4-2}$$

However, this expression does not ensure that $\bar{\boldsymbol{B}}'_e(t_2)$ is unimodular. Motivated by [Rubin and Attia 1996], the expression (4-2) is used to obtain an equation for the deviatoric part $\bar{\boldsymbol{B}}''_e(t_2)$ of $\bar{\boldsymbol{B}}'_e(t_2)$ of the form

$$\bar{\boldsymbol{B}}''_e(t_2) = \frac{1}{1+\Delta t\, \Gamma} \Big(\boldsymbol{B}'_e(t_1) + \Delta t\, \Gamma \boldsymbol{C}'^{-1}_r(t_2) - \tfrac{1}{3}\big[\big(\boldsymbol{B}'_e(t_1) + \Delta t\, \Gamma \boldsymbol{C}'^{-1}_r(t_2) \big) \cdot \boldsymbol{I} \big] \boldsymbol{I} \Big). \tag{4-3}$$

Then, the final value $\bar{\boldsymbol{B}}'_e(t_2)$ can be obtained using the procedure discussed in [Rubin and Attia 1996] to determine the scalar $\bar{\alpha}$, in the expression

$$\bar{\boldsymbol{B}}'_e(t_2) = \tfrac{1}{3}\bar{\alpha}\boldsymbol{I} + \bar{\boldsymbol{B}}''_e(t_2), \tag{4-4}$$

by the condition that $\bar{\boldsymbol{B}}'_e(t_2)$ is unimodular.

Algorithm 2. Typically, for elastic-viscoplastic response an equation like (1-13) is integrated in two steps. First, the elastic trial value $\bar{\boldsymbol{B}}'^{*}_e(t)$ is determined by the evolution equation (which is (1-13) with $\Gamma = 0$) and initial condition

$$\dot{\boldsymbol{B}}'^{*}_e = \boldsymbol{L}\boldsymbol{B}'^{*}_e + \boldsymbol{B}'^{*}_e\boldsymbol{L}^T - \tfrac{2}{3}(\boldsymbol{D}\cdot\boldsymbol{I})\boldsymbol{B}'^{*}_e, \quad \boldsymbol{B}'^{*}_e(t_1) = \boldsymbol{B}'_e(t_1). \tag{4-5}$$

Then, the value $\boldsymbol{B}'_e(t_2)$ at the end of the time step is determined by relaxing the elastic trial value $\boldsymbol{B}'^{*}_e(t_2)$ at constant total deformation with the help of an approximation of the evolution equation (1-13).

In particular, using the formulation (1-17) and (1-18) it is easy to see that the exact solution of (4-5) is given by

$$\boldsymbol{B}'^{*}_e(t_2) = \boldsymbol{F}'_r(t_2)\boldsymbol{B}'_e(t_1)\boldsymbol{F}'^{T}_r(t_2), \tag{4-6}$$

so that the solution of (1-13) can be written in the form

$$\boldsymbol{B}'_e(t_2) = \boldsymbol{B}'^{*}_e(t_2) - \boldsymbol{F}'_r(t_2)\bigg(\int_{t_1}^{t_2} \Gamma \boldsymbol{F}'^{-1}_r \boldsymbol{A}_p \boldsymbol{F}'^{-T}_r\, dt \bigg) \boldsymbol{F}'^{T}_r(t_2). \tag{4-7}$$

An elastic trial of the type (4-6) was used in [Simo 1992] for a nonunimodular tensor and in [Simo and Hughes 1998, p. 315] updating a unimodular tensor. In approximating the integral in (4-7) it is important to ensure that the approximation remains properly invariant under SRBM. Here, the values of $\{\boldsymbol{A}_p, \boldsymbol{F}_r'\}$ are approximated by their values at the end of the time step and use is made of the fully implicit form

$$\boldsymbol{B}_e'(t_2) = \boldsymbol{B}_e'^*(t_2) - \Delta t\,\Gamma\Big(\boldsymbol{B}_e'(t_2) - \frac{3}{\boldsymbol{B}_e'^{-1}(t_2)\cdot\boldsymbol{I}}\,\boldsymbol{I}\Big). \tag{4-8}$$

