Journal of
Mechanics of
Materials and Structures

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Volume 6, No. 1-4 January–June 2011

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 6, No. 1-4, 2011

A REMARKABLE STRUCTURE OF LEONARDO AND A HIGHER-ORDER INFINITESIMAL MECHANISM

TIBOR TARNAI AND ANDRÁS LENGYEL

Dedicated to the memory of Marie-Louise Steele

This paper is concerned with the static and kinematic behavior of two chain-like bar-and-joint assemblies which have the same topology. One is a structure which is both statically and kinematically indeterminate, and constitutes a higher-order infinitesimal mechanism. The other is a structure which is both statically and kinematically determinate, introduced by Leonardo da Vinci in the *Codex Madrid*. Proceeding along the internal joints from the bottom to the top of the assembly, the lateral components of the displacements of the internal joints of the infinitesimal mechanism show an exponential decay, and the forces in the internal bars of Leonardo's structure show an exponential growth. It is pointed out that, in the elastic model of Leonardo's structure, the propagation of displacements of internal joints and the propagation of forces in internal bars also show an exponential character in a modified form. This work also provides some hints for overcoming difficulties arising in higher-order infinitesimal mechanisms, and corrects minor mistakes made by Leonardo.

1. Introduction

Cable nets, cable domes, and tensegrity structures, from the mechanical point of view, are usually bar-and-joint assemblies that constitute infinitesimal mechanisms. The answer to the important question of whether self-stress may impart first-order stiffness to such a structure depends on whether or not it is an infinitesimal mechanism of first or higher order [Pellegrino and Calladine 1986]. (An infinitesimal mechanism is of the n-th order ($n = 1, 2, \dots$) if it involves no elongation of any bar up to and including the n-th order but exhibits an elongation of order $n + 1$ in at least one bar.)

In [Tarnai 1989], we discussed problems of definition and determination of the order of an infinitesimal mechanism, and presented, among other things, an example of a higher-order infinitesimal mechanism where the propagation of the displacements of the joints shows exponential decay. On the other hand, Nielsen [1998] called attention to a drawing of a structure by Leonardo da Vinci (see page 597) taken from the *Codex Madrid* [Leonardo 1493], which contains notes and drawings of his mechanical studies, and which was believed to be lost until the stunning announcement, in 1967, of its accidental rediscovery in the National Library in Madrid. In this structure, the propagation of forces in the bars shows an exponential growth.

In this paper, the accidental similarity between the two structures, as well as the duality between the exponential decay of the displacements and the exponential growth of the bar forces, are investigated.

Research supported by the Hungarian Scientific Research Fund (OTKA) grant no. K81146.
Keywords: bar-and-joint structure, both statically and kinematically indeterminate structure, rigidity, infinitesimal mechanism, force propagation, Leonardo da Vinci.

2. Higher-order infinitesimal mechanisms

A structure is called an infinitesimal mechanism if it has only infinitesimal free motions. Koiter defined

> "an infinitesimal mechanism of the first order by its property that any infinitesimal displacement of the mechanism is accompanied by second order elongations of at least some of the bars. An infinitesimal mechanism is called of second (or higher) order if there exists an infinitesimal motion such that no bar undergoes an elongation of lower than the third (or higher) order" [Koiter 1984].

Koiter's definition can be mathematically formulated as follows. Consider a bar-and-joint assembly which contains b bars and consider a system of infinitesimal displacements of the joints. Let us denote an infinitesimal displacement component of a characteristic joint by u and the elongation of the bar k due to u by e_k. Write the power-series expansion of e_k in u:

$$e_k = a_{1k}u + a_{2k}u^2 + a_{3k}u^3 + \cdots \quad (k = 1, 2, \ldots, b). \tag{1}$$

For an infinitesimal mechanism, there always exists a system of infinitesimal displacements of joints such that, in (1), $a_{1k} = 0$ for $k = 1, 2, \ldots, b$. Such a system of infinitesimal displacements is obtained as the solution of a set of homogeneous compatibility equations. The displacement vector coordinates depend on the chosen coordinate system but the positions of the displaced joints do not. However, it is advantageous if one axis of the coordinate system is chosen in the direction of the displacement of the characteristic joint.

Definition 1 [Tarnai 1984]. An infinitesimal mechanism is *of order at least n* ($n \geq 1$) if there exists a system of infinitesimal displacements of joints such that, in (1), $a_{1k} = a_{2k} = \cdots = a_{nk} = 0$ for $k = 1, 2, \ldots, b$. An infinitesimal mechanism is *of order n* if it is of order at least n, but not of order at least $n + 1$.

Consider, for example, the mechanism in Figure 1, left. If we move it as in the middle figure, the lengthening of the horizontal bars is quadratic in the displacement u (for small u; see next section for details). The length of the vertical bars does not change. Therefore, for this displacement, all the coefficients a_{1k} in (1) vanish, so the system has order at least 1.

However, we can also consider the displacement in Figure 1, right. Here, if the central joint moves by u keeping the length of the bars attached to it constant and the far ends of these bars along the center line, the far ends move up and down by an amount proportional to u^2, and so the length of the horizontal bars increases by an amount proportional to u^4. For this displacement, then, a_{1k}, a_{2k} and a_{3k} vanish for all k. So in fact this system has order at least 3.

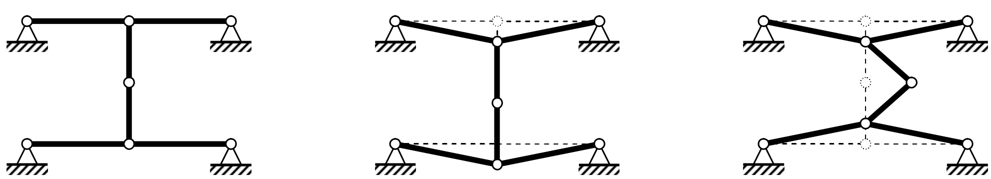

Figure 1. A two-DOF infinitesimal mechanism in the plane. Left: initial state. Middle: a displacement where the horizontal bars undergo second-order elongation. Right: a displacement where the same bars undergo fourth-order elongation.