Again, since this result does not ensure that $\boldsymbol{B}_e'(t_2)$ is unimodular, the solution is determined by the deviatoric part $\boldsymbol{B}_e''(t_2)$ of $\boldsymbol{B}_e'(t_2)$ given by

$$\boldsymbol{B}_e''(t_2) = \frac{1}{1+\Delta t\,\Gamma}\,\boldsymbol{B}_e''^*(t_2), \tag{4-9}$$

where $\boldsymbol{B}_e''^*(t_2)$ is the deviatoric part of the elastic trial $\boldsymbol{B}_e'^*(t_2)$

$$\boldsymbol{B}_e''^*(t_2) = \boldsymbol{B}_e'^*(t_2) - \tfrac{1}{3}\big(\boldsymbol{B}_e'^*(t_2)\cdot\boldsymbol{I}\big)\,\boldsymbol{I}. \tag{4-10}$$

Then, the final value $\boldsymbol{B}_e'(t_2)$ is obtained using the procedure discussed in [Rubin and Attia 1996] to determine the scalar α in the expression

$$\boldsymbol{B}_e'(t_2) = \tfrac{1}{3}\alpha\boldsymbol{I} + \boldsymbol{B}_e''(t_2), \tag{4-11}$$

which has the same invariance properties under SRBM as the exact value of $\boldsymbol{B}_e'(t_2)$.

Algorithm 3. Algorithms 1 and 2 integrate the evolution equation (1-13) including coupling of the rates of deformation and spin $\{\boldsymbol{D}, \boldsymbol{W}\}$ through the expressions for the velocity gradient $\boldsymbol{L}$ and the relative deformation gradient $\boldsymbol{F}_r$. For Algorithm 3 this evolution equation is reformulated in terms of the Jaumann derivative which focuses on spin. Specifically, (1-13) can be written in the form

$$\overset{W}{\boldsymbol{B}_e'} = \hat{\boldsymbol{A}}(\boldsymbol{B}_e', \boldsymbol{D}) - \Gamma\boldsymbol{A}_p, \quad \hat{\boldsymbol{A}}(\boldsymbol{B}_e', \boldsymbol{D}) = \boldsymbol{D}\boldsymbol{B}_e' + \boldsymbol{B}_e'\boldsymbol{D} - \tfrac{2}{3}(\boldsymbol{D}\cdot\boldsymbol{I})\boldsymbol{B}_e'. \tag{4-12}$$

Then, the solution is given by (4-9) and (4-10), where the elastic trial $\boldsymbol{B}_e'^*(t_2)$ is determined by the evolution equation and initial condition

$$\overset{W}{\boldsymbol{B}_e'^*} = \hat{\boldsymbol{A}}(\boldsymbol{B}_e'^*, \boldsymbol{D}), \quad \boldsymbol{B}_e'^*(t_1) = \boldsymbol{B}_e'(t_1). \tag{4-13}$$

Next, using the Hughes–Winget algorithm the solution of (4-13) is approximated by

$$\boldsymbol{B}_e'^*(t_2) = \boldsymbol{\Lambda}_W\big(\boldsymbol{B}_e'(t_1) + \Delta t\,\hat{\boldsymbol{A}}(\boldsymbol{B}_e'(t_1), \boldsymbol{D})\big)\boldsymbol{\Lambda}_W^T, \tag{4-14}$$

where $\boldsymbol{\Lambda}_W$ is defined by (2-11). Furthermore, use is made of the approximation (3-8) to obtain the deviatoric tensor

$$\boldsymbol{B}_e''^*(t_2) = \boldsymbol{\Lambda}_W\big(\tilde{\boldsymbol{B}}_e' - \tfrac{1}{3}(\tilde{\boldsymbol{B}}_e'\cdot\boldsymbol{I})\boldsymbol{I}\big)\boldsymbol{\Lambda}_W^T, \quad \tilde{\boldsymbol{B}}_e' = \boldsymbol{B}_e'(t_1) + \boldsymbol{\gamma}\boldsymbol{B}_e'(t_1) + \boldsymbol{B}_e'(t_1)\boldsymbol{\gamma} - \tfrac{2}{3}(\boldsymbol{\gamma}\cdot\boldsymbol{I})\boldsymbol{B}_e'(t_1), \tag{4-15}$$

which is then used in (4-9) to obtain the final value (4-11) for this algorithm.

5. Example of steady-state simple shear at constant shear rate

The algebra required to obtain the solutions discussed in the example in this and the next sections is rather heavy so use has been made of the symbolic program Maple to derive the results.

This section presents the example of steady-state simple shear at constant shear rate to analytically analyze the robustness and accuracy of Algorithms 1, 2 and 3. For this problem the deformation gradient $\boldsymbol{F}$ and the constant velocity gradient $\boldsymbol{L}$ are specified by

$$\boldsymbol{F} = \boldsymbol{I} + \gamma\Gamma t(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2), \quad \boldsymbol{L} = \gamma\Gamma(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2), \tag{5-1}$$

where the constant scalar γ should not be confused with the tensor $\boldsymbol{\gamma}$ in (2-10). It then follows that the relative deformation tensor associated with (5-1) is given by

$$\boldsymbol{F}_r(t_2) = \boldsymbol{I} + \kappa(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2), \quad \boldsymbol{F}_r^{-1}(t_2) = \boldsymbol{I} - \kappa(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2), \quad \kappa = \Delta t\, \gamma\Gamma. \tag{5-2}$$

Moreover, since this deformation is isochoric, $\boldsymbol{F}_r$ is a unimodular tensor with

$$\boldsymbol{F}_r' = \boldsymbol{F}_r. \tag{5-3}$$

Exact solution. The initial value $\boldsymbol{B}_e'(t_1)$ is a steady-state solution of the evolution equation (1-13) provided that it satisfies the equation

$$\boldsymbol{L}\boldsymbol{B}_e'(t_1) + \boldsymbol{B}_e'(t_1)\boldsymbol{L}^T - \tfrac{2}{3}(\boldsymbol{D}\cdot\boldsymbol{I})\boldsymbol{B}_e'(t_1) - \Gamma\boldsymbol{A}_p(t_1) = 0. \tag{5-4}$$

The exact solution of this algebraic equation can be written in the form

$$\boldsymbol{B}_e'(t_1) = a^2(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1) + b^2(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + c^2(\boldsymbol{e}_3 \otimes \boldsymbol{e}_3) + d(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1), \tag{5-5}$$

where $\{a, c\}$ take the forms

$$a = \frac{\sqrt{1+d^2b^2}}{b^2}, \quad c = b, \tag{5-6}$$

and the constants $\{b, d\}$ attain the values $\{b_e, d_e\}$, respectively, given by

$$b_e = \frac{1}{(1+\gamma^2)^{1/6}}, \quad d_e = \frac{\gamma}{(1+\gamma^2)^{1/3}}. \tag{5-7}$$

Solution of Algorithm 1. The initial value $\boldsymbol{B}_e'(t_1)$ is a steady-state solution of Algorithm 1 if $\bar{\boldsymbol{B}}_e'(t_2)$ and its deviatoric part $\bar{\boldsymbol{B}}_e''(t_2)$ are given by

$$\bar{\boldsymbol{B}}_e'(t_2) = \boldsymbol{F}_r'^{-1}(t_2)\boldsymbol{B}_e'(t_1)\boldsymbol{F}_r'^{-T}(t_2), \quad \bar{\boldsymbol{B}}_e''(t_2) = \bar{\boldsymbol{B}}_e'(t_2) - \tfrac{1}{3}\big(\bar{\boldsymbol{B}}_e'(t_2)\cdot\boldsymbol{I}\big)\boldsymbol{I}. \tag{5-8}$$

More specifically, substitution of (5-8) into (4-3) and taking $\boldsymbol{B}_e'(t_1)$ in the form (5-5), with the condition (5-6), yields a system of algebraic equations for $\{b, d\}$. These equations can be solved to obtain the values $\{b_1, d_1\}$, given by

$$d_1 = (\gamma+\kappa)b_1^2 - \kappa, \tag{5-9}$$

where b_1 is the positive real root of the equation

$$(1+\gamma^2+\gamma\kappa)b_1^6 + \kappa^2 b_1^4 - \kappa^2 b_1^2 - 1 = 0. \tag{5-10}$$

Solution of Algorithm 2. The initial value $\boldsymbol{B}'_e(t_1)$ is a steady-state solution of Algorithm 2 if $\boldsymbol{B}''_e(t_2)$ equals the deviatoric part $\boldsymbol{B}'_e(t_1)$, given by

$$\boldsymbol{B}''_e(t_2) = \boldsymbol{B}''_e(t_1) = \boldsymbol{B}'_e(t_1) - \tfrac{1}{3}\big(\boldsymbol{B}'_e(t_1)\cdot\boldsymbol{I}\big)\boldsymbol{I}. \tag{5-11}$$