Remark. We see from this example that, for multi-degree of freedom (DOF) infinitesimal mechanisms, displacements corresponding to different DOFs can result in different numbers n; but according to Definition 1, the maximum n will be the order of the infinitesimal mechanism. Thus, the definition can be reformulated as follows. For a system of infinitesimal displacements, consider the exponent of the first nonvanishing term in the series (1) for each bar, and take the minimum of these exponents. Then determine this minimum for each possible system of infinitesimal displacements, and take the maximum of these minima. The number obtained is one more than the order n of the infinitesimal mechanism.

Thus, in the assembly of Figure 1, we can say that the order is at least 1, based on the DOF whose displacement is illustrated in the middle diagram. But another DOF, illustrated in the diagram on the right, ensures that the order is at least 3. Since we have to take the maximum, we can declare that the assembly in Figure 1, left is a third-order infinitesimal mechanism. (It can be shown that no displacement leads to a higher order.)

In mathematics, the term "n-th order infinitesimal mechanism" is not used. Mathematicians usually deal with n-th order rigidity and n-th order flexes instead [Connelly 1980].

3. A series of infinitesimal mechanisms

Consider the bar-and-joint assembly in Figure 2a, all bars being of unit length. As shown in the figure, we denote by x the horizontal component of the displacement of joint B and by $2y$ the vertical component of the displacement of joint A. We constrain the displacements so that bars 1 and 2 undergo the same elongation, denoted by e, and bars 3 and 4 are inextensional (elongation zero). We can express the displacement component y in terms of x, and the elongation e in terms of y, as

$$y = 1 - \sqrt{1 - x^2} \quad \text{and} \quad e = \sqrt{1 + (2y)^2} - 1.$$

Taking the binomial series expansion of these functions, we get

$$y = \tfrac{1}{2}x^2 + \tfrac{1}{8}x^4 + \tfrac{1}{16}x^6 + \cdots , \tag{2}$$

$$e = \tfrac{1}{2}(2y)^2 - \tfrac{1}{8}(2y)^4 + \cdots . \tag{3}$$

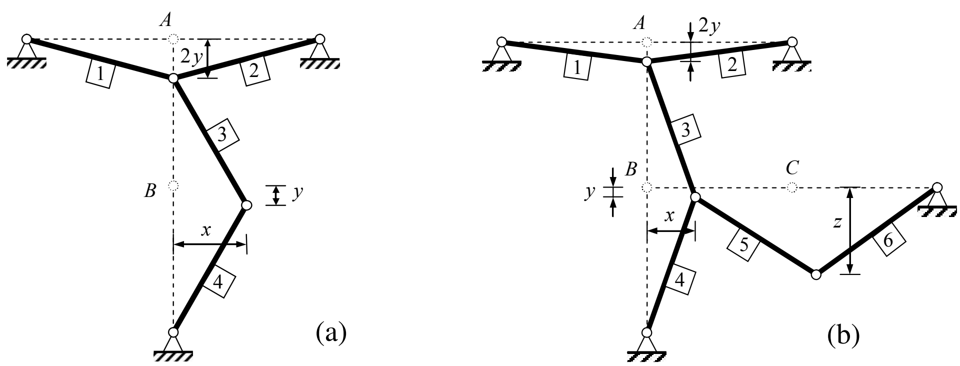

Figure 2. A planar assembly composed of (a) two and (b) three three-pinned frame units of collinear joints, and their infinitesimal free displacements.

Introduction of (2) into (3) yields

$$e = \tfrac{1}{2}x^4 + \tfrac{1}{4}x^6 + \tfrac{1}{32}x^8 + \cdots,$$

that is, fourth-order elongations arise in bars 1 and 2. Bars 3 and 4 have no elongation, by assumption. Thus, $n + 1 = 4$, and so $n = 3$: the assembly in Figure 2a is an infinitesimal mechanism of order at least 3. It can be shown that no displacement leads to a higher order, so the assembly in Figure 2a is a third-order infinitesimal mechanism. (A similar calculation applied to the rightmost configuration of the structure in Figure 1 justifies our earlier assertion that that structure has order at least 3.)

Now let us supplement the bar-and-joint assembly in Figure 2a with a horizontal three-pinned frame of collinear joints, connected to joint B, as in Figure 2b. All bars have unit length, and bars 3, 4, 5, and 6 are taken as inextensional. Let us apply a vertical displacement z at joint C. In this case, the displacement components x and y shown in Figure 2b and the elongation e of bars 1 and 2 are related by

$$x = \sqrt{1-(1-y)^2}, \qquad (z-y)^2 + \bigl(2 - x - \sqrt{1-z^2}\,\bigr)^2 = 1, \qquad e = \sqrt{1+(2y)^2} - 1.$$

Using Maple to eliminate x and y and to expand the resulting formula for e in powers of z, we obtain for the elongation of bars 1 and 2 the series

$$e = \tfrac{1}{2}z^8 + \tfrac{1}{2}z^{10} - z^{11} + \tfrac{11}{16}z^{12} + \cdots;$$

that is, eighth-order elongations arise in bars 1 and 2. Thus, the assembly in Figure 2b is an infinitesimal mechanism of order at least 7. It can be shown that no displacement leads to a higher order, so the assembly in Figure 2b is a seventh-order infinitesimal mechanism.

More generally, it can be shown [Tarnai 1989] that an the assembly composed of M three-pinned frame units with collinear joints, as in Figure 3, is an infinitesimal mechanism of order $2^M - 1$.