More specifically, substitution of (5-11) into (4-9) and taking $\boldsymbol{B}'_e(t_1)$ in the form (5-5), with the condition (5-6), yields a system of algebraic equations for $\{b, d\}$. These equations can be solved to obtain the values $\{b_2, d_2\}$ given by

$$b_2 = \frac{1}{(1+\gamma^2+\gamma\kappa)^{1/6}}, \quad d_2 = \frac{\gamma}{(1+\gamma^2+\gamma\kappa)^{1/3}}. \tag{5-12}$$

Solution of Algorithm 3. The initial value $\boldsymbol{B}'_e(t_1)$ will be a steady-state solution of Algorithm 3 if $\boldsymbol{B}''_e(t_2)$ in (4-9) satisfies (5-11) with the deviatoric part $\boldsymbol{B}''^{*}_e(t_2)$ of the elastic trial given by (4-15). The solution of this system of equations has the form (5-5), where $\{a, b, c, d\}$ obtain the values $\{a_3, b_3, c_3, d_3\}$ given by

$$a_3 = b_3\sqrt{\frac{256+512\gamma^2+512\gamma\kappa+16(2-\gamma^2)\kappa^2-16\gamma\kappa^3+\kappa^4}{256+16(2+\gamma^2)\kappa^2+16\gamma\kappa^3+\kappa^4}}, \quad b_3 = \left(\frac{N_3(\gamma,\kappa)}{D_3(\gamma,\kappa)}\right)^{1/6},$$
$$d_3 = \frac{(512-112\kappa^2+\kappa^4)b_3^2-(80-\kappa^2)\kappa^2 a_3^2}{512+256\gamma\kappa+64\kappa^2+2\kappa^4}\gamma, \quad c_3 = \sqrt{\tfrac{1}{2}(a_3^2+b_3^2)-\gamma d_3}. \tag{5-13}$$

In these expressions the functions N_3 and D_3 are polynomials of their arguments that can be obtained analytically by requiring $\boldsymbol{B}'_e(t_1)$ to be unimodular. Also, it is noted that the values of $\{a, c\}$ no longer satisfy the conditions (5-6) of the exact solution. Since this formulation uses the approximations (2-11) and (3-8) it is not properly invariant under SRBM.

Discussion. In these solutions the parameter γ is a normalized loading rate and the parameter κ is a normalized time increment. Figure 1 plots the exact solution as a function of the loading rate γ. From this figure it can be seen that the elastic distortional deformation is large for large values of γ. It can also be shown that in the limit that κ approaches zero all three algorithms reproduce the exact steady-state values (5-7) for all values of γ. However, for positive values of κ Algorithms 1 and 2 reproduce the exact result that ($c_1 = b_1$, $c_2 = b_2$), whereas Algorithm 3 predicts that c_3 is different from b_3.

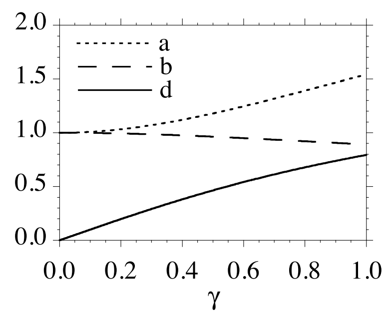

Figure 1. Simple shear: Exact steady-state solution.

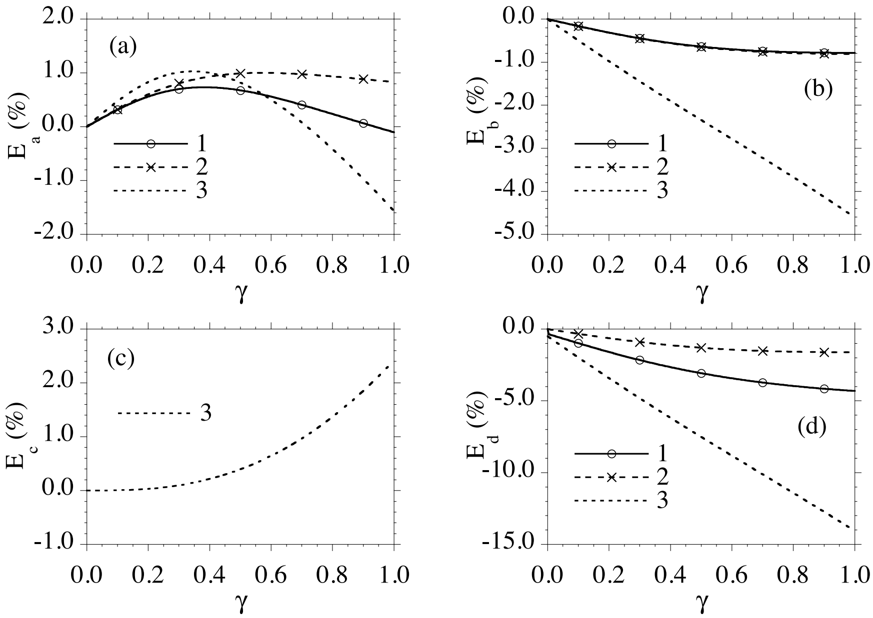

Figure 2. Simple shear: Errors in the solutions predicted by Algorithms 1, 2 and 3 for the normalized time increment $\kappa = 0.1$.

To compare the predictions of the algorithms with the exact solution it is convenient to define the errors

$$E_a = \frac{a}{a_e} - 1, \quad E_b = \frac{b}{b_e} - 1, \quad E_c = \frac{c}{c_e} - 1, \quad E_d = \frac{d}{d_e} - 1, \tag{5-14}$$

where $\{a, b, c, d\}$ are the values predicted by each of the algorithms, with the values of $\{a, c\}$ given by (5-6) for Algorithms 1 and 2. Figure 2 shows these errors as functions of γ for the relatively large normalized time increment $\kappa = 0.1$. From these results it can be seen that Algorithm 1 is slightly more accurate than Algorithm 2 for E_a. On the other hand, Algorithm 2 is more accurate than Algorithm 1 for E_d. Also, both Algorithms 1 and 2 are more accurate than Algorithm 3. Overall, it is concluded that both Algorithms 1 and 2 predict relatively robust results for the steady-state solution of simple shear at constant rate of deformation.

6. Example of steady-state isochoric extension relative to a rotating coordinate system

In order to better understand the implications of an algorithm not being strongly objective, consider the case of isochoric extension relative to a rotating coordinate system. Specifically, let $\boldsymbol{e}'_i$ be an orthonormal triad of vectors which rotates with constant angular velocity $\omega\Gamma$ about the fixed $\boldsymbol{e}_3$ direction

$$\boldsymbol{e}'_1 = \cos(\omega\Gamma t)\boldsymbol{e}_1 + \sin(\omega\Gamma t)\boldsymbol{e}_2, \quad \boldsymbol{e}'_2 = -\sin(\omega\Gamma t)\boldsymbol{e}_1 + \cos(\omega\Gamma t)\boldsymbol{e}_2, \quad \boldsymbol{e}'_3 = \boldsymbol{e}_3. \tag{6-1}$$

For this deformation field it is convenient to introduce the orthogonal tensor $\boldsymbol{Q}$ defined by

$$\begin{gathered} \boldsymbol{Q}(t) = \boldsymbol{e}'_i \otimes \boldsymbol{e}_i, \quad \dot{\boldsymbol{e}}'_i = \Omega \boldsymbol{e}'_i, \quad \Omega = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T, \quad \Omega = \omega\Gamma(-\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1), \\ \boldsymbol{Q} = \cos(\omega\Gamma t)(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + (\boldsymbol{e}_3 \otimes \boldsymbol{e}_3) + \sin(\omega\Gamma t)(-\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1), \end{gathered} \tag{6-2}$$

where the usual summation convention is used for repeated indices. Moreover, the deformation gradient $\boldsymbol{F}$ and the velocity gradient $\boldsymbol{L}$ are specified by

$$\begin{gathered}\boldsymbol{F}=\boldsymbol{Q}\big(\lambda(\boldsymbol{e}_1\otimes\boldsymbol{e}_1)+\lambda^{-1/2}(\boldsymbol{e}_2\otimes\boldsymbol{e}_2+\boldsymbol{e}_3\otimes\boldsymbol{e}_3)\big),\quad \lambda=\exp(\gamma\Gamma t),\quad J=1,\\ \boldsymbol{L}=\gamma\Gamma\,\boldsymbol{Q}\big((\boldsymbol{e}_1\otimes\boldsymbol{e}_1)-\tfrac{1}{2}(\boldsymbol{e}_2\otimes\boldsymbol{e}_2+\boldsymbol{e}_3\otimes\boldsymbol{e}_3)\big)\,\boldsymbol{Q}^T+\Omega,\end{gathered}\tag{6-3}$$

where λ is the stretch of a material line element which in the reference configuration (at $t=0$) was oriented in the $\boldsymbol{e}_1$ direction, and $\gamma\Gamma$ is the constant logarithmic stretch rate. It then follows that the relative deformation tensor associated with (6-3) is given by

$$\boldsymbol{F}_r(t_2)=\boldsymbol{Q}(t_2)\big(e^{\kappa}(\boldsymbol{e}_1\otimes\boldsymbol{e}_1)+e^{-\kappa/2}(\boldsymbol{e}_2\otimes\boldsymbol{e}_2+\boldsymbol{e}_3\otimes\boldsymbol{e}_3)\big)\,\boldsymbol{Q}^T(t_1),\quad \kappa=\Delta t\,\gamma\Gamma.\tag{6-4}$$