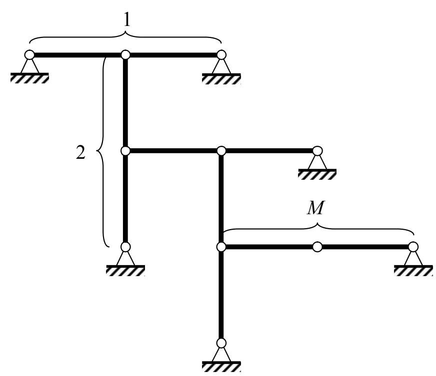

Figure 3. A planar assembly composed of M three-pinned frame units of collinear joints.

4. A chain-like infinitesimal mechanism

Let the value of M be equal to 9; thus we have the assembly in Figure 4, which is both statically and kinematically indeterminate, constituting an infinitesimal mechanism of order $n = 2^9 - 1 = 511$. We selected $M = 9$ because later we want to compare this infinitesimal mechanism to the similar structure by Leonardo, mentioned in the Section 1. Let each bar be 6 m long. We suppose that all bars are inextensional except the uppermost two, where extensions are allowed to develop freely. Let us investigate

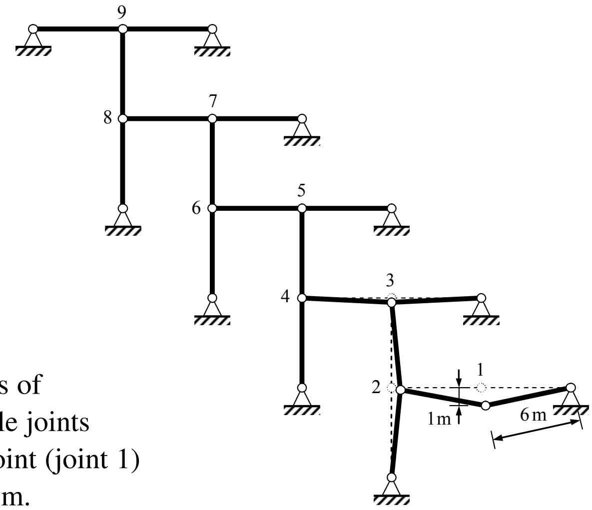

Figure 4. A chain-like planar assembly composed of nine three-pinned frame units of collinear joints, and displacements of middle joints (numbered 1 through 9) when the lowest joint (joint 1) has a prescribed vertical displacement of 1 m.

the motion of the assembly if the lowermost joint (joint 1) has a vertical displacement of 1 m. In this investigation, we apply the approximation $\sqrt{1+a} \approx 1 + \frac{1}{2}a$, and consider only the lateral displacement of the middle joint of a three-pinned frame unit, that is, the vertical displacement for a horizontal three-pinned unit and the horizontal displacement for a vertical three-pinned unit. The axial displacements of the middle joints are neglected. If the lateral displacement of joint i is δ_i then, from this, a displacement δ_{i+1} arises at joint $i+1$, and the directions of the two displacements are perpendicular to each other. According to the approximations above, we have

$$\delta_{i+1} = 2\left(6 - \sqrt{6^2 - \delta_i^2}\right),$$

which yields $\delta_{i+1} = \frac{1}{6}\delta_i^2$, that is,

$$\delta_i = \left(\tfrac{1}{6}\right)^{2^{i-1}-1} \quad i = 1, 2, \ldots, 9. \tag{4}$$

It is easy to see that, proceeding from the bottom to the top of the assembly, the displacements are *exponentially decreasing*.

It is also possible to calculate the exact position and displacement of the joints, taking into account the displacement in the axial direction in addition to that in the lateral direction. In fact, the numbered joints move along circular arcs centered at the supported joints (not numbered) to which they are connected by 6-meter bars. Once the position of joint 1 is defined, the next one can be calculated geometrically as the intersection of two circles centered at joint 1 and the next supported joint, respectively. The procedure goes on this way for all the numbered joints. The results are only slightly different from those of the approximate calculations.

The displacements δ_i obtained both by approximate calculations, according to (4), and by exact ones are given for the first few joints in Table 1. Thus the 1 m displacement of the first joint decreases to approximately 5 mm at the third joint, and 4 μm at the fourth joint. The displacement of the fifth joint is only 2 pm, less than the size of an atom. (Atomic radii lie in the range 50–200 pm.) From a practical point of view, this means that from the fourth joint on there is basically no displacement. This surprising result illustrates the most important property of higher-order infinitesimal mechanisms: mathematically

i	δ_i^{approx} (m)	δ_i^{exact} (m)
1	1	1
2	$6^{-1} = 1.667 \times 10^{-1}$	1.674×10^{-1}
3	$6^{-3} = 4.630 \times 10^{-3}$	4.674×10^{-3}
4	$6^{-7} = 3.572 \times 10^{-6}$	3.641×10^{-6}
5	$6^{-15} = 2.127 \times 10^{-12}$	2.210×10^{-12}

Table 1. Lateral displacements of the middle joints of three-pinned units of the assembly in Figure 4. (At joint i, the approximate value δ_i^{approx} and the exact value δ_i^{exact} of the lateral displacement are given).

they have only infinitesimal free motions, but physically they behave locally like finite mechanisms. The motion of mechanisms of this kind was also investigated in [Hortobágyi 2000].

5. Leonardo's structure

5.1. ***Structure in the Codex Madrid.*** On folio 75R of the Codex Madrid [Leonardo 1493], which we reproduce in Figure 5, there is a drawing of a chain-like structure composed of 12-unit-long cables. Each cable is V-shaped, and the distance between the midpoint of the cable and the straight line connecting the end points of the cable is 1 unit. The straight lines connecting the end points of the cables are alternately horizontal and vertical. (The bar-and-joint model of Leonardo's structure shown in Figure 6 is both statically and kinematically determinate, unlike the infinitesimal mechanism studied in Section 4.)