Moreover, since this deformation is isochoric, the tensor $\boldsymbol{F}_r$ is unimodular as was the case in (5-3).

Exact solution. The tensor $\boldsymbol{B}'_e(t)$ is a steady-state solution relative to the rotating basis $\boldsymbol{e}'_i$ of the evolution equation (1-13) provided that it satisfies the equation

$$\Omega\boldsymbol{B}'_e+\boldsymbol{B}'_e\Omega^T=\boldsymbol{L}\boldsymbol{B}'_e+\boldsymbol{B}'_e\boldsymbol{L}^T-\tfrac{2}{3}(\boldsymbol{D}\cdot\boldsymbol{I})\boldsymbol{B}'_e-\Gamma\boldsymbol{A}_p.\tag{6-5}$$

The exact solution of this algebraic equation can be written in the form

$$\boldsymbol{B}'_e(t)=a^2(\boldsymbol{e}'_1\otimes\boldsymbol{e}'_1)+\frac{1}{a}(\boldsymbol{e}'_2\otimes\boldsymbol{e}'_2+\boldsymbol{e}'_3\otimes\boldsymbol{e}'_3),\tag{6-6}$$

where a takes the value a_e given by

$$a_e=\Big(\frac{1+\gamma}{1-2\gamma}\Big)^{1/3}\quad\text{for } \gamma<\tfrac{1}{2}.\tag{6-7}$$

It can be seen from this solution that no steady-state solution of this form exists if the rate of extension is too large ($\gamma\geq\frac{1}{2}$).

Solution of Algorithm 1. The values of $\{\boldsymbol{B}'_e(t_1),\,\boldsymbol{B}'_e(t_2)\}$ correspond to a steady-state solution relative to the rotating basis $\boldsymbol{e}'_i$ of Algorithm 1 if they have the forms

$$\boldsymbol{B}'_e(t_1)=\boldsymbol{Q}(t_1)\hat{\boldsymbol{B}}'_e\boldsymbol{Q}^T(t_1),\quad \boldsymbol{B}'_e(t_2)=\boldsymbol{Q}(t_2)\hat{\boldsymbol{B}}'_e\boldsymbol{Q}^T(t_2),\quad \hat{\boldsymbol{B}}'_e=a^2(\boldsymbol{e}_1\otimes\boldsymbol{e}_1)+\frac{1}{a}(\boldsymbol{e}_2\otimes\boldsymbol{e}_2+\boldsymbol{e}_3\otimes\boldsymbol{e}_3),\tag{6-8}$$

where the stretch a needs to be determined. Next, using (1-17) and (6-4) it follows that

$$\bar{\boldsymbol{B}}'_e(t_2)=\boldsymbol{F}_r'^{-1}\boldsymbol{B}'_e(t_2)\boldsymbol{F}_r'^{-T}=\boldsymbol{Q}(t_1)\Big(a^2e^{-2\kappa}(\boldsymbol{e}_1\otimes\boldsymbol{e}_1)+\frac{e^{\kappa}}{a}(\boldsymbol{e}_2\otimes\boldsymbol{e}_2+\boldsymbol{e}_3\otimes\boldsymbol{e}_3)\Big)\,\boldsymbol{Q}^T(t_1).\tag{6-9}$$

These expressions will satisfy (4-3) provided that a takes the value a_1, which is the real positive root of the equation

$$\big(\gamma-(\gamma+\kappa)e^{-2\kappa}\big)a_1^3-\kappa(e^{\kappa}-e^{-2\kappa})a_1+\big((\gamma+\kappa)e^{\kappa}-\gamma\big)=0\tag{6-10}$$

closest to unity. It can be shown that in the limit that $\kappa\to 0$ the real positive solution of (6-10) yields the exact result (6-7) for all possible values of γ. For finite values of κ, a maximum value of γ exists beyond which the real solution of (6-10) becomes negative. Thus, the maximum value of γ which produces a physical solution is a function of the size of the time step though the value of κ.

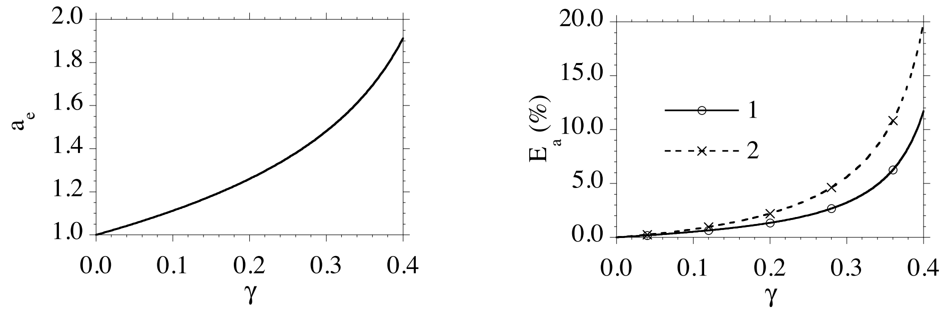

Figure 3. Isochoric extension: exact solution (left) and errors in the solutions predicted by Algorithms 1 and 2 for the normalized time increment $\kappa = 0.1$.

Solution of Algorithm 2. The values of $\{\boldsymbol{B}_e'(t_1), \boldsymbol{B}_e'(t_2)\}$ correspond to a steady-state solution relative to the rotating basis $\boldsymbol{e}_i'$ of Algorithm 2 if they have the forms (6-8). Next, using these expressions, the elastic trial value (4-6), and equating the estimate (4-9) with the deviatoric part $\boldsymbol{B}_e''(t_2)$ of $\boldsymbol{B}_e'(t_2)$ in (6-8) yields an algebraic equation for a which is solved to obtain the value a_2 given by

$$a_2 = \left(\frac{\gamma + \kappa - \gamma e^{-\kappa}}{\gamma + \kappa - \gamma e^{2\kappa}}\right)^{1/3}. \tag{6-11}$$

This solution has the same character as the solution of Algorithm 1.

Figure 3, left, plots the exact solution a_e in (6-7) as a function of the normalized stretch rate γ. The right half of the figure shows the error E_a in (5-14) predicted by Algorithms 1 and 2 for the normalized time increment $\kappa = 0.1$. From this figure is it easy to see that Algorithm 1 is more accurate than Algorithm 2. Moreover, we emphasize that both algorithms are strongly objective because the solutions (6-10) and (6-11) are uninfluenced by rate of rotation $\omega\Gamma$ of the axes $\boldsymbol{e}_i'$, which can be interpreted as a superposed rate of rigid body rotation.

Discussion of Algorithm 3. To discuss Algorithm 3 it is convenient to introduce the two tensors

$$\tilde{\boldsymbol{\Lambda}} = \boldsymbol{Q}(t_2)^T \boldsymbol{\Lambda}_W \boldsymbol{Q}(t_1), \quad \tilde{\boldsymbol{\gamma}} = \boldsymbol{Q}(t_1)^T \big(\boldsymbol{\gamma} \boldsymbol{B}_e'(t_1) + \boldsymbol{B}_e'(t_1)\boldsymbol{\gamma}\big) \boldsymbol{Q}(t_1). \tag{6-12}$$

It then follows from (4-9), (4-15) and (6-8) that

$$\begin{gathered} \tilde{\boldsymbol{B}}_e' = \boldsymbol{Q}(t_1)\big(\hat{\boldsymbol{B}}_e' + \tilde{\boldsymbol{\gamma}} - \tfrac{2}{3}(\boldsymbol{\gamma}\cdot\boldsymbol{I})\hat{\boldsymbol{B}}_e'\big)\boldsymbol{Q}^T(t_1), \\ \hat{\boldsymbol{B}}_e' - \tfrac{1}{3}(\hat{\boldsymbol{B}}_e'\cdot\boldsymbol{I})\boldsymbol{I} = \frac{1}{1+\Delta t\,\Gamma}\tilde{\boldsymbol{\Lambda}}\Big(\big(\hat{\boldsymbol{B}}_e' + \tilde{\boldsymbol{\gamma}} - \tfrac{2}{3}(\boldsymbol{\gamma}\cdot\boldsymbol{I})\hat{\boldsymbol{B}}_e'\big) - \tfrac{1}{3}\big[\big(\hat{\boldsymbol{B}}_e' + \tilde{\boldsymbol{\gamma}} - \tfrac{2}{3}(\boldsymbol{\gamma}\cdot\boldsymbol{I})\hat{\boldsymbol{B}}_e'\big)\cdot\boldsymbol{I}\big]\boldsymbol{I}\Big)\tilde{\boldsymbol{\Lambda}}^T. \end{gathered} \tag{6-13}$$