Leonardo investigated the propagation of forces in the structure if the bottom cable is loaded at its midpoint with a weight of one pound. He made an approximate analysis. He considered the cables to be inextensional and neglected the fact that a cable segment, joining another cable at its midpoint, has an inclination: he calculated the forces in the cables as if the direction of the joining cable segment were perpendicular to the straight line connecting the end points of the cable. In this way, Leonardo obtained equal forces in both segments of a cable. Let φ denote the angle of inclination of cable segments. From the geometrical data it follows that

$$\sin\varphi = \tfrac{1}{6}.$$

By resolving the forces horizontally and vertically at the node loaded by the one-pound weight, force S_1 in the lowermost cable is obtained: $S_1 = 3$ pounds. For the second cable, this member force of 3 pounds acts as a load and, because of the above-mentioned approximation, the force in the second cable is $S_2 = 3S_1$. The force in the k-th cable from below is $S_k = 3S_{k-1}$, that is,

$$S_k = 3^k \text{ pounds.}$$

Therefore, the propagation of forces in cables is *exponentially increasing*: the forces form a geometric progression with ratio 3. From the 1 pound load applied at the midpoint of the lowermost cable, a force S_9 of 19,683 pounds ($S_9 = 3^9$ pounds) arises in the uppermost cable. (Leonardo wrote 19,530 pounds for that force; indeed, we see in Figure 5 that in two steps of the progression his multiplications by 3 are slightly off: 27 → 80, instead of 81, and 2160 → 6510, instead of 6480. The reason for this is not known.)

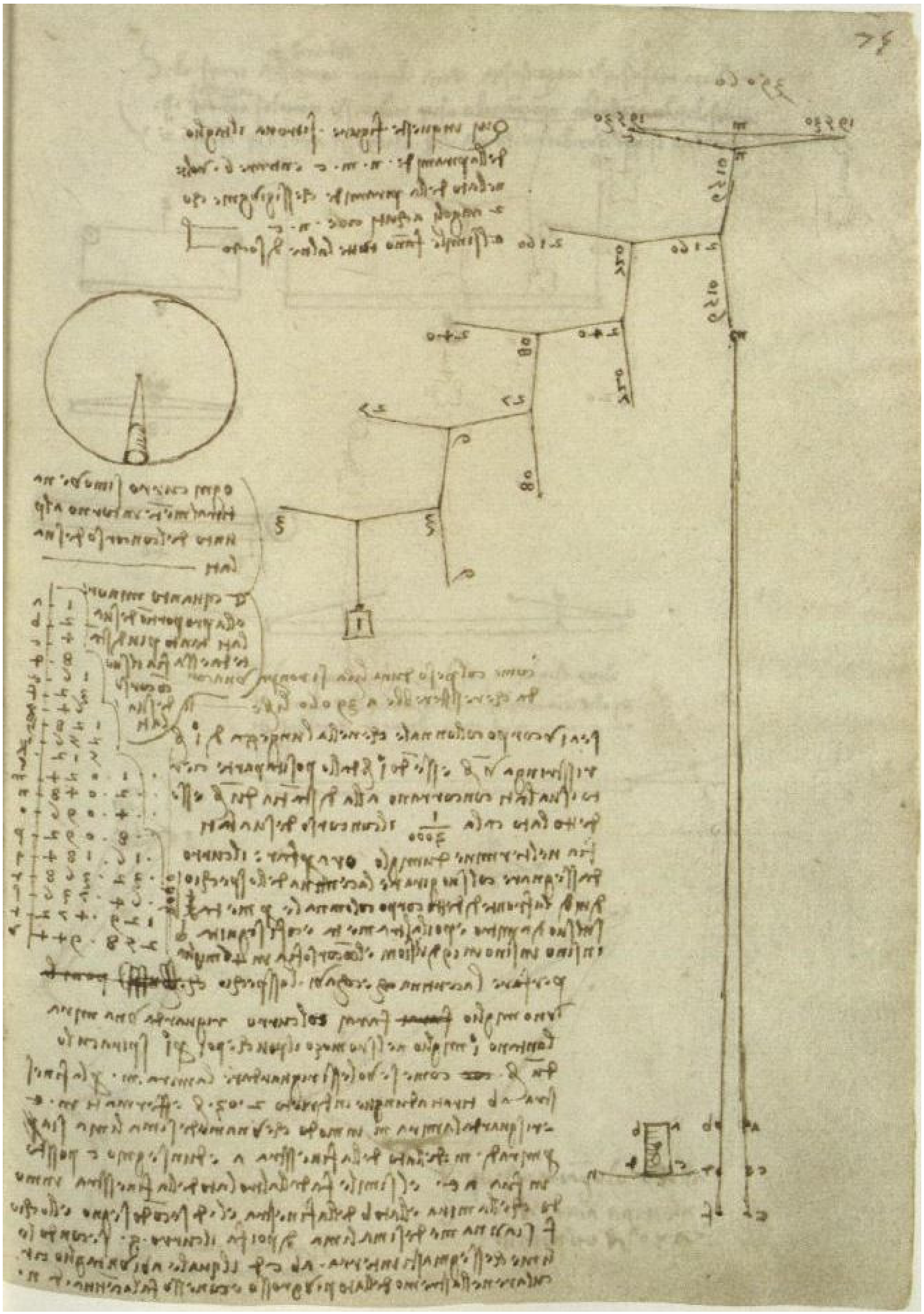

Figure 5. Leonardo da Vinci's drawing of a planar assembly [Leonardo 1493, folio 75R]. The forces in the cables, arising from a one-pound load at the lowest cable, are written on the respective cables. Leonardo, as usual, made this note in mirror image.