Consequently, Algorithm 3 will admit solutions of the form (6-8) if the tensors $\{\tilde{\boldsymbol{\Lambda}}, \tilde{\boldsymbol{\gamma}}\}$ are independent of the value of the angular velocity $\omega\Gamma$ and if they are diagonal tensors with respect to the fixed basis $\boldsymbol{e}_i$. Specifically, it can be shown that

$$\tilde{\boldsymbol{\Lambda}}\cdot(\boldsymbol{e}_1\otimes\boldsymbol{e}_2) = -\tilde{\boldsymbol{\Lambda}}\cdot(\boldsymbol{e}_2\otimes\boldsymbol{e}_1) = -\frac{4\cos\dfrac{\kappa}{2}\sinh^2\dfrac{\kappa}{4}\sin\dfrac{\omega\kappa}{\gamma}}{\cosh\kappa - 8\sinh^2\dfrac{\kappa}{4}\sin^2\dfrac{\omega\kappa}{2\gamma} + \cos\dfrac{\omega\kappa}{\gamma}}$$

and

$$\tilde{\boldsymbol{\gamma}} \cdot (\boldsymbol{e}_1 \otimes \boldsymbol{e}_2) = \tilde{\boldsymbol{\gamma}} \cdot (\boldsymbol{e}_2 \otimes \boldsymbol{e}_1) = \frac{2(a^3+1)\sinh\dfrac{3\kappa}{4}\sin\dfrac{\omega\kappa}{\gamma}}{a\left(\cosh\dfrac{\kappa}{4}+\cosh\dfrac{3\kappa}{4}\cos\dfrac{\omega\kappa}{\gamma}\right)}. \tag{6-14}$$

Not only are these quantities nonzero but they depend explicitly on the angular velocity $\omega\Gamma$, which is a direct consequence of the algorithm not being strongly objective.

7. Conclusions

Evolution equations based on the Jaumann derivative like (1-2) or (4-13) necessarily require three types of approximations. One for the specification of the time dependence of the spin tensor $\boldsymbol{W}$, one for the integration algorithm of the evolution equation (1-3) for the rotation tensor $\boldsymbol{\Lambda}_W$, and one for the integration algorithm of the evolution equation for an auxiliary variable like $\overline{\boldsymbol{T}}_W$ in (1-5). In contrast, when evolution equations are formulated in terms of the velocity gradient $\boldsymbol{L}$, the evolution equations (1-9) for the relative deformation gradient $\boldsymbol{F}_r$ and (1-16) for the unimodular part $\boldsymbol{F}_r'$ of the relative deformation tensor can be integrated exactly in terms of the positions of material points at the beginning and end of the time step, independently of the time dependence of $\boldsymbol{L}$ during the time step. Consequently, such integration algorithms have the advantage that approximations are limited to the algorithms for the evolution equations of the auxiliary tensors like (1-11) for $\overline{\boldsymbol{T}}_L$ and (1-18) for $\overline{\boldsymbol{B}}_e'$. Moreover, since these auxiliary tensors are unaffected by SRBM it is simple to develop algorithms which are strongly objective in the sense discussed in [Papes and Mazza 2009].

Focusing attention to the response of a simple elastic-viscoplastic material, examples of steady-state simple shear at constant shear rate and steady-state isochoric extension relative to a rotating coordinate system are considered to assess the robustness and accuracy of the three algorithms presented in Section 4. Algorithms 1 and 2 are strongly objective, but Algorithm 3, which is based on the Hughes–Winget algorithm, is only weakly incrementally objective and exhibits unphysical dependence on the arbitrary rotation tensor $\boldsymbol{Q}$ in SRBM when the rate of deformation is nonzero. Algorithms 1 and 2 give robust and relatively accurate predictions of these steady-state solutions for reasonably large time increments. From a practical point of view, Algorithm 2 is perhaps the easiest to implement. It presents an improvement over the simple formulation in [Rubin and Attia 1996] in that here the elastic trial value $\boldsymbol{B}_e'^*(t_2)$ is evaluated exactly.

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