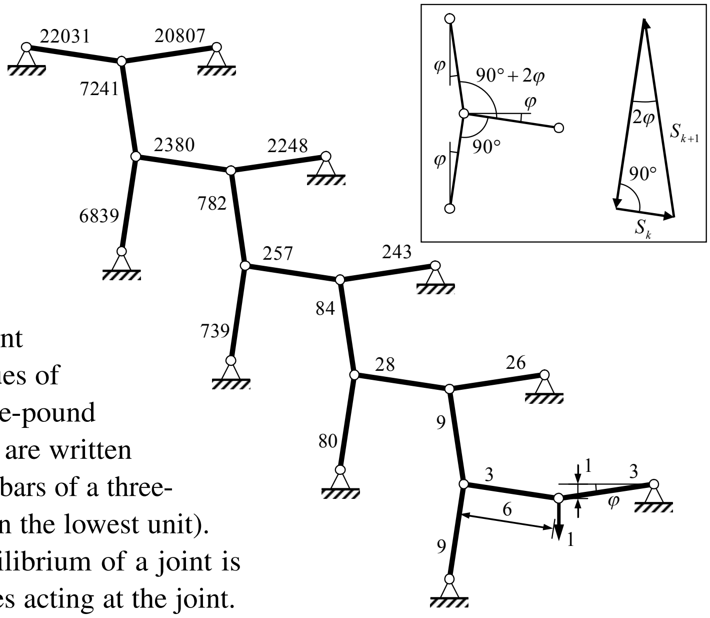

Figure 6. The inextensional bar-and-joint model of Leonardo's structure. The values of the forces in the bars, arising from a one-pound vertical load at the lowest internal joint, are written on the respective bars. Forces in the two bars of a three-pinned frame unit are different (except in the lowest unit). In the upper right-hand corner, the equilibrium of a joint is shown by the vector triangle of bar forces acting at the joint.

5.2. ***Structure with rigid members subject to unit load.*** These minor inaccuracies compel us to analyse the equilibrium of Leonardo's structure in detail. We consider all nodes in their exact positions and all members in their exact directions. We call the middle points of the three-pinned frame units internal joints and the bars connecting two internal joints internal bars. Regarding all members as infinitely rigid, one can obtain the bar forces at all nodes from a vector triangle. A typical equilibrium system is shown in the upper right-hand corner of Figure 6. The bar forces are displayed at their corresponding locations rounded to the nearest integer. Note that the exponential growth of the forces in the internal bars follows the quotient

$$\frac{S_{k+1}}{S_k} = \frac{1}{\sin 2\varphi} = \frac{1}{2\sin\varphi\cos\varphi} = \frac{1}{2\frac{1}{6}\frac{\sqrt{35}}{6}} = \frac{18}{\sqrt{35}} = 3.042555\ldots,$$

which is slightly larger than 3.

5.3. ***Structure with elastic members subject to a load.*** Leonardo's discovery that a single load at the lowermost node produces an approximately twenty thousand times magnified cable force at the top indicates that the elastic deformations of the members ought to be considered.

Let all members of length L be linearly elastic with Young's modulus E and cross-sectional area A. Let us apply a single load at the lowermost node similarly to Leonardo's drawing. We expect an exponential growth in the bar forces upwards. The elastic elongations $e_k = S_k L/(EA)$ of the members due to bar forces S_k result in increasing nodal displacements downwards in a twofold way. On the one hand, longer bars of three-pinned frame units result in larger heights, and, on the other hand, they produce a shorter base length (and consequently a larger height) in the next three-pinned frame unit attached. Thus, this double exponential change in the bar forces and the displacements implies that a surprisingly small force can produce large displacements even if the structure is quite stiff.

EA (kN)	D (mm)	$S_{9,10}$ (kN)	$\sigma_{9,10}$ (MPa)	δ_1 (m)
10^0	7.979×10^{-2}	4.410×10^{-3}	8.819×10^2	4.596×10^0
10^1	2.523×10^{-1}	1.721×10^{-2}	3.441×10^2	4.293×10^0
10^2	7.979×10^{-1}	6.306×10^{-2}	1.261×10^2	3.847×10^0
10^3	2.523×10^0	2.201×10^{-1}	4.402×10^1	3.253×10^0
10^4	7.979×10^0	7.365×10^{-1}	1.473×10^1	2.533×10^0
10^5	2.523×10^1	2.326×10^0	4.652×10^0	1.728×10^0
10^6	7.979×10^1	6.538×10^0	1.308×10^0	9.295×10^{-1}
10^7	2.523×10^2	1.427×10^1	2.853×10^{-1}	3.183×10^{-1}
10^8	7.979×10^2	2.038×10^1	4.076×10^{-2}	5.493×10^{-2}
10^9	2.523×10^3	2.184×10^1	4.368×10^{-3}	6.096×10^{-3}

Table 2. Extremal values of bar forces and nodal lateral displacements under the vertical load $F = 1$ N applied at joint 1. The cross-section diameter D, the maximal force $S_{9,10}$, the normal stress $\sigma_{9,10}$ in bar {9, 10} (according to the numbering of Figure 7), and the maximal vertical displacement δ_1 of joint 1 are given for different values of the normal stiffness EA of the bars.

In order to bring this structure closer to reality and have a better picture of its behavior, let us choose steel as our material, with Young's modulus $E = 200$ GPa, apply a small unit load ($F = 1$ N) at the lowermost node, and consider a series of different values of tensile stiffness: $EA = 10^0, 10^1, \ldots, 10^9$ kN. Let the 6 m-long bars be of circular cross-section. Table 2 shows the cross-section diameter, the largest bar force (at the top of the structure), the stress, and the largest vertical displacement (at the lowermost node) as functions of the tensile stiffness. As the tensile stiffness increases, the bar forces also increase but at a lesser rate. Consequently, the stresses and deformations decrease, resulting in smaller displacements. The larger the tensile stiffness, the more closely the force-displacement system approximates that of the infinitely rigid structure of Section 5.2. The maximal force in the table also gives a hint of the rate of the force propagation in the chain, which is less than that in the case of the rigid structure. This is because when the displacements are large the vector triangles change significantly.

However, a relatively good match (for example, in the range $EA = 10^8$–10^9 kN) can only be obtained for unrealistically large cross-sections, which means that this cable structure with ordinary cross-sections (such as $EA = 10^3$–10^4 kN) indeed undergoes large displacements even for a small load. Figure 7 shows the deformed structure with tensile stiffness $EA = 10^4$ kN. The member forces $S^F_{i,i+1}$ and the lateral components δ^F_i of the nodal displacements are listed in Table 3. The results were obtained by a large displacement iterative analysis.

According to Table 3, the forces $S^F_{i,i+1}$ in internal bars form a monotonically increasing sequence where the ratio $S^F_{i+1,i+2}/S^F_{i,i+1}$ is monotonically increasing with an increase in i ($i = 1, 2, \ldots, 8$). Thus, the bar forces show an exponential-like increase, but the change is faster than in a geometric progression. A similar observation can be made for the displacements, but in an inverse manner. The lateral components δ^F_i of the displacements of the internal joints form a monotonically decreasing sequence where the ratio $\delta^F_{i+1}/\delta^F_i$ is monotonically decreasing with an increase in i. Thus, the lateral components of the displacements show an exponential-like decrease, but the change is faster than in a geometric progression.

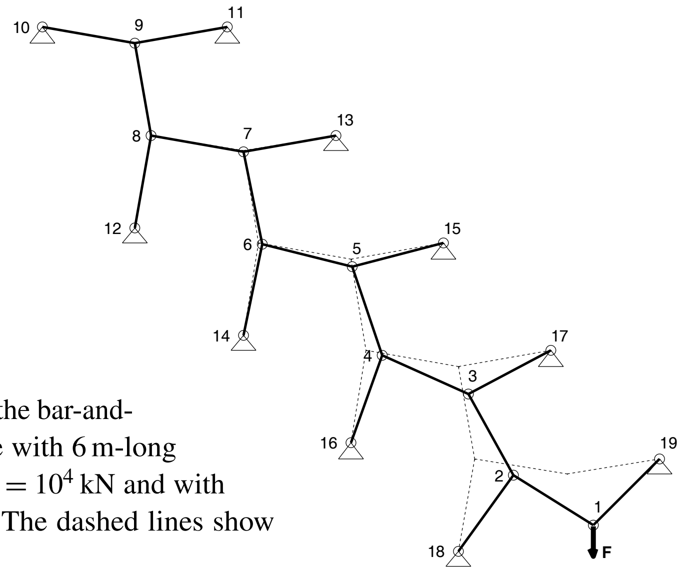

Figure 7. The equilibrium shape of the bar-and-joint model of Leonardo's structure with 6 m-long linearly elastic bars of stiffness $EA = 10^4$ kN and with a vertical load $F = 1$ N at joint 1. The dashed lines show the structure at the rest position.

$\{i, i+1\}$	$S^F_{i,i+1}$ (kN)	$S^{\Delta T}_{i,i+1}$ (kN)	i	δ^F_i (m)	$\delta^{\Delta T}_i$ (m)	$\delta^{\Delta T\,\mathrm{lim}}_i$ (m)
$\{1, 2\}$	8.884×10^{-4}	7.365×10^{-4}	1	2.533×10^{0}	-1.011×10^{0}	-1.000×10^{0}
$\{2, 3\}$	1.130×10^{-3}	2.630×10^{-3}	2	1.756×10^{0}	-1.621×10^{-1}	-1.636×10^{-1}
$\{3, 4\}$	1.804×10^{-3}	8.357×10^{-3}	3	1.040×10^{0}	-4.381×10^{-2}	-4.931×10^{-2}
$\{4, 5\}$	3.741×10^{-3}	2.568×10^{-2}	4	5.088×10^{-1}	-8.882×10^{-3}	-1.579×10^{-2}
$\{5, 6\}$	9.526×10^{-3}	7.806×10^{-2}	5	2.078×10^{-1}	2.275×10^{-3}	-5.139×10^{-3}
$\{6, 7\}$	2.707×10^{-2}	2.365×10^{-1}	6	7.509×10^{-2}	5.880×10^{-3}	-1.675×10^{-3}
$\{7, 8\}$	8.041×10^{-2}	7.154×10^{-1}	7	2.555×10^{-2}	6.931×10^{-3}	-5.409×10^{-4}
$\{8, 9\}$	2.427×10^{-1}	2.162×10^{0}	8	8.432×10^{-3}	6.864×10^{-3}	-1.684×10^{-4}
$\{9, 10\}$	7.365×10^{-1}	6.542×10^{0}	9	2.574×10^{-3}	5.595×10^{-3}	-4.255×10^{-5}

Table 3. Forces in internal bars and lateral displacements of internal joints of the structure with bar stiffness $EA = 10^4$ kN. Force $S^F_{i,i+1}$ in bar $\{i, i+1\}$, the lateral component δ^F_i of displacement of internal joint i for a vertical load $F = 1$ N applied at joint 1, force $S^{\Delta T}_{i,i+1}$ in bar $\{i, i+1\}$, and the lateral component $\delta^{\Delta T}_i$ of the displacement of internal joint i for temperature drop $\Delta T = -40°$ C are given. $\delta^{\Delta T\,\mathrm{lim}}_i$ provides the lateral component of displacement of internal joint i in the limit state due to a decrease in temperature. Downward and rightward displacement components are considered positive.

5.4. ***Thermal effect.*** We have seen that Leonardo's structure is extremely sensitive to small deformations of the members. Let us now examine the effects of the shortening of the elements contrary to the elongations due to tensile forces.

Consider a simultaneous shortening of the lengths L as a kinematic load, for example, due to uniform cooling of all bars. Denoting the thermal expansion coefficients by α [1/°C], all bars undergo elongation $e = L \cdot \alpha \cdot \Delta T$ due to a temperature rise ΔT [°C] (or shortening due to a temperature drop, as in our case).

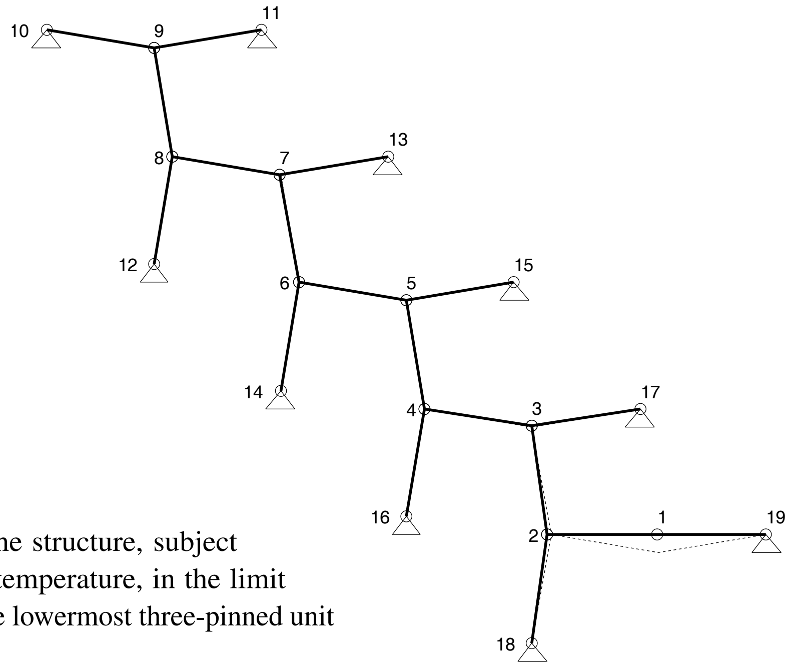

Figure 8. The shape of the structure, subject to a uniform decrease in temperature, in the limit state where the joints of the lowermost three-pinned unit become collinear.

The new position of the internal nodes can be calculated, starting with the uppermost, proceeding to the one below it, and then to the one to the right, and so on. The shortening of the bars makes the height of a three-pinned frame unit smaller, and at the same time the base length of the next unit longer. Thus in each step the nodal displacements are magnified. This phenomenon is similar to the one described in Section 5.3 but in the opposite direction. This means that the internal nodes now move towards the base of the frame units, that is, upwards and leftwards, alternately.

Note that none of the base lengths can exceed $2L$, which sets a limit on the measure of the cooling. This limit occurs at a small $e = -0.00000709091\ldots$ m shortening (which is equivalent to a $\Delta T = -0.0984849\ldots{}^\circ$ C change in temperature in the case of steel with $\alpha = 1.2 \cdot 10^{-5} 1/{}^\circ$ C). Now the joints of the last three-pinned frame unit are collinear (as shown in Figure 8). This is a limit state where the originally both statically and kinematically determinate structure becomes a statically and kinematically indeterminate structure. Until this decrease in temperature, the assembly is free of stresses.

At this limit state, the lateral displacements $\delta_i^{\Delta T\,\lim}$ of all the internal nodes are summarized in the last column of Table 3. The direction is emphasized by the use of the minus signs before the displacement values. Proceeding from the bottom to the top of the structure, the lateral displacements are exponentially decreasing, which can be easily illustrated in a semilogarithmic scale plot (Figure 9). The line in the figure is approximately linear except at the first and last nodes. The ratio of decrease is around $3.1^{-1} = 0.32\ldots$.

If the cooling continues, stresses appear in the structure. In further calculations, the elastic properties of the structure, considered in Section 5.3, should be taken into account. The same algorithm has been applied to calculate the equilibrium shape as in Section 5.3. Our calculations show that the structure basically tightens up, at this point having only tensile forces, and only minor displacements occur beyond that. The shape of the structure is similar to the one shown in Figure 8. The bar forces and the stresses naturally increase with the change in temperature but the elongations are countered with the thermal

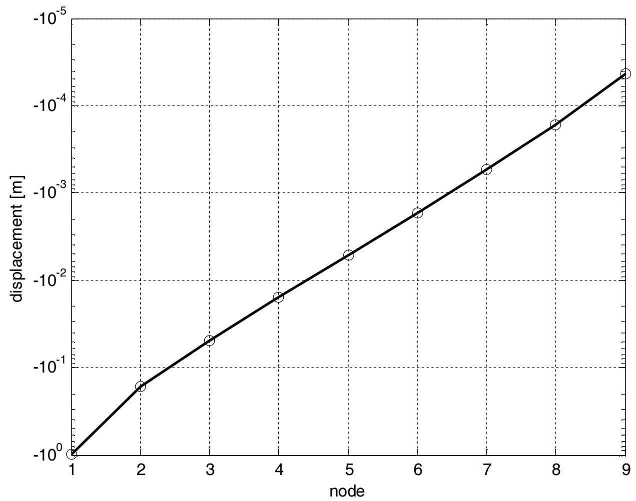

Figure 9. Lateral displacement of the internal nodes in the limit state.

shortenings. The member forces $S_{i,i+1}^{\Delta T}$ and the lateral components $\delta_i^{\Delta T}$ of the nodal displacements are listed in Table 3 for $\Delta T = -40°$ C. In accordance with our earlier notation, the positive displacement values refer to rightward or downward motion, while the negative displacement values refer to leftward or upward motion. In the progression of the forces in the internal bars, the ratio is decreasing from $3.572\ldots$ to $3.0265\ldots$; in a longer range, it is around 3.03, that is, close to the value of $3.0425\ldots$ found for the infinitely rigid structure loaded with a concentrated force at joint 1.

6. Concluding remarks

It is ascertained that, proceeding from bottom to top, the lateral components of the displacements of internal joints of the inextensional mechanism in Figure 4 show an exponential decay, where the displacements decrease much faster than a geometric progression, and the forces in the internal bars of Leonardo's infinitely rigid structure in Figure 6 show an exponential growth according to a geometric progression with common ratio $3.0425\ldots$. If the members in Leonardo's structure are linearly elastic equal bars, then elastic deformations change the structural behavior. Though the lateral components of the displacements of internal joints and the forces in internal bars show exponential-like decrease and increase, respectively, the change is not as fast as in the case of the inextensional members. In the investigated structure (with $EA = 10^4$ kN), the sequence of the lateral components of nodal displacements has a ratio monotonically decreasing from 1.4426^{-1} to 3.2761^{-1}, and the sequence of bar forces has a ratio monotonically increasing from 1.2721 to 3.0352, that is, the ratio of change in both is approximately the same. If the force unit is Newtons instead of pounds, then the force in bar {9, 10} of the infinitely rigid structure is 22,031 N (Figure 6). As a consequence of the elastic properties of the structure, however, this value is decreased to 736 N (Table 3), that is, *to approximately one thirtieth of its value*.

With numerical calculation we found that, with a decrease in the magnitude of the load F on the elastic structure, while keeping monotonicity, the ratio in the sequence of forces in the internal bars increases, and converges to $3.042555\ldots$, the common ratio of the geometric progression of bar forces in the infinitely rigid structure. For $F = 10^{-5}$ N, for instance, the ratio varies from 3.0368... to 3.042553.... The nodal displacements are less regular: the ratio monotonically decreases from 3.0336^{-1} to 3.2891^{-1}.

The determination of the order of a chain-like infinitesimal mechanism was straightforward. This can give the impression that the determination of the order of an infinitesimal mechanism is an easy task. Unfortunately, this is not the case. Connelly called attention to difficulties [Tarnai 1989; Connelly and Servatius 1994] which partly come from the definitions of higher-order infinitesimal mechanisms. One problem can be if the elongation functions are not analytic at a point, for instance, where there is a cusp in the configuration space. This particular problem can be resolved by using series expansion with fractional exponents [Gáspár and Tarnai 1994]. Another, quite general mathematical problem is the parametrization. The displacement vector of any joint can be given as a function of parameter u, but we can define a new displacement vector as a function of parameter u^2. The result of this reparametrization is that the order of the infinitesimal mechanism will be twice as large as for the original parametrization. Consequently, the order can be arbitrarily large. This problem is not yet settled. A very promising approach was presented recently by [Stachel 2007] who suggested that the definition of an n-th order infinitesimal mechanism must be based on irreducible representations of flexes. Although the notion of a *higher-order infinitesimal mechanism* is a purely geometrical term, to simplify the analysis Koiter suggested the introduction of elasticity in the bars, that is, the use of the elastic energy function of the assembly and interpretation of the infinitesimal mechanism as the buckling mode of the assembly under zero loading. Koiter's idea overcomes the difficulties mentioned by Connelly, and, indeed, was successfully applied, for instance, by [Salerno 1992].

Our analysis in Section 5.2 has shown that the forces in Leonardo's structure show an exponential growth with ratio 3.0425..., which is different from Leonardo's own value of 3. However, it is possible to introduce a modification to the original structure to fit this number (see Figure 10). Now each cable

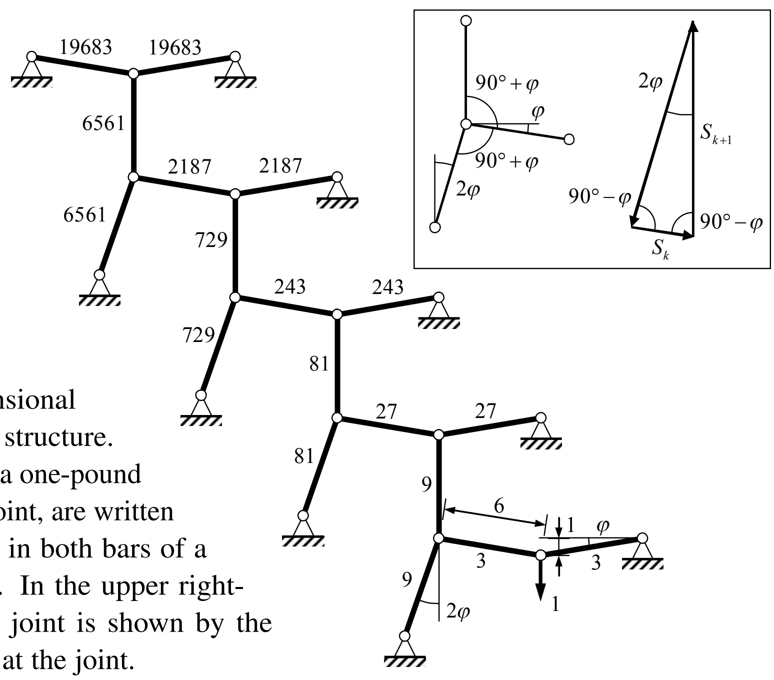

Figure 10. The modified inextensional bar-and-joint model of Leonardo's structure. The forces in the bars, arising from a one-pound vertical load at the lowest internal joint, are written on the respective bars. The forces in both bars of a three-pinned frame unit are equal. In the upper right-hand corner, the equilibrium of a joint is shown by the vector triangle of bar forces acting at the joint.

is connected to the middle point of the next one at an angle of $90° + \varphi$ in a symmetric way as shown in the inset in the upper right-hand corner.

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Received 29 Mar 2010. Revised 29 Jun 2010. Accepted 7 Jul 2010.

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